Ultrametric probe of the spin-glass state in a field

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We study the ultrametric structure of phase space of one-dimensional Ising spin glasses with random power-law interaction in an external random field. Although in zero field the model in both the mean-field and non-mean-field universality classes shows an ultrametric signature [Phys. Rev. Lett. **102**, 037207 (2009)], when a field is applied ultrametricity seems only present in the mean-field regime. These results agree with data for spin glasses studied within the Migdal-Kadanoff approximation. Our results therefore suggest that the spin-glass state might be fragile to external fields below the upper critical dimension.

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I. INTRODUCTION

Spin glasses[1,2] are paradigmatic model systems that find wide applicability across disciplines. Although studied intensely over the last four decades, our understanding of some of their fundamental aspects is still in its infancy. In particular, the understanding of the nature of the spin-glass state remains controversial and active discussion has emerged recently.[3–13] In particular, it is unclear if the mean-field replica symmetry breaking (RSB) picture[14] of Parisi describes the nonmean-field behavior of spin-glasses in an externally-applied field best. While the droplet theory[15–18] states that there is no spin-glass state in a field for short-range systems, the mean-field RSB picture[2,14,19–21] states that for low enough temperatures T and fields H, i.e., below the de Almeida-Thouless line,[22] a stable spin-glass state emerges. The question lies at the core of theoretical descriptions and is of immediate importance to applications in fields ranging from, e.g., sociology to economics.

One way to settle the applicability of the RSB picture to short-range spin glasses in a field while avoiding technical difficulties when measuring observables in a field[12,13] is by testing if the phase space is ultrametric (UM). Unfortunately, the existence of an UM phase structure for short-range spin glasses on hypercubic lattices remains elusive,[23] mainly because only small systems can be studied numerically. Recent results in zero field[23] suggest that short-range systems are not UM, whereas other opinions exist.[24–27]

More recently[12] results on one-dimensional Ising models with power-law interactions showed that short-range spin glasses might be UM after all. Therefore, a natural probe for the spin-glass state in a field is to study the UM response of one-dimensional Ising models with power-law interactions when an external field is applied. The model has the advantage in that by tuning the exponent of the power law, the universality class can be tuned between a mean-field and a non-mean-field universality class. In addition, large linear system sizes can be simulated whereas for hypercubic lattices[23] only small linear system sizes can be studied.

Our results show that for this model in a field the phase space has an UM structure in the mean-field regime. However, in the nonmean-field regime, when an external field is applied, the UM structure of phase space seems to be much weaker for the studied system sizes, suggesting that the spin-glass state for short-range systems is fragile with respect to externally-applied fields. These results are compared to studies of spin glasses within the Migdal-Kadanoff (MK) approximation.

The paper is organized as follows. In Sec. II we introduce the model studied, followed by details on the numerical method in Sec. III. Our probe for UM behavior is outlined in Sec. IV, followed by results (Section V) and conclusions.

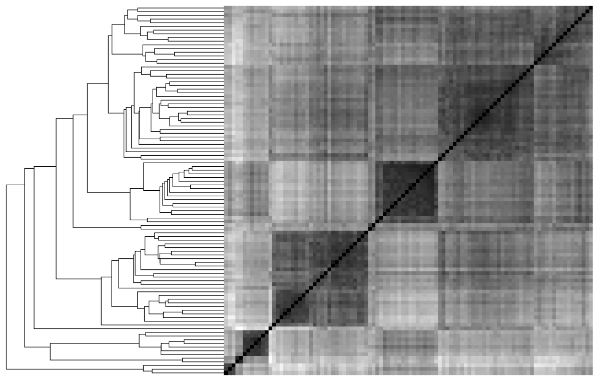

FIG. 1: Dendrogram obtained by clustering 100 configurations (see text) for a sample system with $\sigma = 0.0$ (Sherrington-Kirkpatrick model) and $L = 512$ at $T = 0.36$, together with the matrix $d_{\alpha\beta}$ (grey scale, distance 0 is black). The order of the states is given by the leaves of the dendrogram (figure rotated clockwise by 90°).

II. MODEL

The one-dimensional Ising chain with long-range power-law interactions[17,28–30] is described by the Hamiltonian

$$\mathcal{H} = -\sum_{i<j} J_{ij} S_i S_j - \sum_i h_i S_i \; ; \qquad J_{ij} = c(\sigma)\frac{\epsilon_{ij}}{{r_{ij}}^{\sigma}} \, , \quad (1)$$

where $S_i \in \{\pm 1\}$ are Ising spins and the sum ranges over all spins in the system. The L spins are placed on a ring to ensure periodic boundary conditions and $r_{ij} = (L/\pi)\sin(\pi|i-j|/L)$

is the geometric distance between the spins. ϵ_{ij} are Gaussian random couplings. In the range of interest $0 \leq \sigma \leq 2$, the constant $c(\sigma)$ is chosen[30] such that the mean-field transition temperature to a spin-glass phase is $T_c^{\rm MF}(\sigma \leq 0.5, L, H = 0) = 1$. In Eq. (1), the spins couple to site-dependent random fields h_i chosen from a Gaussian distribution with zero mean and standard deviation $[h_i^2]_{\rm av}^{1/2} = H$.

The model has a rich phase diagram when the exponent σ is changed:[30] Both the universality class and the range of the interactions can be continuously tuned. In particular, $\sigma = 0$ gives the Sherrington-Kirkpatrick (SK) model,[31,32] whose solution is the mean-field theory for spin glasses and where a spin-glass state in a field is expected, i.e., an UM signature for low enough H and temperatures T. More importantly,[28] for $1/2 < \sigma < 2/3$ the critical behavior is mean-field-like, while for $2/3 < \sigma \leq 1$ it is non-mean field like.

Here we study in a field $H = 0.10$ the SK model $[\sigma = 0]$ to test our analysis protocol, as well as the one-dimensional chain for $\sigma = 0.60$ (also mean-field like), as well as $\sigma = 0.75$ ($T_c \sim 0.69$, roughly corresponding to four space dimensions) outside the mean-field regime. We choose two values of $\sigma \neq 0$ to be able to discern any trends when the effective dimensionality[33] is reduced. In general $d_{\rm eff} = (2 - \eta)/(2\sigma - 1)$, where η is the critical exponent η for the short-range model at space dimension $d = d_{\rm eff}$, which is zero in the mean-field regime and, for example, $-0.275(25)$ in four space dimensions.[34]

III. NUMERICAL METHOD AND EQUILIBRATION

We generate spin-glass configurations by first equilibrating the system at low temperatures and an external random field of average strength $H = 0.1$ using the exchange Monte Carlo method.[35] Once the system is in thermal equilibrium we record states ensuring that these are well separated in the Markov process and thus not correlated by measuring auto-correlation times. In practice, if we equilibrate the system for $\tau_{\rm eq}$ Monte Carlo sweeps, we generate for each disorder realization 10^3 states separated by $\tau_{\rm eq}/10$ Monte Carlo sweeps. We test equilibration using the method presented in Ref. 11. Simulation parameters are listed in Table I.

The presented data are all for $T = 0.36$. In Ref. 36 we fixed $T \approx 0.4T_c$ for all values of σ studied to ensure that we are deep in the spin-glass phase. However, it is unclear if one-dimensional spin glasses with power-law interactions have a spin-glass state for $\sigma > 2/3$.[11,13,37] Using the T_c estimates of Leuzzi *et al*.[13] at zero and finite field ($H = 0.1$) for the *diluted* version of the model we estimate that if a spin-glass state exists for $H = 0.1$ it should suppress the zero-field T_c by approximately 17%. For $\sigma = 0.75$ it is known that $T_c(H = 0) \approx 0.69(1)$,[11] which is in the nonmean-field regime. Therefore $T = 0.36$ corresponds roughly to $0.6T_c(H = 0.1)$, i.e., deep in the putative spin-glass phase.

For comparison, we also study spin glasses within the standard MK approximation,[38] i.e., spin glasses on hierarchical lattices. Due to the simple lattice structure, the phase space is also expected to be simple. We used a variation of the standard MK recursion where, starting from one bond, iteratively each bond is replaced by 2^d bonds and 2^{d-1} spins ($d = 3$). For details, see, e.g., Refs. 39 and 40.

TABLE I: Simulation parameters for the one-dimensional chain with $H = 0.1$ and different power-law exponents σ. L is the system size, $N_{\rm sa}$ is the number of disorder realizations and $\tau_{\rm eq}$ is the number of equilibration sweeps. For the parallel tempering simulations $T_{\rm min} = 0.36$ and $T_{\rm max} = 1.40$ with a total of 16 temperatures.

σ			L	$N_{\rm sa}$	$\tau_{\rm eq}$
0.00	0.60	0.75	32	4 000	20 000
0.00	0.60	0.75	64	4 000	150 000
0.00	0.60	0.75	128	4 000	500 000
0.00	0.60	0.75	256	4 000	1 000 000
0.00	0.60	0.75	512	4 000	1 000 000

IV. ULTRAMETRICITY

Ultrametricity appears in different fields or research ranging from linguistics to the taxonomy of animal species and is a key component of Parisi's mean-field solution of the SK model.[1,14,41] Therefore, if a spin-glass model has no UM phase-space structure there is a strong indication that Parisi's mean-field picture might not work for this system.

In an UM space[42] the triangle inequality $d_{\alpha\gamma} \leq d_{\alpha\beta} + d_{\beta\gamma}$ is replaced by a stronger condition where $d_{\alpha\gamma} \leq \max\{d_{\alpha\beta}, d_{\beta\gamma}\}$, i.e., the two longer distances must be equal and the states lie on an isosceles triangle. Here, $d_{\alpha\beta}$ represents the distance between two points α and β in phase space.

We use the approach developed in Ref. 12 which is closely related to the one used by Hed *et al.* in Ref. 23. For each disorder realization we produce $M = 10^3$ equilibrium configurations. These are sorted using the average-linkage agglomerative clustering algorithm.[43] The clustering procedure starts with M clusters containing each exactly one configuration. Distances are measured in terms of the hamming distance $d_{\alpha\beta} = (1 - |q_{\alpha\beta}|)$, where $q_{\alpha\beta} = N^{-1}\sum_i S_i^\alpha S_i^\beta$ is the spin overlap between configurations $\{S^\alpha\}$ and $\{S^\beta\}$. Iteratively the two closest clusters C_a and C_b are merged into one cluster C_d, reducing the number of clusters by one. The distances of the new cluster C_d to the other remaining clusters have to be calculated: The distance between two clusters is the average distance between all pairs of members of the clusters. The iterative procedure stops when only one cluster remains, the results are then typically structured in a tree-like structure called a dendrogram (see Fig. 1). To probe for a putative UM space structure, we randomly select three configurations from the hierarchical cluster structure (see Ref. 23), resulting in three mutual distances. Next, we sort these hamming distances $d_{\rm max} \geq d_{\rm med} \geq d_{\rm min}$ and compute the correlator

$$K = (d_{\rm max} - d_{\rm med})/\varrho(d), \qquad (2)$$

where $\varrho(d)$ is the width of the distance distribution. If the phase space is UM, then we expect $d_{\rm max} = d_{\rm med}$ for $L \to \infty$. Thus $P(K) \to \delta(K = 0)$ for $L \to \infty$ and the for the variance of the distribution $\mathrm{Var}(K) \to 0$ for $L \to \infty$.

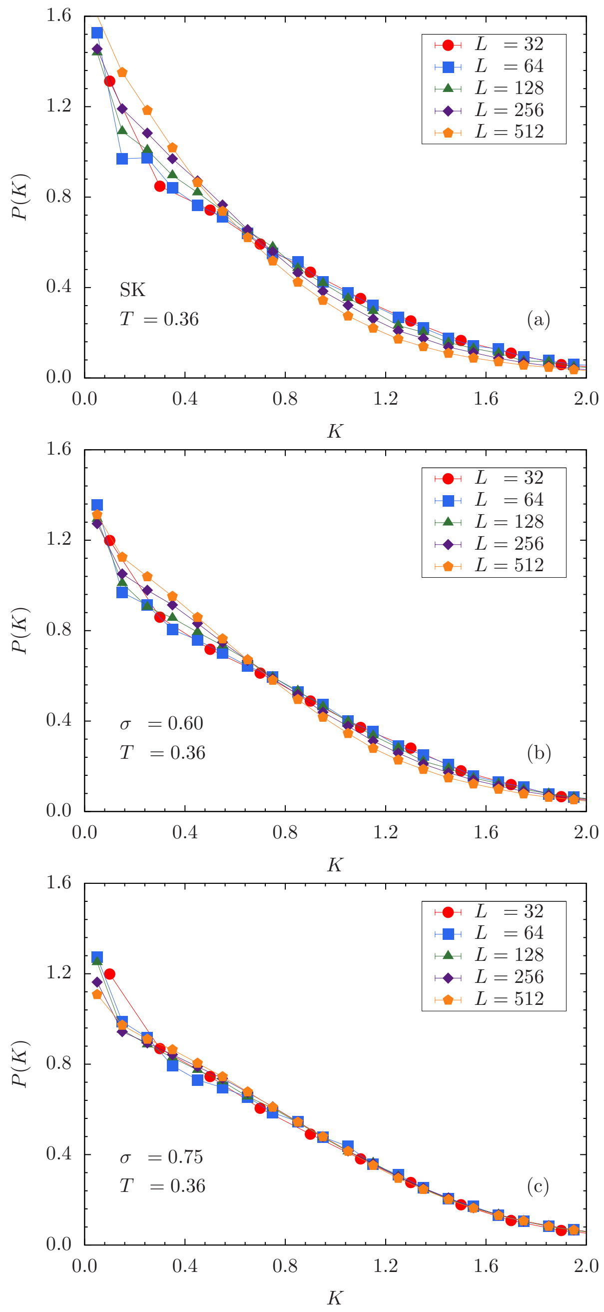

FIG. 2: (Color online) Distribution $P(K)$ for different system sizes (all panels have the same horizontal and vertical scale) and an external random field $H = 0.1$. (a) Data for the SK model. The distribution diverges very slightly for $K \to 0$ and $L \to \infty$ thus signaling an UM phase structure. (b) Data for $\sigma = 0.60$ (mean-field universality class). There is still a weak hint of a divergence for $K \to 0$. (c) Data for $\sigma = 0.75$ (nonmean-field universality class). There is no sign of a divergence in $P(K)$ for $K \to 0$. Note that when $H = 0$ data for $\sigma = 0.75$ show a clear signature for UM behavior.[30]

V. RESULTS

Figure 2(a) shows the distribution $P(K)$ for the SK model ($\sigma = 0$) and $T = 0.36$ and $H = 0.10$. There is a slight hint

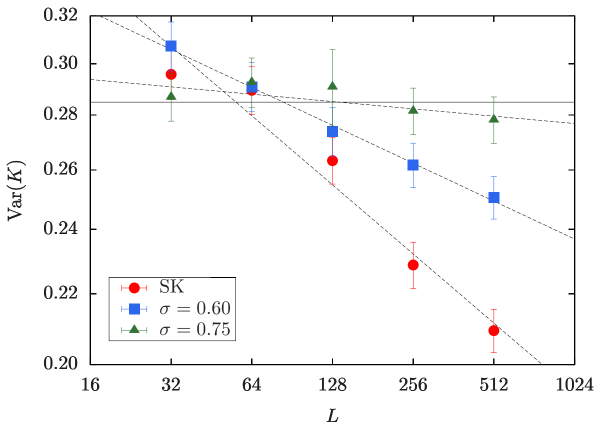

FIG. 3: (Color online) Variance $\mathrm{Var}(K)$ of $P(K)$ as a function of system size L for different values of σ. The data can be fit to a power law (dashed lines). In the mean-field regime (SK and $\sigma = 0.6$) a fit to a constant is unlikely (see text). The power-law decay of the variance as a function of system size suggests a divergence in $P(K)$ for $K \to 0$. For $\sigma = 0.75$ the data are compatible with a constant (solid line) or a very weak power-law behavior, suggesting that there might not be an UM phase space structure.

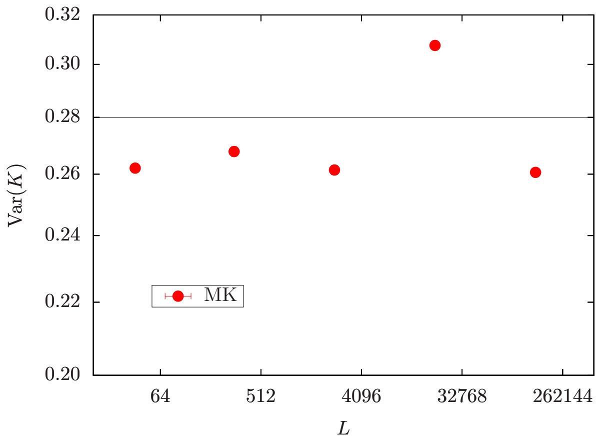

FIG. 4: (Color online) Variance $\mathrm{Var}(K)$ of $P(K)$ as a function of system size L for spin glasses on hierarchical lattices. The data are compatible with a constant behavior, showing that there is no UM phase space structure for spin glasses within the MK approximation. The solid line is a guide to the eye.

for a divergence for $K \to 0$. Similar results are found for the mean-field regime with $\sigma = 0.60$ [Figure 2(b)]. The UM signature in a field is considerably weaker than when no field is applied.[12] While for the SK model there is still a faint sign of a divergence, for larger values of σ it is hard to see if the distributions diverge for $K \to 0$ and $L \to \infty$. Figure 2(c) shows data for $\sigma = 0.75$, $T = 0.36$ and $H = 0.10$ where no visible sign of a divergence is present, suggesting that phase space might not be UM outside the mean-field regime. A better probe is given by the variance $\mathrm{Var}(K)$ of the distribution as a function of system size, Fig. 3. The variance of the distribution for the SK model clearly decays with a power law (Q-factor of the fit ~ 0.28).[44] A fit of a constant gives $Q = 0$.

Similar results are obtained for $\sigma = 0.60$ where a fit to a power law is very likely with $Q = 0.989$. Again, a fit to a constant gives $Q < 10^{-5}$. However, for $\sigma = 0.75$ both a fit to a very weak power law [$\mathrm{Var}(K) \sim L^{-\gamma}$ with $\gamma = 0.014(6)$] and a constant are equally probable with Q values in the vicinity of 0.8 — 0.9. Although larger systems would be needed to fully discern the behavior, the data are compatible with a constant. Either ultrametricity in the nonmean-field regime is completely lost in a field or strongly weakened.

Within the MK approximation the distributions $P(K)$ also show no divergence for $K \to 0$. Figure 4 shows the variance of the distributions as a function of the system size. There is no discernible decrease with an increasing number of spins, i.e., no UM structure of phase space. This is to be expected because the model is defined on a hierarchical lattice. However, a direct comparison to the results for $\sigma = 0.75$ strengthens the evidence of a non-UM structure for the latter case.

VI. SUMMARY AND CONCLUSION

We have studied numerically the low-temperature configuration landscape of long-range spin-glasses with power-law interactions. By tuning the exponent σ that governs the decay of the power-law interactions and therefore their range we can tune the system out of the mean-filed universality class. Using a hierarchical clustering method and analyzing the resulting distance matrices we show that when an external field is applied the system is only UM in the mean-field regime, unlike in the zero-field case where an UM signal was found for values of σ that correspond to space dimensions above and below the upper critical dimension. These results are in agreement with calculations on MK hierarchical lattices. Therefore, the spin-glass state is fragile with respect to externally-applied fields below the upper critical dimension.

Acknowledgments

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