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# Modeling the out-of-equilibrium dynamics of bounded rationality and economic constraints

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**Abstract:** The mathematical analogies between economics and classical mechanics can be extended from constrained optimization to constrained dynamics by formalizing economic (constraint) forces and economic power in analogy to physical (constraint) forces in Lagrangian mechanics. In a differential-algebraic equation framework, households, firms, banks, and the government employ forces to change economic variables according to their desire and their power to assert their interest. These ex-ante forces are completed by constraint forces from unanticipated system constraints to yield the ex-post dynamics. The out-of-equilibrium model combines Keynesian concepts such as the balance sheet approach and slow adaptation of prices and quantities with bounded rationality (gradient climbing) discussed in behavioral economics and agent-based models. Depending on the power relations and adaptation speeds, the model converges to a neoclassical equilibrium or not. The framework integrates different schools of thought and overcomes some restrictions inherent to optimization approaches, such as the problem of aggregating individual behavior into macroeconomic relations and the assumption of markets operating in or close to equilibrium.

**Keywords:** Simultaneous Equation Models; Stability of Equilibrium; Balance Sheet Approach; Constrained Dynamics; Out-of-equilibrium Dynamics, Lagrangian mechanics.

**JEL:** A12; B13; C30; C62; E10; E70.

## 1. Introduction

Dynamic economic models have to describe the time evolution of stocks, flows and other variables subject to economic constraints. Models based on general equilibrium, Keynesian disequilibrium or agent-based interaction differ in their assumptions about rationality, heterogeneity and adaptation speeds within the economy (Section 2.1). Introducing a novel out-of-equilibrium framework that tries to bridge some methodological gaps between these approaches, Glötzl et al. (2019) extend the historical analogies between general equilibrium models and Newtonian physics (Section 2.2): Similar to the forces of interacting 'bodies' under constraint from Lagrangian mechanics, the modeling approach depicts the economy from the perspective of economic forces and economic power. Economic force corresponds to the desire of agents to change certain variables, while economic power captures their ability to assert their interest to change them. Optimization is replaced by a gradient seeking approach in line with bounded rationality discussed in behavioral economics. The introduction of constraint forces, i. e. forces arising from system constraints, allows for a consistent assessment of ex-ante and ex-post dynamics of the dynamical system. The model presented in section 3 is based on a Keynesian balance sheet approach in which quantities adjust gradually and prices react slowly on supply-demand mismatches. It contains two households, two production sectors with input–output relations, banks and the government. A stability analysis reveals the conditions and power relations under which convergence to the usual neoclassical equilibrium is achieved. Section 5 concludes.

## 2. Modeling dynamics subject to constraints

### 2.1. Literature review

In general, a dynamic economic model is described by $J$ agents and $I$ variables $x_i(t)$ that can correspond to any stocks or flows of commodities, resources, financial liabilities, or any other variables or parameters such as prices or interest rates. The *structure* of the model consists of $K$ economic constraints that remove many degrees of freedom. Constraints can be identities, relations "that hold by definition" (Allen, 1982, p. 4) such as the national income account identity or accounting constraints in balance sheets. In material flow analysis (Brunner and Rechberger, 2004), constraints include laws of nature such as conservation of mass and energy as 'first laws' of chemistry and thermodynamics. Input–output relations or production functions imply certain technological limitations, while budget constraints are derived from the behavioral assumption that nobody is giving away money without an equivalent remuneration. The respect for identities is "the beginning of wisdom" in economics, but they must not be "misused to imply causation" (Tobin, 1995, p. 11). To derive causal arguments, a 'closure' has to be chosen that combines individual agency and the constraints: If the $I$ variables were influenced by $I$ behavioral equations, the system of equations would be overdetermined because of the additional $K$ constraints. The schools of economic thought differ in their ways of making this system of equations solvable (Taylor, 1991), which will be discussed comparing (1) general equilibrium, (2) Keynesian disequilibrium and (3) agent-based models in the following.

In most general equilibrium models, each agent fully controls and voluntarily adapts all the stocks and flows directly affecting him (such as individual working hours or savings), resulting in various individual first-order conditions. Satisfying the $K$ system constraints of market exchange can only be guaranteed by letting $K$ prices adapt that make all the individual plans compatible with each other (neoclassical closure). Interacting via price signals, constraints imposed by other agents or system properties can be fully anticipated by the agents (Arrow and Hahn, 1971). The behavioral core of most Dynamic Stochastic General Equilibrium (DSGE) models is based on a representative agent with rational expectations that solves an intertemporal optimization problem subject to the constraints. The properties of utility and production functions, the Euler equation that describes the intertemporal trade-off, and the transversality condition as infinite time boundary condition guarantee that a unique and stable equilibrium path exists. External shocks combined with various frictions that slow down the return to equilibrium can create deviations from this optimum (Christiano et al., 2018; Lindé, 2018; Becker, 2008; Colander, 2009; Kamihigashi, 2008). While recent DSGE models also include some heterogeneity among households and firms (Kaplan et al., 2016; Christiano et al., 2018), many aspects of heterogeneity have to be left out. Galí (2018, p. 101) justifies this with “tractability”, but this is not only a question of complexity, but necessary to use this approach at all. Every optimization approach requires one single function to be optimized. Therefore, a society of utility maximizers has to be aggregated into a single social welfare function. Unfortunately, the assumptions made about individual rationality are “not enough to talk about social regularities”, but it is necessary that “macro-level assumptions … restrict the *distribution* of preferences or endowments” to guarantee a unique equilibrium (Rizvi, 1994, p. 359–63). Aggregation is possible if and only if demand is independent of the distribution of income among the agents (Gorman, 1961; Stoker, 1993; Kirman and Koch, 1986; Kirman, 1992), which Rizvi (1994, p. 363) calls an “extremely special situation”. If agents had heterogeneous rates for discounting or intertemporal substitution, this condition would not be satisfied, and no unique stable equilibrium path would exist. These mathematical reasons restrict the integration of broader heterogeneity and social influences into DSGE models.

Keynesian disequilibrium models depart from the assumption that price adaptations can clear markets sufficiently fast. Departing from equilibrium assumptions implies that the *ex-ante* (planned) behavior does not necessarily respect the economic constraints. The *ex-post* (actual) dynamics are influenced by both system constraints and the agency of others. The quantities of demand or supply do not necessarily coincide, and terms such as “forced saving” or “involuntary unemployment” (Barro and Grossman, 1971) imply that agents cannot have complete control over the variables affecting them. For example, in some Keynesian disequilibrium models quantities of voluntary exchange are rationed by the ‘short-side’: Depending on market conditions, demand is limited by insufficient supply or otherwise (Benassy, 1975; Malinvaud, 1977). In contrast, some Post-Keynesian models consider the labor market to be purely demand-led and employees have no influence on working times: The $K$ constraints that guarantee stock-flow consistency are satisfied by simply dropping $K$ behavioral equations (Godley and Lavoie, 2012; Caverzasi and Godin, 2015). This one-sided ‘drop closure’ is justified if and only if exactly $K$ stocks or flows are unaffected by agency, but only determined by the constraints (for a critique, see Richters and Glötzl, 2020).

Agent-based models (ABM) assume that individuals cannot solve infinitely dimensional optimization problems, but use bounded rationality instead. Interactions between heterogeneous agents matter beyond market prices, and social interaction, social norms, power relations or institutions influence economic choices. Compared to selfish utility maximizers, this corresponds to a broader version of methodological individualism (Gallegati and Richiardi, 2009). ABM describe how quantities and prices can converge to a (statistical) equilibrium, but also discontinuities, tipping points, lock-ins or path dependencies can be studied (Kirman, 2010). The aggregate dynamics cannot be deduced from individual behavior that is often modeled as a sequence of simple rules. ABM lack a common core, and different coordinating mechanisms such as price adaptations, auctions, matching algorithms or quantity rationing are implemented to account for the economic constraints (Tesfatsion, 2006; Gintis, 2007; Gallegati and Richiardi, 2009; Ballot et al., 2014; Riccetti et al., 2015). Unfortunately, many ABM fail to actually satisfy stock-flow consistency: For example, in the exit-entry process of firms, defaulted firms are often simply recapitalized "ex-nihilo", violating economic identities and leading to logically incoherent flows and stocks evolutions. This lead to calls for stock-flow consistent agent-based modeling (Caiani et al., 2016; Caverzasi and Russo, 2018).

In the following, this paper presents a novel out-of-equilibrium framework that tries to bridge some methodological gaps between general equilibrium, disequilibrium and agent-based models. Compared to DSGE, the model goes back two steps and does without infinite intertemporal optimization and stochastic shocks, but removes the restriction that all utility functions can be aggregated into a social welfare function. It describes the interaction of bounded rational agents that exert economic forces to improve their situation (gradient climbing) subject to the economic constraints. The simultaneous processes of trade and price adaptation may dynamically converge towards equilibrium.

### 2.2. General Constrained Dynamics framework

The paper carries on an "unfinished business" of the early neoclassicals such as Irving Fisher or Vilfredo Pareto (Leijonhufvud, 2006, pp. 26–30): Inspired by the description of stationary states in classical mechanics, they derived an economic theory of static equilibrium (Pikler, 1955; Mirowski, 1989; Grattan-Guinness, 2010; Glötzl et al., 2019). Despite some efforts, they were unable to describe analoguously the adaptive *processes* that were thought to converge to the states analyzed in *static* theory (Donzelli, 1997; McLure and Samuels, 2001; Leijonhufvud, 2006).

Glötzl (2015) and Glötzl et al. (2019) took up this old challenge, introducing an economic framework inspired by the concept of constrained dynamics from Lagrangian mechanics (Lagrange, 1788; Flannery, 2011). In the General Constrained Dynamics framework, each agent seeks to change the existing configuration in the direction of his desires, but is subject to external constraints that can typically be written as:

$$0 = Z_k(x, \dot{x}), \quad k \in \{1, \dots K\}. \tag{1}$$

The dynamics of the model are the result of *economic forces* and *economic power*: An economic force $f_{ji}$ corresponds to the desire of agent $j$ to change a certain variable $x_i$. Economic power $\mu_{ji}$ captures the ability of an agent $j$ to assert its interest to change variable $x_i$.[1] The total impact on the variable $x_i$ is the product of economic force and power $\mu_{ji} f_{ji}$, i. e. the product of *desire* and *ability*:

$$\dot{x}_i = \sum_{j=1}^{J} \mu_{ji} f_{ji}(x). \tag{2}$$

All agents are unable to calculate infinite dimensional intertemporal optimization problems based on rational expectations about the reactions of the other market participants. Instead, they base their decisions on how much to work, invest, consume or save on the observation of current marginal utilities, profits, productivities and prices. They do not jump to the point of highest utility as rational utility *maximizers*, but instead try to 'climb up the utility hill' gradually by pushing the economy in the direction of highest marginal utility. In a continuous time framework, this can be modeled by defining the forces exerted by the agents as gradients

[1] The economic power factors $\mu_{ji}$ as 'ability to change' a variable correspond to the inverse of the mass in the Newtonian equations, in which mass is the 'resistance' to a change of velocity (Estola, 2017, p. 382; Glötzl et al., 2019).

of their utility functions. This corresponds to bounded rationality described by Lindenberg (2001, p. 248) as the "general desire to improve one's condition." With this gradient seeking approach, agents still satisfy the definition of rationality by Mankiw (2008, p. 6): "A rational decision maker takes an action if and only if the marginal benefit of the action exceeds the marginal cost." One might say that the agents in the economy are as rational as shortsighted first year economics students.

To guarantee consistency, Glötzl et al. (2019) proposed a 'Lagrangian closure' based on analogies to constraint forces in physics:[2] If all the variables $x_i$ in a constraint $Z_k$ are affected by agency, additional constraint forces $z_{ki}$ are added to the time evolution of $x_i$, which together with the forces $f_{ji}$ applied by all agents with power factors $\mu_{ji}$ creates the ex-post dynamics:

$$\dot{x}_i = \sum_{j=1}^{J} \mu_{ji} f_{ji}(x) + \sum_{k=1}^{K} z_{ki}(x, \dot{x}), \tag{3}$$

$$0 = Z_k(x, \dot{x}). \tag{4}$$

The constraint forces lead to unintended deviations of the actual time evolution from the planned one. In economics, the magnitude of the constraint forces $z_{ki}$ cannot be derived from laws of nature, but reflect assumptions about adaptation processes within the economic system. In physics (Flannery, 2011; Glötzl et al., 2019), the time-dependent constraint forces $z_{ki}$ can be calculated as

$$z_{ki}(x, \dot{x}) = \lambda_k \frac{\partial Z_k}{\partial x_i}, \tag{5}$$

or, if $\partial Z_k / \partial x_i \equiv 0$, as

$$z_{ki}(x, \dot{x}) = \lambda_k \frac{\partial Z_k}{\partial \dot{x}_i}. \tag{6}$$

[2] The 'Newtonian Microeconomics' approach by Estola and Dannenberg (2012) and Estola (2017) is similar in the formalization of 'economic forces', but they accept that supply and demand differ not only ex-ante, but also ex-post (Estola, 2017, pp. 222, 386). This violation of economic identities occurs because they lack a formalization of economic constraint forces.

The additional variable $\lambda_k$ ('Lagrangian multiplier') is introduced to make the model solvable. This rule from mechanics is a plausible choice also in economics (Glötzl et al., 2019), and the static version of these constraint forces is known from optimization exercises such as maximizing $U(x_1, x_2)$ subject to a budget constraint $0 = M - p_1 x_1 - p_2 x_2$. The first order condition $0 = \frac{\partial U}{\partial x_1} - \lambda p_1$ means that the 'utility force' and the constraint force cancel out, the latter given by the derivative of the constraint with a Lagrangian multiplier $\lambda$ similar to Eqs. (5–6). The system of differential-algebraic equations (Eqs. 3–6) can be solved numerically for $x(t)$ and $\dot{x}(t)$.

Introducing this dynamic framework, Glötzl et al. (2019) presented a microeconomic Edgeworth box exchange model with two agents and two commodities and slow price adaptation that converges to the neoclassical contract curve for most parameters. Richters and Glötzl (2020) described a simple post-Keynesian stock-flow consistent disequilibrium model of the macroeconomic monetary circuit in this framework. This paper extends these ideas to a complex macroeconomic model.

## 3. The Model

### 3.1. Model structure: the constraints

The model studies the interaction of two households, two production sectors, a bank, and the government. They trade two consumer goods, labor and capital, financed by bank credit or equity. All agents show bounded rationality and try to increase their utility with a gradient climbing approach. Prices react slowly on demand–supply mismatches. As depicted in Fig. 1, the model consists of 42 economic variables:

- 11 financial balance sheet entries: $M_a$, $M_b$, $V_a$, $V_b$, $E_{bank}$, $E_{f1}$, $E_{f2}$, $D_{f1}$, $D_{f2}$, $D_g$, $V_g$,
- 4 stocks of real capital and inventories: $K_{f1}$, $K_{f2}$, $S_{f1}$, $S_{f2}$,
- 8 prices: $r_{f1}$, $r_{f2}$, $r_g$, $r_M$, $p_1$, $p_2$, $w_1$, $w_2$,
- 6 flows of labor: $L_{a1}$, $L_{a2}$, $L_{b1}$, $L_{b2}$, $L_{f1}$, $L_{f2}$,
- 8 flows of goods: $C_{a1}$, $C_{a2}$, $C_{b1}$, $C_{b2}$, $G_{g1}$, $G_{g2}$, $A_{12}$, $A_{21}$,

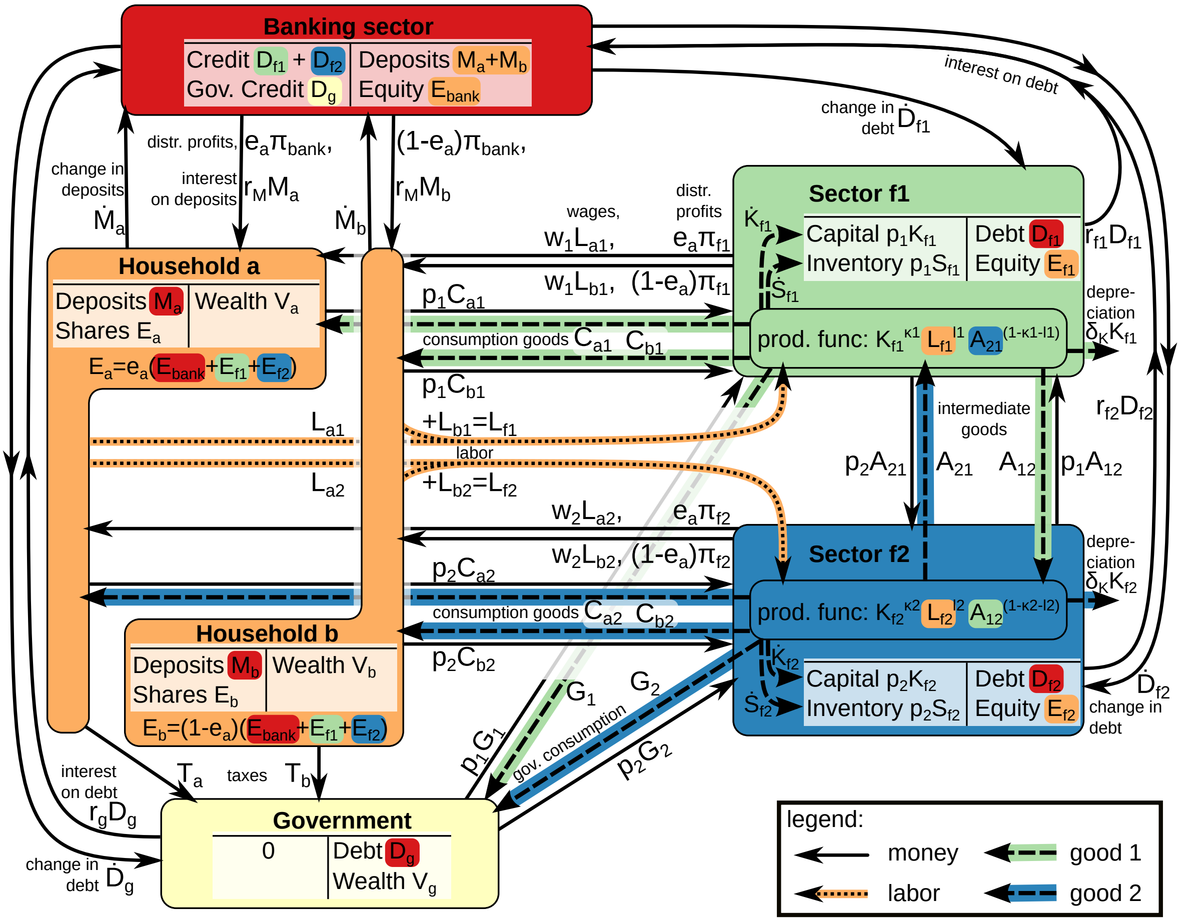


Figure 1: Model structure: The diagram depicts the balance sheets of the different sectors and the flows of money (black arrows) and goods (colored arrows) within the economy. The interconnectedness of the balance sheets is depicted by background colors: For example, the liability of sector $f1$ towards the bank has a red background, while the corresponding claim in the bank's balance sheet has a green background. Distinct flows that share an arrow are separated by commas.
The six balance sheets provide the constraints in Eqs. (7–12). Consistency of money flows provides the budget constraints (Eqs. 13–18). Eqs. (19–20) reflect consistency of labor flows, while consistency of good 1 and good 2 provides Eqs. (21–22).

- 5 flows of money: $\pi_{f1}$, $\pi_{f2}$, $\pi_{bank}$, $T_a$, $T_b$.

They are related by 16 constraints (Eqs. 7–22).

The consistency of double-entry bookkeeping in each of the six sectors provides six mathematical constraints:

$$0 = p_1(K_{f1} + S_{f1}) - D_{f1} - E_{f1}, \tag{7}$$

$$0 = p_2(K_{f2} + S_{f2}) - D_{f2} - E_{f2}, \tag{8}$$

$$0 = D_{f1} + D_{f2} + D_g - M_a - M_b - E_{bank}, \tag{9}$$

$$0 = M_a + e_a(E_{f1} + E_{f2} + E_{bank}) - V_a, \tag{10}$$

$$0 = M_b + (1 - e_a)(E_{f1} + E_{f2} + E_{bank}) - V_b, \tag{11}$$

$$0 = 0 - D_g - V_g. \tag{12}$$

The balance sheets are interconnected, because every financial claim has a corresponding liability, depicted by the colored background of the entries in Fig. 1. Household $a$ holds a fraction $e_a$ of the shares of the firm and banking sectors, while $1 - e_a$ is left for household $b$. They cannot trade their stakes in the firms. Eqs. (7–12) are used as definitions for $E_{f1}$, $E_{f2}$, $E_{bank}$, $V_a$, $V_b$ and $V_g$. Therefore, no Lagrangian multipliers are needed to guarantee consistency. Summing all these equations yields $V_a + V_b + V_g = p_1(K_{f1} + S_{f1}) + p_2(K_{f2} + S_{f2})$, thus the actual wealth consists of real stocks of capital and inventories, because the debt relations cancel out. In the following, the equations for household $b$ and sector $f2$ are provided, but explanations refer to household $a$ and sector $f1$ only.

Six budget constraints track the flow of money for each agent. Household $a$ consumes an amount $C_{a1}$ at price $p_1$ from sector $f1$ and $C_{a2}$ at price $p_2$ from sector $f2$. It works an amount $L_{a1}$ for wage $w_1$ in sector $f1$ and $L_{a2}$ for wage $w_2$ in sector $f2$, but has to pay taxes, for simplicity only on labor income, with an exogenous tax rate $\theta$: Additional to wages, it receives a share $e_a$ of the total distributed profits of firms and banks, while the deposits $M_a$ earn him a yearly interest of $r_M M_a$. The budget constraints are:

$$\begin{aligned} Z_a = 0 = \dot{M}_a + p_1 C_{a1} + p_2 C_{a2} - (1-\theta)\left(w_1 L_{a1} + w_2 L_{a2}\right) \\ \ldots - e_a(\pi_{f1} + \pi_{f2} + \pi_{bank}) - r_M M_a, \end{aligned} \tag{13}$$

$$\begin{aligned} Z_b = 0 = \dot{M}_b + p_1 C_{b1} + p_2 C_{b2} - (1-\theta)\left(w_1 L_{b1} + w_2 L_{b2}\right) \\ \ldots - (1-e_a)(\pi_{f1} + \pi_{f2} + \pi_{bank}) - r_M M_b. \end{aligned} \tag{14}$$

The government $g$ pays interest $r_g$ on government debt $D_g$ and buys goods from the two sectors $G_{g1}$ and $G_{g2}$ at price $p_1$ and $p_2$. It levies wage taxes with a constant tax rate $\theta$, which results in the following budget constraint:

$$Z_g = 0 = p_1 G_{g1} + p_2 G_{g2} - \theta\left(w_1 L_{a1} + w_2 L_{a2} + w_1 L_{b1} + w_2 L_{b2}\right) + r_g D_g - \dot{D}_g. \tag{15}$$

Sector $f1$ (and equivalently $f2$) has to pay a wage $w_1$ per unit of work, a price $p_2$ for intermediate goods $A_{21}$ used in production, and interest $r_{f1}$ on debt $D_{f1}$. Money inflows arise from selling goods at price $p_1$ to households, the government, and sector $f2$. The difference between money inflows and outflows is distributed as profits $\pi_{f1}$ or changes the stock of debt $\dot{D}_{f1}$, implying the following budget constraints:

$$Z_{f1} = 0 = w_1 L_{f1} + p_2 A_{21} + r_{f1} D_{f1} - p_1(C_{a1} + C_{b1} + G_{g1} + A_{12}) + \pi_{f1} - \dot{D}_{f1}, \tag{16}$$

$$Z_{f2} = 0 = w_2 L_{f2} + p_1 A_{12} + r_{f2} D_{f2} - p_2(C_{a2} + C_{b2} + G_{g2} + A_{21}) + \pi_{f2} - \dot{D}_{f2}. \tag{17}$$

The banking sector receives interest payments on credits and pays interest $r_M M_a$ and $r_M M_b$ to households. The difference between money inflows and outflows is distributed as profits $\pi_{bank}$ or changes the stock of equity $\dot{E}_{bank}$, implying the following budget constraint:

$$Z_{bank} = 0 = r_M\left(M_a + M_b\right) - r_{f1} D_{f1} - r_{f2} D_{f2} - r_g D_g + \pi_{bank} + \dot{E}_{bank}. \tag{18}$$

Note that the constraints $Z_a$, $Z_b$, $Z_g$, $Z_{f1}$, $Z_{f2}$ and $Z_{bank}$ are linearly dependent with the time derivative of Eq. (9) – as in every stock-flow consistent model, one budget constraint is redundant (Godley, 1999, p. 395). Consequently, Eq. (9) can be dropped and will just serve as an initial condition for $t = 0$, resulting in 15 linearly independent constraints.

Labor input $L_{f1}$ of sector $f1$ has to be identical to the amount of work in this sector by households $a$ and $b$, interrelating the variables of those agents:

$$Z_{L1} = 0 = L_{a1} + L_{b1} - L_{f1}, \tag{19}$$

$$Z_{L2} = 0 = L_{a2} + L_{b2} - L_{f2}. \tag{20}$$

As households and firms influence all these variables, these constraints cannot be treated as definitions for one variable. Consequently, constraint forces with Lagrangian multipliers $\lambda_{L1}$ and $\lambda_{L2}$ are added to the time evolution of these variables to ensure consistency. $\lambda_{L1}$ is negative if the desired change in variables would lead to ex-ante excess supply for labor in sector $f1$. It will show up as constraint force in the time evolution of $L_{a1}$, $L_{b1}$ and $L_{f1}$ (Eqs. 25, 27, 47). (Note: the index $_i$ is identical for the Lagrangian multipliers $\lambda_i$ and the corresponding constraints $Z_i$ throughout the paper).

A constraint within sector $f1$ is that total production given by a Cobb-Douglas production function depending on capital $K_{f1}$, labor $L_{f1}$ and intermediate input $A_{21}$ has to be equal to consumption by households $C_{a1} + C_{b1}$, government consumption $G_{g1}$, deliveries to sector $f2$ as intermediate goods $A_{12}$, gross investment $\delta_K K_{f1} + \dot{K}_{f1}$ and change in inventory $\dot{S}_{f1}$. Sector $f2$ is constructed symmetrically, assuming a circular-horizontal production structure.

$$Z_{P1} = 0 = {K_{f1}}^{\kappa_1} {L_{f1}}^{l_1} {A_{21}}^{1-\kappa_1-l_1} - \dot{K}_{f1} - \delta_K K_{f1} - C_{a1} - C_{b1} - G_{g1} - \dot{S}_{f1} - A_{12}, \tag{21}$$

$$Z_{P2} = 0 = {K_{f2}}^{\kappa_2} {L_{f2}}^{l_2} {A_{12}}^{1-\kappa_2-l_2} - \dot{K}_{f2} - \delta_K K_{f2} - C_{a2} - C_{b2} - G_{g2} - \dot{S}_{f2} - A_{21}. \tag{22}$$

These identities will be guaranteed by the Lagrangian multipliers $\lambda_{P1}$ and $\lambda_{P2}$.

### 3.2. Agents' behavior

Given 42 variables and 15 linearly independent constraints, only 27 behavioral equations could be chosen without the concept of Lagrangian closure. To show the flexibility of the framework, both the Lagrangian closure and the drop closure will be used for different variables. In the latter case, the behavior is implemented as an algebraic equation, not a differential equation. The model considers behavioral influences on 34 variables, which results in 49 equations for 42 variables. Therefore, 7 Lagrangian multipliers have to be added, one for each constraint in which all the variables are influenced by behavior. The following sections explain the constraints and behavioral assumptions in detail for households, government, firms and banks.

#### 3.2.1. Households

The households are assumed to derive utility from consumption and leisure. In each variable, the utility functions $U_a$ and $U_b$ satisfy the Inada conditions:[3]

$$U_{a(t)} = C_{a1(t)}{}^{\alpha_{C1}} C_{a2(t)}{}^{\alpha_{C2}} + (1 - L_{a1(t)} - L_{a2(t)})^{\alpha_L} \,, \tag{23}$$

$$U_{b(t)} = C_{b1(t)}{}^{\beta_{C1}} C_{b2(t)}{}^{\beta_{C2}} + (1 - L_{b1(t)} - L_{b2(t)})^{\beta_L} \,. \tag{24}$$

Ex-post, households' decisions must be consistent with the budget constraints (Eqs. 13–14). The constraint forces are proportional to the Lagrangian multiplier $\lambda_a$ times the derivative of the constraint $Z_a$ with respect to the particular variable (see section 2.2).

For work $L$, the derivative of the budget constraint yields $\frac{\partial Z_a}{\partial L_{a1}} = -(1-\theta)w_1$, $\frac{\partial Z_a}{\partial L_{a2}} = -(1-\theta)w_2$. Additionally, the structural equations (19–20) for labor have to be satisfied. To avoid that total labor in a sector is different from the sum of work performed by the two households in this sector, an additional constraint force is added. Following the Lagrangian closure, the constraint forces are proportional to the derivative, $\frac{\partial Z_{L1}}{\partial L_{a1}} = \frac{\partial Z_{L2}}{\partial L_{a2}} = +1$ and $\frac{\partial Z_{L1}}{\partial L_{f1}} = -1$, which implies that all these variables are adjusted by the same amount. If

[3] $U$ is strictly increasing, strictly concave, continuously differentiable and $U'(0) = \infty$ and $U'(\infty) = 0$ in every argument.

$\dot{L}_{a1} + \dot{L}_{b1} > \dot{L}_{f1}$ ex-ante, the constraint force reduces $\dot{L}_{a1}$ and $\dot{L}_{b1}$ while increasing $\dot{L}_{f1}$ until consistency is reached. Instead, a post-Keynesian economist may assume that firms' demand fully determines households' supply of labor, which illustrates that the choice of constraint forces reflects assumption about power relations within the economy. Taken together, the gradient forces from the utility function and the constraint forces yield the following time evolution:

$$\dot{L}_{a1(t)} = -\mu_{aL1} \cdot \alpha_L \left(1 - L_{a1(t)} - L_{a2(t)}\right)^{\alpha_L - 1} - \lambda_{a(t)} w_{1(t)} (1-\theta) + \lambda_{L1(t)}, \tag{25}$$

$$\dot{L}_{a2(t)} = -\mu_{aL2} \cdot \alpha_L \left(1 - L_{a1(t)} - L_{a2(t)}\right)^{\alpha_L - 1} - \lambda_{a(t)} w_{2(t)} (1-\theta) + \lambda_{L2(t)}, \tag{26}$$

$$\dot{L}_{b1(t)} = -\mu_{bL1} \cdot \beta_L \left(1 - L_{b1(t)} - L_{b2(t)}\right)^{\beta_L - 1} - \lambda_{b(t)} w_{1(t)} (1-\theta) + \lambda_{L1(t)}, \tag{27}$$

$$\dot{L}_{b2(t)} = -\mu_{bL2} \cdot \beta_L \left(1 - L_{b1(t)} - L_{b2(t)}\right)^{\beta_L - 1} - \lambda_{b(t)} w_{2(t)} (1-\theta) + \lambda_{L2(t)}. \tag{28}$$

For consumer goods, Eqs. (21–22) have to be satisfied, guaranteeing that goods produced are identical to those consumed, invested, stored or delivered to the other sector. Any ex-ante mismatch is compensated by adding constraint forces with factor $\frac{\partial Z_{P1}}{\partial C_{a1}} = \frac{\partial Z_{P2}}{\partial C_{a2}} = -1$ and Lagrangian multipliers $\lambda_{P1}$ and $\lambda_{P2}$ to the equation of motion. The derivative of the budget constraint yields $\frac{\partial Z_a}{\partial C_{a1}} = p_1$, $\frac{\partial Z_a}{\partial C_{a2}} = p_2$. The time evolution is given by:

$$\dot{C}_{a1(t)} = \mu_{aC1} \cdot \alpha_{C1} C_{a1(t)}{}^{\alpha_{C1}-1} C_{a2(t)}{}^{\alpha_{C2}} + \lambda_{a(t)} p_{1(t)} - \lambda_{P1(t)}, \tag{29}$$

$$\dot{C}_{a2(t)} = \mu_{aC2} \cdot \alpha_{C2} C_{a1(t)}{}^{\alpha_{C1}} C_{a2(t)}{}^{\alpha_{C2}-1} + \lambda_{a(t)} p_{2(t)} - \lambda_{P2(t)}, \tag{30}$$

$$\dot{C}_{b1(t)} = \mu_{bC1} \cdot \beta_{C1} C_{b1(t)}{}^{\beta_{C1}-1} C_{b2(t)}{}^{\beta_{C2}} + \lambda_{b(t)} p_{1(t)} - \lambda_{P1(t)}, \tag{31}$$

$$\dot{C}_{b2(t)} = \mu_{bC2} \cdot C_{b1(t)}{}^{\beta_{C1}} \beta_{C2} C_{b2(t)}{}^{\beta_{C2}-1} + \lambda_{b(t)} p_{2(t)} - \lambda_{P2(t)}. \tag{32}$$

An extension to 'positional' or 'conspicuous' consumption (Stiglitz, 2008; Dutt, 2009) could be modeled by adding a positive influence of household $b$ on consumption decisions by household $a$.

The desired change in deposits held by households $\dot{M}_a$ and $\dot{M}_b$ reflects an intertemporal choice, but note that the bounded rational households cannot solve infinite optimization problems. We assume that households value additional saving by the possible gain in leisure after a short period of time discounted by a factor $\rho_a$, at an average wage $(1-\theta)(w_1+w_2)/2$. Combining this behavioral force with with power factor $\mu_{aM}$ and the constraint force from the budget constraint with factor $\frac{\partial Z_a}{\partial M_a}=1$ leads to:

$$\dot{M}_{a(t)} = \mu_{aM}(1+\alpha_r(r_{M(t)}-\rho_a))\frac{2\alpha_L\left(1-L_{a1(t)}-L_{a2(t)}\right)^{\alpha_L-1}}{(1-\theta)(w_{1(t)}+w_{2(t)})} + \lambda_{a(t)}, \tag{33}$$

$$\dot{M}_{b(t)} = \mu_{bM}(1+\beta_r(r_{M(t)}-\rho_b))\frac{2\beta_L\left(1-L_{b1(t)}-L_{b2(t)}\right)^{\beta_L-1}}{(1-\theta)(w_{1(t)}+w_{2(t)})} + \lambda_{b(t)}. \tag{34}$$

The parameter $\alpha_r$ captures how strongly household $a$ considers this intertemporal choice. For an alternative specification with a simple 'money in the utility function' approach (Sidrauski, 1967), see Richters and Glötzl (2020).

### 3.2.2. Government

In this simple model, the government does not own assets or accumulates a stock of capital, but simply finances government consumption by tax income and debt. The government derives utility from buying goods and has a disutility that grows with government debt. The following utility function is chosen to illustrate that a utility function independent from the households choice can be specified, because there is no need to aggregate the individual utilities to a social welfare function before solving the model:

$$U_{g(t)} = {G_{g1(t)}}^{\gamma_{G1}} + {G_{g2(t)}}^{\gamma_{G2}} - \gamma_D(1+\gamma_r r_{g(t)})\left(\frac{D_{g(t)}}{p_{1(t)}+p_{2(t)}}\right)^2. \tag{35}$$

Further assets and roles for the government such as redistribution, market stabilization or provision of public goods may be implemented in the future.

The government tries to improve its utility. Together with constraint forces proportional to $\frac{\partial Z_g}{\partial G_{g1}}$, $\frac{\partial Z_g}{\partial G_{g2}}$ and $\frac{\partial Z_g}{\partial D_g}$, this yields:

$$\dot{D}_{g(t)} = -\mu_{gD}\cdot\gamma_D(1+\gamma_r r_{g(t)})\frac{2D_{g(t)}}{p_{1(t)}+p_{2(t)}} - \lambda_{g(t)}, \tag{36}$$

$$\dot{G}_{g1(t)} = \mu_{gG1} \cdot \gamma_{G1} G_{g1(t)}{}^{\gamma_{G1}-1} + \lambda_{g(t)} p_{1(t)} - \lambda_{P1(t)}, \tag{37}$$

$$\dot{G}_{g2(t)} = \mu_{gG2} \cdot \gamma_{G2} G_{g2(t)}{}^{\gamma_{G2}-1} + \lambda_{g(t)} p_{2(t)} - \lambda_{P2(t)}. \tag{38}$$

As in Eqs. (29–32) for households, the constraint forces $\lambda_{P1}$ and $\lambda_{P2}$ correspond to ex-ante mismatches of supply and demand for goods.

As discussed above, the government sets taxation as proportional to labor income:

$$0 = \theta \left( w_{1(t)} L_{a1(t)} + w_{2(t)} L_{a2(t)} \right) - T_{a(t)}, \tag{39}$$

$$0 = \theta \left( w_{1(t)} L_{b1(t)} + w_{2(t)} L_{b2(t)} \right) - T_{b(t)}. \tag{40}$$

These algebraic equations are equivalent to adding summands $(\theta(w_1 L_{a1} + w_2 L_{a2}) - T_a)^2$ and $(\theta(w_1 L_{b1} + w_2 L_{b2}) - T_b)^2$ to the utility function $U_g$, and this 'desire' being pursued with infinite power $\mu_{gT}$ (see Richters and Glötzl, 2020).

### 3.2.3. Firms

The firms in sector $f1$ hold inventories $S_{f1}$ that act as a buffer stock against unexpected changes in demand. From a modeling perspective, these buffer stocks are important as they avoid the system of equations to become stiff and unsolvable.

Sector $f1$ produces consumption goods for households $C_{a1} + C_{b1}$ and the government $G_{g1}$, change in inventories $\dot{S}_{f1}$, intermediate goods $A_{12}$ to be bought by sector $f2$, and gross investment consisting of replacement investment compensating depreciation $\delta_K K_{f1}$ and net investment $\dot{K}_{f1}$. For tractability, firms' production $P_{f1(t)}$ is given by a Cobb-Douglas function with production inputs capital $K_{f1}$, labor $L_{f1}$ and intermediate goods $A_{21}$, see Eqs. (21–22)

The behavior of firms consists of an inventory and dividend policy, and the goal to increase their profits. The targeted ratio $s_{f1}^{\top}$ of inventories to expected sales (gross investment plus sales to consumers, government, and sector $f2$) is constant. The firms exert a force linearly increasing with the mismatch between targeted and actual inventories. Similar to Eqs. (29–32), a constraint force proportional to $\lambda_{P1}$ with factor $\frac{\partial Z_{P1}}{\partial S_{f1}} = -1$ has to be added, assuming that every part of demand will be negatively affected if ex-ante demand is bigger than ex-ante supply, to guarantee ex-post consistency.

$$\begin{aligned}\dot{S}_{f1(t)} &= \mu_{fS1}\left(s_{f1}^{\top}\left(C_{a1(t)} + C_{b1(t)} + G_{g1(t)} + A_{12(t)} + \delta_K K_{f1(t)} + \dot{K}_{f1(t)}\right) - S_{f1(t)}\right) - \lambda_{P1(t)} \\ &= \mu_{fS1}\left(s_{f1}^{\top}\left({K_{f1(t)}}^{\kappa_1} {L_{f1(t)}}^{l_1} {A_{21(t)}}^{1-\kappa_1-l_1} - \dot{S}_{f1(t)}\right) - S_{f1(t)}\right) - \lambda_{P1(t)}, \end{aligned} \tag{41}$$

$$\dot{S}_{f2(t)} = \mu_{fS2}\left(s_{f2}^{\top}\left({K_{f2(t)}}^{\kappa_2} {L_{f2(t)}}^{l_2} {A_{12(t)}}^{1-\kappa_2-l_2} - \dot{S}_{f2(t)}\right) - S_{f2(t)}\right) - \lambda_{P2(t)}. \tag{42}$$

Concerning the production factors, firms exert forces as gradients of their expected profits as 'utility functions' $U_{f1}$ and $U_{f2}$. Increasing production is costly not only because of direct inputs, but also because additional inventories according to Eqs. (41–42) have to be financed by credit:

$$\begin{aligned} U_{f1} = {} & p_1 {K_{f1}}^{\kappa_1} {L_{f1}}^{l_1} {A_{21}}^{1-\kappa_1-l_1} - p_1 \delta_K K_{f1} - p_2 A_{21} - w_1 L_{f1} \\ & \ldots - r_{f1} p_1 \left(K_{f1} + s_{f1}^{\top}\left({K_{f1}}^{\kappa_1} {L_{f1}}^{l_1} {A_{21}}^{1-\kappa_1-l_1} - \dot{S}_{f1}\right)\right), \end{aligned} \tag{43}$$

$$\begin{aligned} U_{f2} = {} & p_2 {K_{f2}}^{\kappa_2} {L_{f2}}^{l_2} {A_{12}}^{1-\kappa_2-l_2} - p_2 \delta_K K_{f2} - p_1 A_{12} - w_2 L_{f2} \\ & \ldots - r_{f2} p_2 \left(K_{f2} + s_{f2}^{\top}\left({K_{f2}}^{\kappa_2} {L_{f2}}^{l_2} {A_{12}}^{1-\kappa_2-l_2} - \dot{S}_{f2}\right)\right). \end{aligned} \tag{44}$$

Taking profits as basis for decision-making is similar to optimization approaches, but the difference is that firms do not jump directly to the point of highest profits by fully anticipating the reactions of households to changes in goods prices or wages. Instead, firms try to increase their profits using a gradient seeking approach, only fully aware of the current marginal productivities and prices without any expectation about future sales. The time evolution of the input factors consists of these profit driven forces and an additional constraint force with Lagrangian multiplier $\lambda_{P1}$ to satisfy the production equations (21–22) ex-post.

For capital, the economic force exerted by the firms is given by $\mu_{fK1} \frac{\partial U_{f1}}{\partial K_{f1}}$, while the prefactor for the Lagrangian multiplier is calculated as $\frac{\partial Z_{P1}}{\partial K_{f1}}$:

$$\begin{aligned} \dot{K}_{f1(t)} = {} & \mu_{fK1} \cdot p_{1(t)} \left(\left(1 - r_{f1(t)} s_{f1}^{\top}\right) \kappa_1 {K_{f1(t)}}^{\kappa_1 - 1} {L_{f1(t)}}^{l_1} {A_{21(t)}}^{1-\kappa_1-l_1} - \delta_K - r_{f1(t)}\right) \\ & \ldots + \lambda_{P1(t)} \kappa_1 {K_{f1(t)}}^{\kappa_1 - 1} {L_{f1(t)}}^{l_1} {A_{21(t)}}^{1-\kappa_1-l_1}, \end{aligned} \tag{45}$$

$$\dot{K}_{f2(t)} = \mu_{fK2} \cdot p_{2(t)} \left( \left(1 - r_{f2(t)} s_{f2}^{\top}\right) \kappa_2 K_{f2(t)}{}^{\kappa_2 - 1} L_{f2(t)}{}^{l_2} A_{12(t)}{}^{1-\kappa_2-l_2} - \delta_K - r_{f2(t)} \right) \\ \ldots + \lambda_{P2(t)} \kappa_2 K_{f2(t)}{}^{\kappa_2-1} L_{f2(t)}{}^{l_2} A_{12(t)}{}^{1-\kappa_2-l_2}. \tag{46}$$

Note that total investment is given by $\dot{K}_{f1} + \delta_K K_{f1}$.

The time evolution of labor demand of firms contains an additional constraint force $-\lambda_{L1}$, added to guarantee consistency with labor supply by households according to Eqs. (19–20).

$$\dot{L}_{f1(t)} = \mu_{fL1} \left( p_{1(t)} \left(1 - r_{f1(t)} s_{f1}^{\top}\right) l_1 K_{f1(t)}{}^{\kappa_1} L_{f1(t)}{}^{l_1-1} A_{21(t)}{}^{1-\kappa_1-l_1} - w_{1(t)} \right) \\ \ldots + \lambda_{P1(t)} l_1 K_{f1(t)}{}^{\kappa_1} L_{f1(t)}{}^{l_1-1} A_{21(t)}{}^{1-\kappa_1-l_1} - \lambda_{L1(t)}, \tag{47}$$

$$\dot{L}_{f2(t)} = \mu_{fL2} \left( p_{2(t)} \left(1 - r_{f2(t)} s_{f2}^{\top}\right) l_2 K_{f2(t)}{}^{\kappa_2} L_{f2(t)}{}^{l_2-1} A_{12(t)}{}^{1-\kappa_2-l_2} - w_{2(t)} \right) \\ \ldots + \lambda_{P2(t)} l_2 K_{f2(t)}{}^{\kappa_2} L_{f2(t)}{}^{l_2-1} A_{12(t)}{}^{1-\kappa_2-l_2} - \lambda_{L2(t)}. \tag{48}$$

If labor is cheap compared to its contribution to production, the labor input is increased, but not instantaneously, and the constraint forces can lead to deviations from this plan.

For intermediate goods $A_{21}$ produced by sector $f2$ and used by sector $f1$, the time evolution contains an additional term $\lambda_{P2}$ with factor $\frac{\partial Z_{P2}}{\partial A_{21}} = -1$ because sector $f1$ is affected if there is insufficient production in sector $f2$:

$$\dot{A}_{21(t)} = \mu_{fA1} \left( p_{1(t)} \left(1 - r_{f1(t)} s_{f1}^{\top}\right) (1 - \kappa_1 - l_1) K_{f1(t)}{}^{\kappa_1} L_{f1(t)}{}^{l_1} A_{21(t)}{}^{-\kappa_1-l_1} - p_{2(t)} \right) \\ \ldots + \lambda_{P1(t)} (1 - \kappa_1 - l_1) K_{f1(t)}{}^{\kappa_1} L_{f1(t)}{}^{l_1} A_{21(t)}{}^{1-\kappa_1-l_1} - \lambda_{P2(t)}, \tag{49}$$

$$\dot{A}_{12(t)} = \mu_{fA2} \left( p_{2(t)} \left(1 - r_{f2(t)} s_{f2}^{\top}\right) (1 - \kappa_2 - l_2) K_{f2(t)}{}^{\kappa_2} L_{f2(t)}{}^{l_2} A_{12(t)}{}^{-\kappa_2-l_2} - p_{1(t)} \right) \\ \ldots + \lambda_{P2(t)} (1 - \kappa_2 - l_2) K_{f2(t)}{}^{\kappa_2} L_{f2(t)}{}^{l_2} A_{12(t)}{}^{1-\kappa_2-l_2} - \lambda_{P1(t)}. \tag{50}$$

The dividend policy is such that distributed profits $\pi_{f1}$ and $\pi_{f2}$ are total production minus input costs, which implies that changes in value of existing capital are not distributed:

$$0 = p_{1(t)} K_{f1(t)}{}^{\kappa_1} L_{f1(t)}{}^{l_1} A_{21(t)}{}^{1-\kappa_1-l_1} - p_{1(t)} \delta_K K_{f1(t)} - p_{2(t)} A_{21(t)} - w_{1(t)} L_{f1(t)} \\ \ldots - r_{f1(t)} D_{f1(t)} - \pi_{f1(t)}, \tag{51}$$

$$0 = p_{2(t)} K_{f2(t)}^{\kappa_2} L_{f2(t)}^{l_2} A_{12(t)}^{1-\kappa_2-l_2} - p_{2(t)} \delta_K K_{f2(t)} - p_{1(t)} A_{12(t)} - w_{2(t)} L_{f2(t)} \\ \ldots - r_{f2(t)} D_{f2(t)} - \pi_{f2(t)}. \tag{52}$$

This is an example of a behavioral equation implemented as an algebraic equation, implying that $\pi_{f1}$ and $\pi_{f2}$ are *not* influenced by constraint forces. Alternatively, a principal–agent dilemma could be modeled by incorporating individual forces of shareholders trying to increase dividends while the management may favor retained earnings (La Porta et al., 2000). Using the accounting and budget constraints in Eqs. (7–8, 16–17), the time evolution of debt and equity can be calculated to be:

$$\dot{D}_{f1(t)} = p_{1(t)} (\dot{K}_{f1(t)} + \dot{S}_{f1(t)}), \tag{53}$$

$$\dot{D}_{f2(t)} = p_{2(t)} (\dot{K}_{f2(t)} + \dot{S}_{f2(t)}), \tag{54}$$

$$\dot{E}_{f1(t)} = \dot{p}_{1(t)} (K_{f1(t)} + S_{f1(t)}), \tag{55}$$

$$\dot{E}_{f2(t)} = \dot{p}_{2(t)} (K_{f2(t)} + S_{f2(t)}). \tag{56}$$

Thus new investment is financed by credit, while changes in value of existing capital changes the equity of firms: $D_{f1}$, $D_{f2}$, $E_{f1}$ and $E_{f1}$ adapt to satisfy the constraints and no Lagrangian multipliers $\lambda_{f1}$ and $\lambda_{f2}$ are necessary to guarantee consistency. Using these assumptions, there is no feedback from net worth on costs or volumes of external finance.

### 3.2.4. Banking sector

The balance sheet and budget constraint of the banking sector are Eqs. (9, 18). Banks are rather passive actors in this model: They lend money 'on demand' at the current interest rates $r_{f1}$, $r_{f2}$ and $r_g$ to firms and the government in line with the concepts of endogenous money creation (see Wray, 1990; Gross and Siebenbrunner, 2019). They pay interest $r_M M_a$ and $r_M M_b$ to households and distribute all their profits $\pi_{bank}$ to the two households, here implemented as an algebraic equation:

$$0 = r_{f1(t)} D_{f1(t)} + r_{f2(t)} D_{f2(t)} + r_{g(t)} D_{g(t)} - r_{M(t)} \left(M_{a(t)} + M_{b(t)}\right) - \pi_{bank(t)}. \tag{57}$$

A richer behavioral model of banks that includes credit rationing or agency costs may be integrated in the future.

### 3.2.5. Price development

The prices react to ex-ante mismatches between supply and demand. If the agents' plans would increase demand stronger than supply, the firms realize that they are unable to change their inventories as desired, which is the case if the Lagrangian multiplier $\lambda_{P1} > 0$. Sector $f1$ slowly increases the price $p_1$ with a linear reaction function to differences between the ex-ante values of supply and demand:

$$\dot{p}_{1(t)} = \mu_{p_1} \lambda_{P1(t)}, \tag{58}$$

$$\dot{p}_{2(t)} = \mu_{p_2} \lambda_{P2(t)}. \tag{59}$$

Similarly, the wages react on a mismatch between ex-ante supply and demand for labor:

$$\dot{w}_{1(t)} = \mu_w \lambda_{L1(t)}, \tag{60}$$

$$\dot{w}_{2(t)} = \mu_w \lambda_{L2(t)}. \tag{61}$$

The interest rates are adapted by the central bank according following a simple inflation targeting rule: If the average price change is above a target $\rho^\top$, interest rates are increased: As the cost for investment are proportional to the price level (Eqs. 53–54), this effectively increases the real interest rate for firms, thus the Taylor principle (Davig and Leeper, 2007) is satisfied.

$$\dot{r}_g = \dot{r}_{f1} = \dot{r}_{f2} = \dot{r}_M = \mu_r \left( \frac{\dot{p}_1/p_1 + \dot{p}_2/p_2}{2} - \rho^\top \right). \tag{62}$$

All the parameters $\mu$ reflect assumptions about power relations and adaptation speeds within the economy.

### 3.3. Time evolution and stationary states

The initial conditions have to satisfy the six balance sheet constraints (Eqs. 7–12), the two labor constraints (Eqs. 19–20) and the five algebraic equations for taxation (Eqs. 39–40) and profit distribution (Eqs. 51–52, 57). No further equilibrium conditions are presupposed.

As an example, Fig. 2 shows the time evolution for the initial conditions, power factors and further parameters summarized in Appendix A. At $t = 0$, plot (c) shows that for household $b$, the marginal utility of leisure divided by the wage $\frac{\partial U_b}{L_{b1}} \frac{1}{(1-\theta)w_1}$ is higher than the marginal utility of consuming good 1 divided by its price $\frac{\partial U_b}{C_{b1}} \frac{1}{p_1}$, and for good $C_{b2}$ this value is even lower. Therefore, the forces of household $b$ try to push the economy towards reducing work and consuming less, particularly of good 2. For household $a$, $\frac{\partial U_a}{C_{a2}} \frac{1}{p_2}$ is higher than $\frac{\partial U_a}{C_{a1}} \frac{1}{p_1}$, thus his forces try increase consumption of good $C_{a2}$ compared to $C_{a1}$. Plot (f) compares the marginal productivities of inputs divided by their respective price. At $t = 0$, the marginal productivity of intermediate inputs $A_{21}$ is high compared to the price $p_2$, while the marginal productivity of capital is lower than the interest rate. This is the reason why profits per equity $\pi_{f1}/E_{f1}$ in sector $f1$ are very low, see plot (d). To improve profits, this sector exerts forces to increase $A_{21}$ and to reduce $K_{f1}$. Sector $f2$ is in the opposite situation. The time evolution created by these ex-ante forces would not satisfy the constraints. For example, the changes in demand and supply for good 1 create a tendency of excess demand ($\lambda_{P1} > 0$). The corresponding constraint forces influence the dynamics such that the constraints are satisfied ex-post. Additionally, the price $p_1$ increases according to Eq. (58), while there is a tendency for excess supply for good 2, leading to a negative slope of $p_2$. The adjustment processes for quantities and prices ultimately converge to a stationary state whose properties can be calculated analytically.

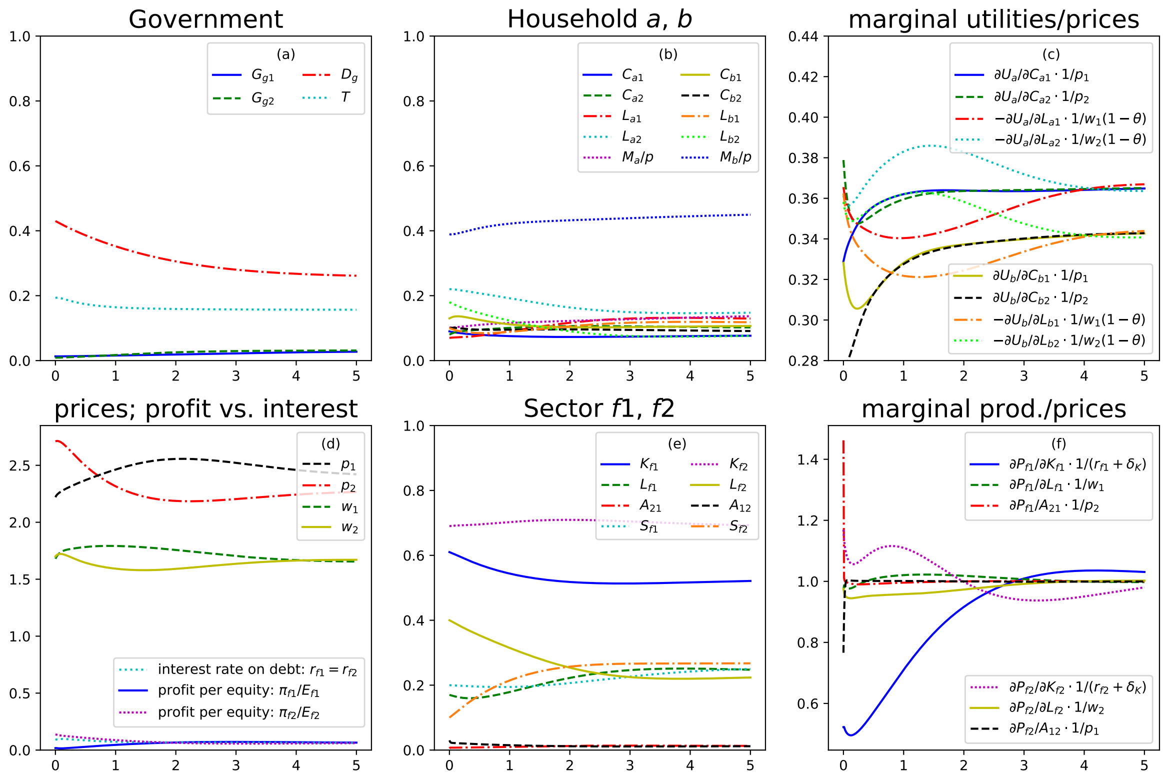


Figure 2: Plots (a), (b), (d) and (e) show the time evolution of the variables for different sectors. Plot (c) shows that for the two households, the marginal utilities for consumption and leisure, divided by their respective price, equalize over time. The budget equation constrains their choices, and the gradient climbing approach converges to the highest reachable level of utility. The same can be stated for plot (f) concerning the marginal input productivies for capital, labor and intermediate goods, divided by their price. Plot (d) shows that in equilibrium, profit paid per unit of equity is identical to the interest rate paid on debt.

## 4. Properties of the stationary state

The fixed point of a model is one with vanishing time derivatives in every variable. To derive the conditions for the stationary state, assume that every power factor is positive. From the price development (Eqs. 58–61), it follows that $\lambda_{P1} = \lambda_{P2} = 0$ and $\lambda_{L1} = \lambda_{L2} = 0$, thus in the stationary state, there is no mismatch between supply and demand for labor and goods.

For sector $f1$, the following conditions hold:

$$0 = p_1 (C_{a1} + C_{b1} + G_{g1} + A_{12}) - A_{21} p_2 - w_1 L_{f1} - r_{f1} D_{f1} - \pi_{f1}, \tag{63}$$

$$0 = s^{\top}_{f1} (C_{a1} + C_{b1} + G_{g1} + A_{12} + \delta_K K_{f1}) - S_{f1}, \tag{64}$$

$$0 = K_{f1}{}^{\kappa_1} L_{f1}{}^{l_1} A_{21}{}^{1-\kappa_1-l_1} - \delta_K K_{f1} - C_{a1} - C_{b1} - G_{g1} - A_{12}, \tag{65}$$

$$0 = (1 - r_{f1} s^{\top}_{f1}) \kappa_1 K_{f1}{}^{\kappa_1 - 1} L_{f1}{}^{l_1} A_{21}{}^{1-\kappa_1-l_1} - \delta_K - r_{f1}, \tag{66}$$

$$0 = p_1 (1 - r_{f1} s^{\top}_{f1}) l_1 K_{f1}{}^{\kappa_1} L_{f1}{}^{l_1 - 1} A_{21}{}^{1-\kappa_1-l_1} - w_1, \tag{67}$$

$$0 = p_1 (1 - r_{f1} s^{\top}_{f1})(1 - \kappa_1 - l_1) K_{f1}{}^{\kappa_1} L_{f1}{}^{l_1} A_{21}{}^{-\kappa_1-l_1} - p_2. \tag{68}$$

This result is independent on the power factors. In this specification of the model, economic power influences the adaptation processes, but not the equilibrium reached. With an inventory target of $s^{\top}_{f1} = 0$, the factor share is identical to the output elasticity, the exponent of the production factor in the Cobb-Douglas function, as in neoclassical competitive equilibrium. With $s^{\top}_{f1} > 0$, a part of total income goes to interest payments related to inventory holding that do not contribute to increased production.

Using the definition (valid because $\dot{K}_{f1} = \dot{S}_{f1} = 0$)

$$P_{f1} = K_{f1}{}^{\kappa_1} L_{f1}{}^{l_1} A_{21}{}^{1-\kappa_1-l_1} = \delta_K K_{f1} + C_{a1} + C_{b1} + G_{g1} + A_{12}, \tag{69}$$

Eqs. (65–68) can be simplified to:

$$p_1 S_{f1} = p_1 s^{\top}_{f1} P_{f1}, \tag{70}$$

$$(r_{f1} + \delta_K)P_{f1} K_{f1} = p_1 (1 - r_{f1} s_{f1}^\top)\kappa_1 P_{f1}\,, \tag{71}$$

$$p_2 A_{21} = p_1 (1 - r_{f1} s_{f1}^\top)(1 - \kappa_1 - l_1)P_{f1}\,, \tag{72}$$

$$w_1 L_{f1} = p_1 (1 - r_{f1} s_{f1}^\top) l_1 P_{f1}\,. \tag{73}$$

The debt is given by:

$$D_{f1} = p_1 (K_{f1} + s_{f1}^\top P_{f1}) - E_{f1}\,. \tag{74}$$

Substituting these results in the definition of profit $\pi_{f1}$ in Eq. (51) yields (see Appendix B):

$$\pi_{f1} = p_1 \left(C_{a1} + C_{b1} + G_{g1} + A_{12}\right) - A_{21} p_2 - w_1 L_{f1} - r_{f1} D_{f1} = r_{f1} E_{f1}\,. \tag{75}$$

This derivation shows that in the stationary state, the profits are a compensation for equity capital $E_{f1}$, and both equity capital and credit have the same rate of return, see Fig. 2(d). This corresponds to the first theorem by Modigliani and Miller (1958), assuming that no financial frictions and no difference in riskiness exists.

For household $a$, we assume that $\mu_{aC1} = \mu_{aC2} = \mu_{aM} = \mu_{aL1} = \mu_{aL2}$, implying that households have the same power to influence all their variables. The following conditions hold:

$$0 = p_1 C_{a1} + p_2 C_{a2} - (1 - \theta)(w_1 L_{a1} + w_2 L_{a2}) - r_M M_a - e_a(\pi_{f1} + \pi_{f2} + \pi_{bank}), \tag{76}$$

$$0 = \alpha_{C1} (C_{a1})^{\alpha_{C1}-1} (C_{a2})^{\alpha_{C2}} + \lambda_a p_1\,, \tag{77}$$

$$0 = (C_{a1})^{\alpha_{C1}} \alpha_{C2} (C_{a2})^{\alpha_{C2}-1} + \lambda_a p_2\,, \tag{78}$$

$$0 = -\alpha_L (1 - L_{a1} - L_{a2})^{\alpha_L-1} - \lambda_a w_1 (1 - \theta), \tag{79}$$

$$0 = -\alpha_L (1 - L_{a1} - L_{a2})^{\alpha_L-1} - \lambda_a w_2 (1 - \theta), \tag{80}$$

$$0 = r_M - \rho_a. \tag{81}$$

The equations imply that total income from wages and capital is equal to taxes and consumption, and that the wages in both sectors have to be identical. Cancelling $\lambda_a$ from all the equations yields the first order conditions for consumers in general equilibrium models:

$$-\frac{\partial U_a/\partial L_{a1}}{(1-\theta)w_1} = -\frac{\partial U_a/\partial L_{a2}}{(1-\theta)w_2} = \frac{\partial U_a/\partial C_{a1}}{p_1} = \frac{\partial U_a/\partial C_{a2}}{p_2}. \tag{82}$$

The ratio of prices equals the ratio of marginal utilities, thus the utility from the last monetary unit spent on each good must be the same and identical to the disutility of increasing working time divided by the wage after tax $(1-\theta)w_1$. In equilibrium, the interest rate on deposits $r_M$ equals the rate of time preference $\rho_a$. Note that this stationary state can be reached if and only if $\rho_a = \rho_b$, as $r_M$ cannot converge to two distinct values simultaneously. If $\rho_a > \rho_b$, household $a$ accumulates debt to finance consumption, as a no-ponzi condition is missing in this model. One way to relax this condition in the future would be to let the bank charge heterogeneous interest rates, depending on the debt-income ratio, which would allow the interest rate on debt for household $a$ to rise to $\rho_a$.

The total income distributed from sector $f1$ to households $a$ and $b$ before taxation is given by (see Appendix B):

$$\pi_{f1} + r_{f1} D_{f1} + w_1 L_{f1} = p_1 P_{f1} - p_2 A_{21} - \delta_K p_1 K_{f1}\,. \tag{83}$$

Total income is equal to production minus intermediate purchases minus depreciation.

For the government, the equations in the stationary state are, assuming $\mu_{gG1} = \mu_{gG2} = \mu_{gD}$:

$$0 = \gamma_{G1} {G_{g1}}^{\gamma_{G1}-1} + \lambda_g p_1\,, \tag{84}$$

$$0 = \gamma_{G_{g2}} {G_{g2}}^{\gamma_{G2}-1} + \lambda_g p_2\,, \tag{85}$$

$$0 = -2\gamma_D(1+\gamma_r r_g) D_g/(p_1 + p_2) - \lambda_g\,, \tag{86}$$

$$0 = r_g D_g + p_1 G_{g1} + p_2 G_{g2} - T_a - T_b\,, \tag{87}$$

$$0 = \theta p_1 (1 - r_{f1} s_{f1}^\top) l_1 P_{f1} + \theta p_2 (1 - r_{f2} s_{f2}^\top) l_2 P_{f2} - T_a - T_b\,. \tag{88}$$

The stationary state is reached if tax income covers government expenditures and interest payments on government debt $D_g$. The disutility of one additional unit of debt is identical to the utility gained by buying goods for this unit.

Overall, the stationary state satisfies all the condition usually presupposed in static neoclassical general equilibrium models.

### 4.1. Local and global stability

The differential-algebraic equation framework poses a challenge for the local stability analysis. Because of the constraints, the variables cannot be varied independently: A change in working hours necessarily implies a change in production, inventories, wage income, saving etc. The six balance sheet constraints (Eqs. 7–12), the two labor constraints (Eqs. 19–20) and the five algebraic equations for taxation (Eqs. 39–40) and profit distribution (Eqs. 51–52, 57) have to be guaranteed even after the shock. Additionally, interest rates have to march in lockstep, $\dot{r}_g = \dot{r}_{f1} = \dot{r}_{f2} = \dot{r}_M$, and $E_{bank} = 0$. These 17 restrictions have to be fulfilled, and I chose $T_a$, $T_b$, $L_{f1}$, $L_{f2}$, $r_{f1}$, $r_{f2}$, $r_M$, $E_{f1}$, $E_{f2}$, $E_{bank}$, $V_a$, $V_b$, $D_g$, $V_g$, $\pi_1$, $\pi_2$ and $\pi_{bank}$ to be determined by constraints, while the remaining 25 values are varied: $x = \{K_{f1}, K_{f2}, L_{a1}, L_{a2}, L_{b1}, L_{b2}, C_{a1}, C_{a2}, C_{b1}, C_{b2}, G_{g1}, G_{g2}, r_g, w_1, w_2, p_1, p_2, S_{f1}, S_{f2}, M_a, M_b, D_{f1}, D_{f2}, A_{12}, A_{21}\}$. The production constraints (Eqs. 21–22) are not problematic because the change in inventories $\dot{S}_{f1}$, $\dot{S}_{f2}$ can absorb the shock. The time evolution $\dot{x} = T(x)$ around the equilibrium $x_{eq}$ can be linearized with the $(25 \times 25)$ Jacobian matrix $J_T$ of all the first-order partial derivatives:

$$J_T(x_{eq}) := \left(\frac{\partial T_i}{\partial x_j}(x_{eq})\right)_{i,j=1,\dots,n} = \begin{pmatrix} \frac{\partial T_1}{\partial x_1}(x_{eq}) & \frac{\partial T_1}{\partial x_2}(x_{eq}) & \dots & \frac{\partial T_1}{\partial x_n}(x_{eq}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial T_n}{\partial x_1}(x_{eq}) & \frac{\partial T_n}{\partial x_2}(x_{eq}) & \dots & \frac{\partial T_n}{\partial x_n}(x_{eq}) \end{pmatrix} \tag{89}$$

The Jacobian matrix contains the reaction of the economy to a shock in equilibrium, as illustrated in Fig. 3.

The relevant quantities for the first order stability of the stationary state are the eigenvalues of the Jacobian $J_T$. An analytical calculation shows that 0 is a double eigenvalue, thus $J_T v_i = 0$ with $v_i$ being two corresponding linearly independent eigenvectors. The first eigenvector

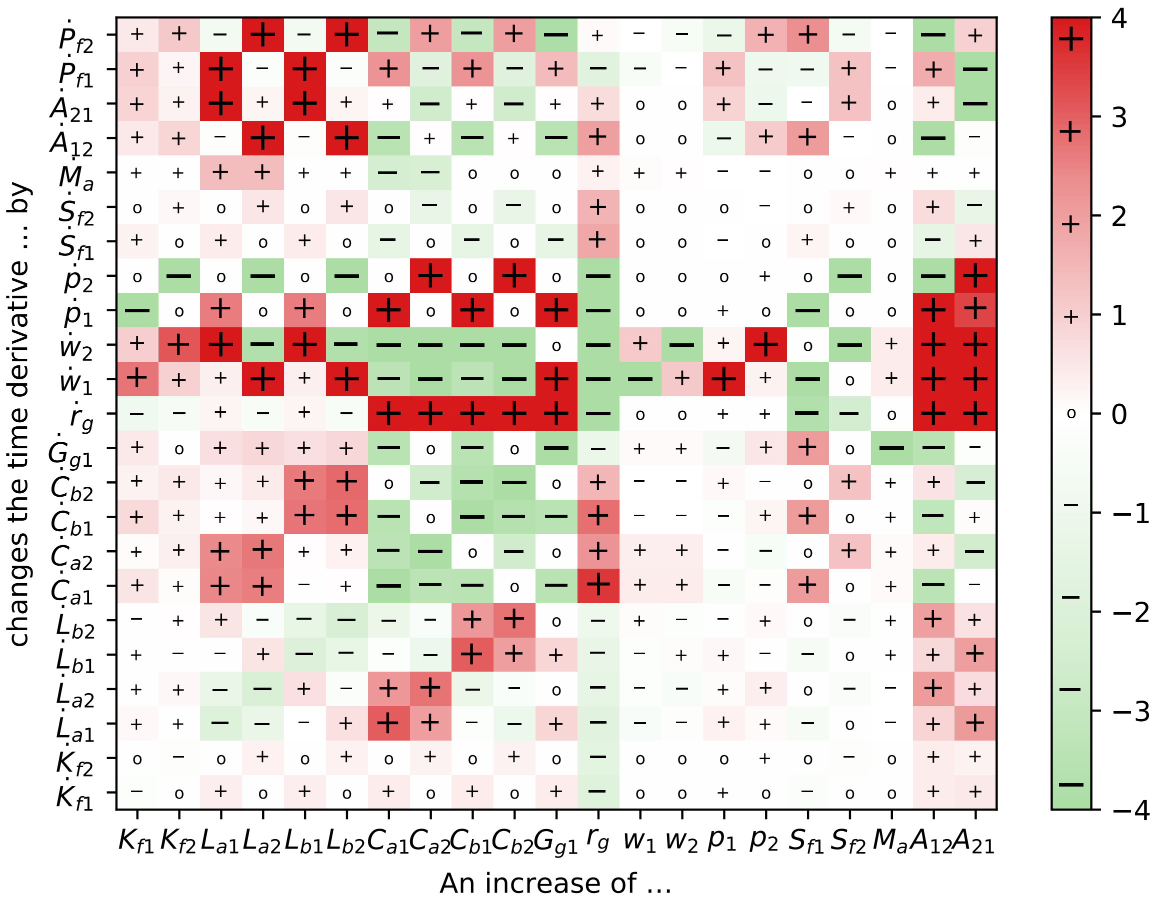

Figure 3: Selection of matrix entries of the Jacobian $J_T$ in Eq. (89) illustrate the impact of a small increase in one of the variables on the time evolution of other. The reactions of the time derivatives to deviations from the equilibrium can be extracted from the diagram.

For example, the penultimate column implies that an increase in intermediate trade $A_{12}$ from sector $f1$ to $f2$ leads to a reduction of the inventory stock ($\dot{S}_{f1} < 0$), which leads to increasing prices $p_1$ an increase in price $\dot{p}_1 > 0$, and increasing inputs $\dot{L}_{a1}, \dot{L}_{b1}, \dot{K}_{f1}, \dot{A}_{21}$ and a negative time evolution of the other sales $\dot{C}_{a1}, \dot{C}_{b1}$ and $\dot{G}_{g1}$. The additional input for sector $f2$ leads to a an increasing inventory stock $\dot{S}_{f2}$, a decreasing price $\dot{p}_2$ and a corresponding increase in households' demand. The other inputs $L_{a2}, L_{b2}, K_{f2}$ grow, because the additional input $A_{12}$ increases their marginal productivities. The additional demand for labor and capital leads to increasing wages and interest rates. These rising costs together with lowered prices $p_2$ will reverse this development in the following and push the economy back to equilibrium.

corresponds to an increase of $L_{a1}$ and $L_{b2}$ by $\Delta L$, while $L_{a2}$ and $L_{b1}$ are reduced by the same amount: Household $a$ works longer in sector $f1$, but shorter in sector $f2$, and household $b$ inversely. The aggregated variables $L_a$, $L_b$, $L_{f1}$ and $L_{f2}$ remain unchanged. The second eigenvector corresponds to an increase of $D_{f1}$ and $E_{f2}$ by $\Delta D$, accompanied by a decrease of $D_{f2}$ and $E_{f1}$ by the same amount. Sector $f1$ is now financed to a larger share by debt instead of equity, while it is the inverse for sector $f2$. Correspondingly, interest payments by sector $f1$ are increased while distributed profits are decreased and inversely for sector $f2$, keeping total equity and total firms' debt unchanged. In both cases, the stationary state is not unique but path dependent in some microscopic variables, but sectoral production, allocation, distribution and consumption remain unchanged.

The other eigenvalues depend on the parameters, particularly the power factors $\mu$, as revealed by the stability analysis in Fig. 4. Starting from the parameters in section 3.3, each power factor related to quantities (such as $\mu_{aC1}$, $\mu_{fK1}$, $\mu_{gD}$ ...) is multiplied by a common factor $\mu_{quantities}$, while power factors related to prices (such as $\mu_w$, $\mu_{p_1}$ ...) are multiplied by a factor $\mu_{prices}$. In the red part on the right, the biggest real part of the eigenvalues is bigger than zero, implying local instability. For $\mu_{quantities}$ big, the quantities react so strongly for example on profit opportunities that the oscillations of the system become unstable. The stationary state in the orange part is locally stable, but the time evolution does not converge to the equilibrium. For example, if $\mu_{quantities} = 0$, the numerical solver aborts because no market forces prevent capital or labor from taking negative values, leading to an undefined value of the production function. In the green part, the time evolution converges to the equilibrium derived in section 4. In the blue part, the system did not converged to a stationary state at $t = 100$.

The reaction functions of the economic actors and the price adaptation cannot as such guarantee global stability. If quantity adjustments are fast, the model becomes unstable, because the bounded rational firms do not anticipate the reactions of the other market participants to their change in production. Instead, they react on supply–demand mismatches by adapting production and prices. If this reaction is very strong, it can lead to growing inventory oscillations as pioneered by Metzler (1941). Faster price adaptation can sometimes improve local stability.

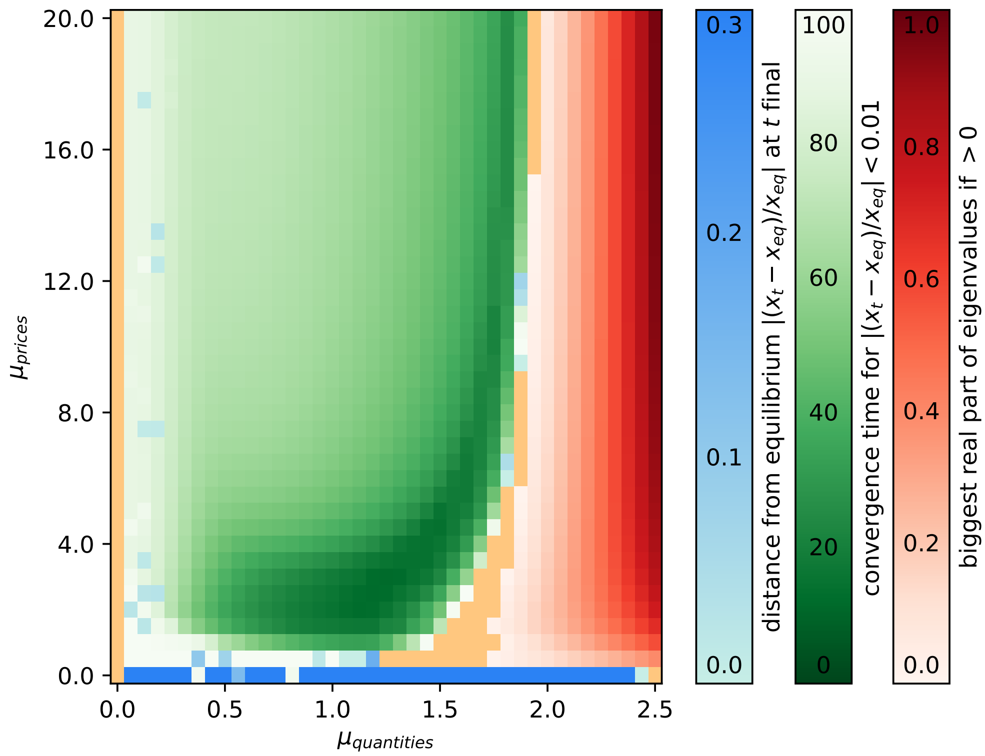


Figure 4: Stability analysis of the system, depending on the scaling factors $\mu_{prices}$ and $\mu_{quantities}$. The red color is used for combinations in which the biggest real part of the eigenvalues of the Jacobian $J_T$ at the stationary state is above zero. Therefore, the system is locally unstable, the model shows explosive behavior and the numerical solver aborts. The orange color indicates the area where the real parts of all the eigenvalues are $\leq 0$, but the numerical solver aborts nevertheless. This part of the system is locally stable, but shows no convergence for the initial conditions. The green part converges to the numerically determined equilibrium. The greener the color, the faster the convergence until $|(x_t - x_{eq})/x_{eq}| < 0.01$. In the blue part, the system did not converge to a stationary state at $t = 100$. The difference $|(x_t - x_{eq})/x_{eq}|$ at $t = 100$ is indicated by the blue color. For $\mu_{quantities} = 0$, the model is unstable because individual influences on quantities are negligible. For $\mu_{prices} = 0$ and $\mu_{quantities} > 0$, the model does not converge to a stationary state as the coordinating influence of price adaptation is missing.

Different from the way 'frictions' are commonly discussed in economic models as slowing down convergence to the equilibrium, very fast adaptations of prices and quantities make the equilibrium unattainable. In this model, intermediate adaptation speeds lead to the fastest convergence to equilibrium.

## 5. Discussion and Conclusions

This paper presented a dynamic modeling approach in continuous time that extends the analogies between mechanics and economics and depicts the economy from the perspective of economic forces and economic power. The conceptual model showed how General Constrained Dynamics can serve as a joint framework for general equilibrium, Keynesian disequilibrium and agent-based models: It includes some Keynesian features such as slow adaptation of prices and quantities or endogenous money creation. Similar to agent-based models, the heterogenous agents have bounded rationality, here modeled as utility improvement by 'gradient seeking'. Nevertheless, in the fixed points of the dynamical system, the first-order conditions of neoclassical general equilibrium solutions are satisfied and the power factors become irrelevant. The latter can be seen in light of the old debate whether control or economic laws determine market outcomes (Böhm-Bawerk, 1914). Different from DSGE models, fast adaptation of quantities and prices does not lead to fast convergence, but can amplify deviations from the equilibrium. As agents do not react optimally to changing conditions and do not anticipate the reaction of others, frictions have a stabilizing effect. It remains open whether this result holds if forward looking expectations of firms and the related intertemporal coordination problem are integrated. If this was the case, political regulation should concentrate on designing market frictions to stabilize markets, instead of eliminating them.

In the future, the model can be extended by additional forces to the time evolution: On the one hand, the approach allows to overcome some known restrictions of DSGE models, namely the aggregation problem, their assumption of rationality, and treatment of situations far from equilibrium. On the other hand, the above model contains many ad hoc assumptions which could be refined depending on the application. This includes not only the principal–agent dilemma of dividend policies or credit rationing by banks, but also the influence of monopolists on prices. Furthermore, political economy issues such as the power relations and influences between politics and firms can be modeled, or households could mutually influence their decisions by ‘positional’ consumption. Stochastic shocks can be integrated, bearing in mind that the shocks have to satisfy the economic constraints. By the choice of the parameters that reflect ‘economic power’ in the sense of the ability to change certain variables and the integration of various social and market forces, the economic and social processes can be modeled in a flexible way.

In this paper, production and utility functions were chosen such that the dynamics converge to stable equilibria for most parameters. Economic models with multiple equilibria typically incorporate incomplete markets due to transaction costs or information asymmetries, increasing returns to scale, or market imperfections such as entry costs or external effects (Benhabib and Farmer, 1999). They were studied to explain issues such as asset bubbles, collateral shortages, liquidity dry-ups, bank runs, or financial crises (Miao, 2016). If multiple equilibria exist, a theory that describes the out-of-equilibrium dynamics is required to determine which of the equilibrium states is reached. A drawback of the GCD approach is that general equilibrium models with multiple markets are tremendously complex in the amount of variables that are simultaneously ‘in equilibrium’. Consequently, providing models able to describe genuine out-of-equilibrium dynamics for all these variables poses a significant challenge. An intermediate approach could combine equilibrium dynamics with out-of-equilibrium processes where necessary. As the concept of Lagrangian closure draws on a mathematical similarity to static optimization models, the General Constrained Dynamics framework is a suitable candidate for this task.

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## Appendix A Initial conditions, power factors and parameters

**Initial conditions:** Household $a$: $L_{a1} = 0.07$; $L_{a2} = 0.22$; $C_{a1} = 0.09$; $C_{a2} = 0.08$; $M_a = 0.25$; Household $b$: $L_{b1} = 0.10$; $L_{b2} = 0.18$; $C_{b1} = 0.13$; $C_{b2} = 0.10$; $M_b = 0.96$;
Firms: production inputs $K_{f1} = 0.61$; $K_{f2} = 0.69$; $L_{f1} = L_{a1} + L_{b1}$; $L_{f2} = L_{a2} + L_{b2}$; $A_{12} = 0.03$; $A_{21} = 0.005$; inventories $S_{f1} = 0.20$; $S_{f2} = 0.10$; debt $D_{f1} = 0.52$; $D_{f2} = 0.26$; equity $E_{f1} = p_1 (K_{f1} + S_{f1}) - D_{f1}$; $E_{f2} = p_2 (K_{f2} + S_{f2}) - D_{f2}$; $E_{bank} = 0$.
Government: expenditures $G_{g1} = 0.013$; $G_{g2} = 0.01$; debt $D_g = M_a + M_b - D_{f1} - D_{f2}$; inflation target $\rho^\top = 0$;
Prices: $w_1 = 1.68$; $w_2 = 1.70$; $p_1 = 2.22$; $p_2 = 2.71$; $r_{f1} = r_{f2} = r_g = 0.09$; $r_M = 0.089$.

**Power factors:** Household $a$: $\mu_{aL1} = \mu_{aL2} = 1; \mu_{aC1} = \mu_{aC2} = 1; \mu_{aM} = 1$.
Household $b$: $\mu_{bL1} = \mu_{bL2} = 1; \mu_{bC1} = \mu_{bC2} = 1; \mu_{bM} = 1$.
Firms: $\mu_{fK1} = 0.5; \mu_{fK2} = 0.5; \mu_{fL1} = 1; \mu_{fL2} = 1; \mu_{fA1} = 1; \mu_{fA2} = 1; \mu_{fS1} = 1.5; \mu_{fS2} = 1$.
Government: $\mu_{gD} = 1; \mu_{gG1} = 1; \mu_{gG2} = 1$.
Price development: $\mu_{p1} = 2.5; \mu_{p2} = 2.5; \mu_w = 2; \mu_r = 1$.

**Parameters:** Household $a$: utility factors $\alpha_r = 2; \rho_a = 0.06; \alpha_L = 0.4; \alpha_{C1} = 0.25; \alpha_{C2} = 0.2$; ownership share $e_a = 0.2$.
Household $b$: utility factors $\beta_r = 2; \rho_b = 0.06; \beta_L = 0.3; \beta_{C1} = 0.25; \beta_{C2} = 0.2$. ownership share $1 - e_a$.
Firms: Cobb-Douglas exponents $\kappa_1 = 0.25; \kappa_2 = 0.3; l_1 = 0.7; l_2 = 0.55$; inventory to sales ratios $s_{f1}^\top = 1; s_{f2}^\top = 1$; depreciation $\delta_K = 0.05$.
Government: utility factors $\gamma_{G1} = 0.6; \gamma_{G2} = 0.4; \gamma_D = 10$; $\gamma_r = 4$; tax rate $\theta = 0.2$.

## Appendix B Derivation of firms profits and households income in the stationary state

Substituting the results from section 4 into Eq. (51) yields:

$$\begin{aligned}
\pi_{f1} &= p_1 (C_{a1} + C_{b1} + G_{g1} + A_{12}) - A_{21} p_2 - w_1 L_{f1} - r_{f1} D_{f1} && \text{(B.1)}\\
&= p_1 (P_{f1} - \delta_K K_{f1}) - A_{21} p_2 - w_1 L_{f1} - r_{f1} \left[p_1 (K_{f1} + s_{f1}^\top P_{f1})) - E_{f1}\right] && \text{(B.2)}\\
&= p_1 P_{f1} \left[1 - (1 - r_{f1} s_{f1}^\top)(1 - \kappa_1)\right] - (\delta_K + r_{f1}) p_1 K_{f1} - r_{f1} p_1 s_{f1}^\top P_{f1} + r_{f1} E_{f1} && \text{(B.3)}\\
&= p_1 P_{f1} \left[1 - (1 - r_{f1} s_{f1}^\top)(1 - \kappa_1)\right] - p_1 (1 - r_{f1} s_{f1}^\top) \kappa_1 P_{f1} - r_{f1} p_1 s_{f1}^\top P_{f1} + r_{f1} E_{f1} && \text{(B.4)}\\
&= p_1 P_{f1} \left[1 - (1 - r_{f1} s_{f1}^\top)(1 - \kappa_1) - (1 - r_{f1} s_{f1}^\top)\kappa_1 - r_{f1} s_{f1}^\top\right] + r_{f1} E_{f1} && \text{(B.5)}\\
&= r_{f1} E_{f1}. && \text{(B.6)}
\end{aligned}$$

Total income distributed to household $a$ and $b$ from sector $f1$ (interest via the banks) is:

$$\begin{aligned}
&\pi_{f1} + r_{f1} D_{f1} + w_1 L_{f1} && \text{(B.7)}\\
&= r_{f1} p_1 (K_{f1} + s_{f1}^\top P_{f1}) + w_1 L_{f1} && \text{(B.8)}\\
&= p_1 (1 - r_{f1} s_{f1}^\top) \kappa_1 P_{f1} - \delta_K p_1 K_{f1} + p_1 r_{f1} s_{f1}^\top P_{f1} + p_1 (1 - r_{f1} s_{f1}^\top) l_1 P_{f1} && \text{(B.9)}\\
&= p_1 P_{f1} - (1 - \kappa_1 - l_1) p_1 P_{f1} (1 - r_{f1} s_{f1}^\top) - \delta_K p_1 K_{f1} && \text{(B.10)}\\
&= p_1 P_{f1} - p_2 A_{21} - \delta_K p_1 K_{f1}. && \text{(B.11)}
\end{aligned}$$

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