Replica theory for Levy spin glasses *

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Infinite-range spin-glass models with Levy-distributed interactions show a spin-glass transition with similarities to both the Sherrington-Kirkpatrick model and to disordered spin systems on finite connectivity random graphs. Despite the diverging moments of the coupling distribution the transition can be analyzed within the replica approach by working at imaginary temperature. Within the replica-symmetric approximation a self-consistent equation for the distribution of local fields is derived and from the instability of the paramagnetic solution to this equation the glass-transition temperature is determined. The role of the percolation of rare strong bonds for the transition is elucidated. The results partly agree and partly disagree with those obtained within the cavity approach. Numerical simulations using parallel tempering are in agreement with the transition temperatures found.

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I. INTRODUCTION

Spin glasses have been one of the most prominent models for disordered systems since the classical paper by Edwards and Anderson [1]. They are built from simple degrees of freedom interacting via random couplings [2]. The ensuing interplay between disorder and frustration gives rise to peculiar static and dynamic properties which made spin-glasses paradigms for complex systems with competing interactions. The concepts and techniques developed for their theoretical understanding became useful also in the quantitative analysis of problems from algorithmic complexity [3–5], game theory [6, 7], artificial neural networks [8, 9], and cryptography [10].

A comprehensive understanding of spin glasses is so far possible on the mean-field level only. Different models of mean-field spin glasses have been introduced and analyzed over the years. The Sherrington-Kirkpatrick (SK) model [11] was designed as generalization of the Weiss model of ferromagnetism. It is the most popular completely connected spin glass model in which each spin interacts with all $\mathcal{O}(N)$ other spins via *weak* couplings $J_{ij} = \mathcal{O}(N^{-1/2})$. The central limit theorem can then be invoked to determine the statistical properties of the local fields and in the simplest situation the distribution of these fields is Gaussian and can be characterized by a single scalar order parameter. The details of the spin glass transition and the intricate nature of the low-temperature phase of this model have been thoroughly elucidated within the framework of the celebrated Parisi solution [12]. Quite recently the main features of this solution were established in a mathematically rigorous way [13].

The variety of mean-field models for spin glasses is, however, by far not exhausted by SK-like systems. The Viana-Bray (VB) model [14] and, more generally, spin glasses on finite-connectivity graphs [15, 16] combine finite coordination number with mean-field behaviour. Here each spin interacts with a finite number of randomly selected other spins through *strong* bonds $J_{ij} = \mathcal{O}(1)$. Accordingly the distribution of local fields is not Gaussian and has to be characterized by all its moments. The analysis of these systems is therefore technically more involved and already in the simplest (replica symmetric) description infinitely many order parameters (or equivalently an order parameter function) have to be introduced. Models of this type often arise in the analysis of complex optimization problems with methods from statistical mechanics [3].

¿From the technical perspective two different methods were developed to analyze spin glasses within the framework of equilibrium statistical mechanics. The replica method [1, 17, 18] centers around averages of integer moments of the partition function of the system. Its crucial step consists in the analytical continuation of the results for the n-th moment of the partition function from integer to real n and the final limit $n \to 0$. Complementary, the cavity method [17, 19, 20] builds on the clustering property of equilibrium states and the stability of the thermodynamic limit $N \to \infty$. Here one considers a system with N spins, adds one additional spin with its couplings to the system and derives self-consistency relations that stem from the fact that the statistical properties of the N and the $N + 1$

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* Dedicated to Professor Thomas Nattermann on the occasion of his 60^{th} anniversary.

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spin system should be similar to each other. Both the SK and the VB model have been analyzed using the replica as well as the cavity method.

Also from the numerical point of view, spin glasses are a challenging problem. Due to the frustrated interactions and since no efficient cluster algorithm exists, it is very hard to equilibrate samples at temperatures below the transition temperature. Hence, other schemes like parallel tempering are used frequently, but still sizes of the order of $N = 1000$ spins are typically the maximum system size one can treat.

In the present paper we investigate the replica-symmetric (RS) theory of an infinite range spin-glass model for which the couplings strengths are drawn from a Levy distribution [21]. The main characteristic of these distributions are power-law tails resulting in diverging moments. Spin glasses with Levy couplings are interesting for several reasons. In real spin glasses with a random distribution of magnetic atoms in a non-magnetic host lattice the RKKY-interaction give rise to a broad spectrum of interaction strengths, in particular for low concentration of magnetic impurities. The coexistence of couplings with vastly different strength is badly represented by a Gaussian distribution as used in the SK model. Also, it is interesting to see whether the concept of frustration which is central to the understanding of spin glasses has to be modified for broad distributions of coupling strengths. Moreover, in a completely connected spin glass with Levy distributed couplings each spin will establish $\mathcal{O}(1)$ strong bonds with other spins whereas the majority of couplings are weak, i.e. tend to zero for $N \to \infty$. Levy spin glasses are hence intermediate between the classes represented by the VB and SK model respectively and it is interesting to see how the spin glass transition is influenced by the percolation of the strong bonds on the one hand and the collective blocking of the many weak bonds on the other hand.

Levy spin glasses also pose new challenges to the theoretical analysis because the diverging second moment of the coupling distribution invalidates the central limit theorem which is at the bottom of many mean-field techniques. Related issues of interest include quantum spin glasses with broad coupling distribution [22], the spectral theory of random matrices with Levy-distributed entries [23, 24], and relaxation and transport on scale-free networks [25]. It is also possible that the peculiar properties of Levy distributions may facilitate mathematically rigorous investigations of spin glasses. In this respect it is interesting to note that the properties of the Cauchy-distribution have recently enabled progress in the mathematically rigorous analysis of matrix games with random pay-off matrices [26].

The Levy spin glass was investigated previously by Cizeau and Bouchaud using the cavity method [21]. Complementary, our main emphasis will be on the application of the replica method to the Levy spin glass. As noted also by Cizeau and Bouchaud a straightforward implementation of the classical version of the replica method for infinite range models [1] is impractical due to diverging order parameters. It is, however, possible to use a variant of the replica method that was developed to deal with non-Gaussian local field distributions characteristic for diluted spin glasses and complex optimization problems [27]. Until now this approach was used only in situations where the local field distribution is inadequately characterized by its second moment alone and higher moments of the distribution are needed for a complete description. Here we show that the method may also be adapted to situations where the moments may not even exist.

The paper is organized as follows. After the precise definition of the model in the next section we recall in section III the main steps of the RS cavity treatment performed by Cizeau and Bouchaud. Section IV comprises our replica analysis including the results for the spin glass transition temperature and the influence of a ferromagnetic bias in the coupling distribution. In section V we describe our numerical simulations and compare their results for the transition temperature with our analytical findings. Finally, section VI gives a short discussion of the results and points out some open problems.

II. THE MODEL

We consider a system of N Ising spins $S_i = \pm 1$, $i = 1, ..., N$ with Hamiltonian

$$H(\{S_i\}) = -\frac{1}{2N^{1/\alpha}} \sum_{(i,j)} J_{ij} S_i S_j \ , \tag{1}$$

where the sum is over all pairs of spins. The couplings $J_{ij} = J_{ji}$ are independent, identically distributed random variables drawn from a symmetric Levy distribution $P_\alpha(J)$. It is defined by its characteristic function [28]

$$\tilde{P}_\alpha(q) := \int dJ \; e^{-iqJ} \; P_\alpha(J) = e^{-|q|^\alpha} \tag{2}$$

with the real parameter α, $0 < \alpha < 2$. A Gaussian distribution of couplings as in the standard SK model is obtained in the limit $\alpha \to 2$.

Levy distributions are stable distributions which roughly means the following. If $x_i, i = 1, ..., N$ are independent random variables drawn from a Levy distribution $P_\alpha(x)$ their sum, $z = \sum_i x_i$, is distributed according to $P_\alpha(z/N^{1/\alpha})$, i.e. z is also Levy distributed with the same parameter α, albeit with a width increased by a factor $N^{1/\alpha}$. Correspondingly the exchange fields

$$h_i^{\text{exch}} := \sum_j \frac{J_{ij}}{N^{1/\alpha}} S_j \tag{3}$$

in a Levy spin glass are Levy distributed and the scaling of the couplings with $1/N^{1/\alpha}$ ensures that they are of order 1 for $N \to \infty$ such that the Hamiltonian (1) is extensive.

¿From the definition (2) we also find the asymptotic form of $P(J)$ for large $|J|$ to be

$$P(J) \sim \frac{1}{|J|^{\alpha+1}} \; . \tag{4}$$

From this asymptotic behaviour and the interval of admissible values of α it is clear that the second and higher moments of Levy distributions do not exist. The long tail of the distribution also implies that the largest among N independent Levy variables is of order $N^{1/\alpha}$, i.e. of exactly the same order as their sum. The sum is hence dominated by its largest summands. Each spin in a Levy spin glass is therefore coupled to the majority of other spins by weak couplings of order $1/N^{1/\alpha}$ and to a few ($\mathcal{O}(1)$ for $N \to \infty$) by strong bonds of order 1.

The thermodynamic properties of the system are described by the ensemble averaged free energy

$$f(\beta) := -\lim_{N\to\infty} \frac{1}{\beta N} \overline{\ln Z(\beta)} \; , \tag{5}$$

with the partition function

$$Z(\beta) := \sum_{\{S_i\}} \exp(-\beta H(\{S_i\})) \; . \tag{6}$$

Here β denotes the inverse temperature and the overbar stands for the average over the random couplings J_{ij}.

III. CAVITY ANALYSIS

The first statistical mechanics analysis of the Levy spin glass was performed 15 years ago by Cizeau and Bouchaud [21] using a variant of the cavity method. In the traditional form of the cavity method for fully connected systems [17] one considers a system of N spins $\{S_i\}$ in a pure equilibrium state and adds N new couplings $J_{0i}, i = 1, ..., N$ between these existing spins and a *cavity* which will later accommodate the $(N+1)$-st spin S_0. For both the couplings J_{ij} in the N-spin system and for the new couplings one particular realization is considered. The exchange field (3) in the cavity

$$h_0^{\text{exch}} = \sum_{i=1}^{N} \frac{J_{0i}}{N^{1/\alpha}} S_i \tag{7}$$

is then a random variable due to the thermal fluctuations of the S_i. The clustering property of pure states of an equilibrium system ensures that the connected correlation functions of the spins tend to zero for $N \to \infty$ [17] and therefore h_0^{exch} is a sum over many, asymptotically independent random variables. If all J_{0i} are of the same order of magnitude this implies a Gaussian distribution of the cavity field uniquely characterized by its variance. Further manipulations generate a self-consistent equation for this variance from which all replica symmetric properties of the system may be derived.

In the case of a Levy spin glass, however, a typical realization of the couplings J_{0i} contains a few very large bonds. This invalidates the central limit theorem (the Lindeberg criterion is not fulfilled, see [30]) and the cavity field distribution is not Gaussian. The traditional form of the cavity method for infinite range models is hence not applicable to the Levy spin glass.

As observed by Cizeau and Bouchaud it is however possible to employ a variant of the cavity method as later used also in the analysis of spin systems on locally tree-like graphs [20] which is known to physicist as Bethe-Peierls approximation [31, 32] and to computer scientists as *belief propagation* [33]. This method builds on the fact that with S_i being a binary quantity its marginal probability distribution

$$P(S_i) = \frac{1}{Z} \sum_{\{S_j\}_{j\neq i}} \exp\left(-\beta H(\{S_j\})\right) \tag{8}$$

can be parametrized by a single variable which we take to be the *local* field h_i defined by

$$h_i := \frac{1}{\beta} \text{artanh}\langle S_i \rangle \ . \tag{9}$$

Accordingly we find

$$P(S_i) = \frac{e^{\beta h_i S_i}}{2\cosh(\beta h_i)} \tag{10}$$

as well as

$$m_i := \langle S_i \rangle = \tanh(\beta h_i) \ . \tag{11}$$

The local field must not be confused with the exchange field (3). Unlike the latter it is not thermally fluctuating. If the cavity distribution is Gaussian the thermal average of the exchange field coincides with the local field [17]. However, in the general case and in particular for the Levy spin glass this does not hold.

For the marginal distribution of the new spin S_0 we have

$$P(S_0) = \frac{e^{\beta h_0 S_0}}{2\cosh(\beta h_0)} \tag{12}$$

as well as

$$P(S_0) = \sum_{\{S_i\}} \exp\left(\beta S_0 \sum_{i=1}^{N} J_{0i} S_i\right) P(\{S_i\}) \ . \tag{13}$$

Using the clustering property in the form

$$P(\{S_i\}) = \prod_{i=1}^{N} \frac{e^{\beta h_i S_i}}{2\cosh(\beta h_i)} \tag{14}$$

a straightforward calculation yields

$$h_0 = \frac{1}{\beta} \sum_{i=1}^{N} \text{artanh}\left(\tanh(\beta h_i) \tanh(\beta \frac{J_{0i}}{N^{1/\alpha}})\right) \ . \tag{15}$$

As observed by Cizeau and Bouchaud one may be tempted to expand in the argument of the second tanh for $N \to \infty$ to find the familiar expression

$$h_0 = \sum_{i=1}^{N} \frac{J_{0i}}{N^{1/\alpha}} m_i \ . \tag{16}$$

However, this would be unjustified since some of the $J_{0i}/N^{1/\alpha}$ are not small.

The local field h_0 as given by (15) is a random quantity both due to its dependence on the old couplings J_{ij} determining the h_i and on the new couplings J_{0i}. As long as $|m_i| = |\tanh(\beta h_i)| < 1$ the non-linearity in (15) suppresses the influence of the few large J_{0i}. As a consequence the second moment of the local field distribution $P(h_0)$ exists:

$$\begin{aligned} Q := \overline{h_0^2} &= \frac{1}{\beta^2} \overline{\sum_{i,j} \text{artanh}(\tanh(\beta h_i) \tanh(\beta \frac{J_{0i}}{N^{1/\alpha}})) \, \text{artanh}(\tanh(\beta h_j) \tanh(\beta \frac{J_{0j}}{N^{1/\alpha}}))} \\ &= \frac{1}{\beta^2} \sum_i \overline{\text{artanh}^2(\tanh(\beta h_i) \tanh(\beta \frac{J_{0i}}{N^{1/\alpha}}))} \\ &= \frac{N}{\beta^2} \int dh P(h) \int dJ P_\alpha(J) \, \text{artanh}^2(\tanh(\beta h) \tanh(\beta \frac{J}{N^{1/\alpha}})) \ . \end{aligned} \tag{17}$$

Cizeau and Bouchaud therefore argue that the central limit theorem may be applied and that $P(h_0)$ is Gaussian [21, 29]. Using translational invariance of the ensemble averaged system (17) may then be written as a self-consistent condition for Q. Using (2) this equation acquires the form

$$Q = \frac{C(\alpha)}{\beta^2} \int \frac{dh}{\sqrt{2\pi Q}} \exp\left(-\frac{h^2}{2Q}\right) \int \frac{dJ}{|J|^{\alpha+1}} \operatorname{artanh}^2(\tanh(\beta h)\tanh(\beta J)) \,, \tag{18}$$

where

$$C(\alpha) := \frac{\Gamma(\alpha+1)\sin(\frac{\alpha}{2}\pi)}{\pi} > 0 \tag{19}$$

is a numerical constant.

Solving (18) numerically one obtains $Q(\beta)$ from which the free energy and all thermodynamics properties may be derived. In particular one easily verifies that the paramagnetic state with $Q = 0$ is always a solution. It is stable for small β and looses its stability at β_c given by

$$1 = C(\alpha) \int \frac{dJ}{|J|^{\alpha+1}} \tanh^2(\beta_c J) \,. \tag{20}$$

IV. REPLICA THEORY

A. General setup

Within the replica approach we employ the replica trick [1] to calculate the average in (5),

$$\overline{\ln Z} = \lim_{n\to 0} \frac{\overline{Z^n} - 1}{n} \,. \tag{21}$$

As usual we aim at calculating $\overline{Z^n}$ for integer n by replicating the system n times, $\{S_i\} \mapsto \{S_i^a\}$, $a = 1, ..., n$, and then try to continue the results to real n in order to eventually perform the limit $n \to 0$.

According to (1) and (6) the partition function is a sum of exponential terms with the exponents linear in the couplings J_{ij}. Due to the algebraic decay $P_\alpha(J) \sim 1/|J|^{\alpha+1}$ of the distribution $P_\alpha(J)$ for large $|J|$ the average $\overline{Z^n(\beta)}$ hence diverges for real β and we cannot proceed in the usual way.

On the other hand, for a purely imaginary temperature, $\beta = -ik$, $k \in \mathbb{R}, k > 0$, we find from the very definition of $P_\alpha(J)$, cf. (2)

$$\overline{Z^n(-ik)} = \sum_{\{S_i^a\}} \exp\Big(-\frac{k^\alpha}{2N} \sum_{i,j} \Big| \sum_a S_i^a S_j^a \Big|^\alpha + \mathcal{O}(1) \Big) \,. \tag{22}$$

Note that the scaling of the interaction strengths with N used in (1) makes the replica Hamiltonian extensive as it should be.

As characteristic for a mean-field system the determination of $\overline{Z^n}$ can now be reduced to an effective single site problem. To this end we use the notation $\vec{S} = \{S^a\}$ for a spin vector with n components. It is then convenient to introduce the variables

$$c(\vec{S}) = \frac{1}{N} \sum_i \delta(\vec{S}_i, \vec{S}) \,, \tag{23}$$

describing the fraction of lattice sites that share one out of the 2^n realizations of the spin vector $\vec{S}$ [27]. Clearly

$$\sum_{\vec{S}} c(\vec{S}) = 1 \,. \tag{24}$$

Because of the identity

$$\frac{1}{N} \sum_{i,j} f(\vec{S}_i, \vec{S}_j) = N \sum_{\vec{S},\vec{S}'} c(\vec{S}) c(\vec{S}') f(\vec{S}, \vec{S}') \tag{25}$$

the exponent in (22) is seen to depend on the spin configuration $\{\vec{S}_i\}$ solely through the variables $c(\vec{S})$. In order to transform the trace over $\{\vec{S}_i\}$ into an integral over the $c(\vec{S})$ we only need to determine the number of spin configurations that realize a given combination of $c(\vec{S})$. A standard calculation yields to leading order in N

$$\sum_{\{S_i^a\}} \prod_{\vec{S}}{}' \delta\Big(\frac{1}{N}\sum_i \delta(\vec{S}_i,\vec{S}) - c(\vec{S})\Big) = \exp\Big(- N\sum_{\vec{S}} c(\vec{S})\ln c(\vec{S})\Big) \tag{26}$$

where the prime at the product denotes that the constraint (24) has to be taken into account.

We may therefore write (22) in the form

$$\overline{Z^n(-ik)} = \int \prod_{\vec{S}} dc(\vec{S})\ \delta(\sum_{\vec{S}} c(\vec{S}) - 1)\ \exp\Big(- N\Big[\sum_{\vec{S}} c(\vec{S})\ln c(\vec{S}) + \frac{k^\alpha}{2}\sum_{\vec{S},\vec{S}'} c(\vec{S})c(\vec{S}')|\vec{S}\cdot\vec{S}'|^\alpha\Big]\Big)\ . \tag{27}$$

In the thermodynamic limit, $N \to \infty$, the integral in (27) can be calculated by the saddle-point method. The corresponding self-consistent equation determining the saddle-point values $c^{(0)}(\vec{\sigma})$ of the $c(\vec{\sigma})$ is given by

$$c^{(0)}(\vec{\sigma}) = \Lambda(n)\exp\Big(- k^\alpha \sum_{\vec{S}} c^{(0)}(\vec{S})|\vec{S}\cdot\vec{\sigma}|^\alpha\Big)\ , \tag{28}$$

where the Lagrange parameter $\Lambda(n)$ enforces the constraint (24).

B. Replica symmetry

Within the replica symmetric approximation one assumes that the solution of (28) is symmetric under permutations of the replica indices. This implies that the saddle-point values $c^{(0)}(\vec{S})$ may only depend on the sum, $s := \sum_a S^a$, of the components of the vector $\vec{S}$. After the limit $n \to 0$ is performed the function $c^{(0)}(s)$ can be related to the replica symmetric distribution $P(h)$ of local magnetic fields (9) via [27]

$$c^{(0)}(s) = \int dh\ P(h)\ e^{-ikhs} \qquad\qquad P(h) = \int \frac{ds}{2\pi}\ e^{ish}\ c^{(0)}(\frac{s}{k})\ . \tag{29}$$

In this way the self-consistent equation (28) may be transformed to a self-consistent equation for $P(h)$.

To proceed along these lines in the present case we use (29) in (28) and perform the following manipulations

$$\begin{aligned} k^\alpha \sum_{\vec{S}} e^{-ikhs}|\vec{S}\cdot\vec{\sigma}|^\alpha &= \int dr\ |kr|^\alpha \sum_{\vec{S}} \delta(r - \vec{S}\cdot\vec{\sigma})\ e^{-ikhs} \\ &= \int \frac{dr\ d\hat{r}}{2\pi}|kr|^\alpha e^{ir\hat{r}} \sum_{\vec{S}} \exp\Big(- ikhs - i\hat{r}\vec{S}\cdot\vec{\sigma}\Big) \\ &= \int \frac{dr\ d\hat{r}}{2\pi}|r|^\alpha e^{ir\hat{r}} \sum_{\vec{S}}\prod_a \exp\Big(- iS^a(kh + k\hat{r}\sigma^a)\Big) \\ &= \int \frac{dr\ d\hat{r}}{2\pi}|r|^\alpha e^{ir\hat{r}}\ [2\cos k(h+\hat{r})]^{\frac{n+\sigma}{2}}\ [2\cos k(h-\hat{r})]^{\frac{n-\sigma}{2}} \\ &\to \int \frac{dr\ d\hat{r}}{2\pi}|r|^\alpha e^{ir\hat{r}} \left[\frac{\cos k(h+\hat{r})}{\cos k(h-\hat{r})}\right]^{\frac{\sigma}{2}}\ , \end{aligned} \tag{30}$$

where the limit $n \to 0$ was performed in the last line and $\sigma := \sum_a \sigma^a$. Using $\Lambda(n) \to 1$ for $n \to 0$ [27] we therefore find from (28) in the replica symmetric approximation

$$c^{(0)}(\sigma) = \exp\Big(- \int dhP(h)\int \frac{dr\ d\hat{r}}{2\pi}|r|^\alpha \exp\Big(ir\hat{r} + \frac{\sigma}{2}\ln\frac{\cos k(h+\hat{r})}{\cos k(h-\hat{r})}\Big)\Big)\ . \tag{31}$$

Using this result in (29) we get

$$P(h) = \int \frac{ds}{2\pi}\exp\Big(ish - \int dh'P(h')\int \frac{dr\ d\hat{r}}{2\pi}|r|^\alpha \exp\Big(ir\hat{r} + \frac{s}{2k}\ln\frac{\cos k(h'+\hat{r})}{\cos k(h'-\hat{r})}\Big)\Big)\ . \tag{32}$$

We are now in the position to continue this result back to real values of the temperature by simply setting $k = i\beta$:

$$P(h) = \int \frac{ds}{2\pi} \exp\left(ish - \int dh' P(h') \int \frac{dr\, d\hat{r}}{2\pi} |r|^\alpha \exp\left(ir\hat{r} - i\frac{s}{2\beta} \ln \frac{\cosh \beta(h' + \hat{r})}{\cosh \beta(h' - \hat{r})} \right) \right) . \tag{33}$$

Finally the r-integral may be performed by using

$$\int \frac{dr\, d\hat{r}}{2\pi} |r|^\alpha e^{ir\hat{r}} f(\hat{r}) = -C(\alpha) \int \frac{d\hat{r}}{|\hat{r}|^{\alpha+1}} \begin{cases} [f(\hat{r}) - f(0)] & \text{if} \quad 0 < \alpha < 1 \\ [f(\hat{r}) - f(0) - \hat{r} f'(0)] & \text{if} \quad 1 < \alpha < 2 \end{cases} , \tag{34}$$

where f' denotes the derivative of f and $C(\alpha)$ is defined in (19). In our case we have $f(0) = 1$ and $f'(0) = 0$ as implied by $P(h) = P(-h)$. Hence no distinction between $\alpha < 1$ and $\alpha > 1$ needs to be made.

We therefore get finally the following self-consistent equation for the replica symmetric field distribution $P(h)$ of a Levy spin glass at inverse temperature β:

$$P(h) = \int \frac{ds}{2\pi} \exp\left(ish + C(\alpha) \int dh' P(h') \int \frac{d\hat{r}}{|\hat{r}|^{\alpha+1}} \left[\exp\left(-i\frac{s}{\beta} \text{artanh}(\tanh \beta h' \tanh \beta \hat{r}) \right) - 1 \right] \right) . \tag{35}$$

The structure of this equation is rather similar to the corresponding equation for the VB model [15]. It is also interesting to look at the second moment of $P(h)$ for which we find

$$\langle h^2 \rangle = \int dh P(h) h^2 = \frac{C(\alpha)}{\beta^2} \int dh P(h) \int \frac{d\hat{r}}{|\hat{r}|^{\alpha+1}} \, \text{artanh}^2(\tanh(\beta h) \tanh(\beta \hat{r})) \tag{36}$$

which is rather similar to (18). However, the $P(h)$ solving (35) is not Gaussian. This can be seen by inserting a Gaussian $P(h')$ in the r.h.s. of (35) which then gets not reproduced on the l.h.s.

C. Spin-glass transition

The paramagnetic field distribution, $P(h) = \delta(h)$, is always a solution of (35). To test its stability we plug into the r.h.s. of (35) a distribution $P_0(h)$ with a small second moment, $\epsilon_0 := \int dh P_0(h)\, h^2 \ll 1$, calculate the l.h.s. (to be denoted by $P_1(h)$) by linearizing in ϵ_0 and compare the new second moment, $\epsilon_1 := \int dh P_1(h)\, h^2$, with ϵ_0. We find $\epsilon_1 > \epsilon_0$, *i.e.* instability of the paramagnetic state, if the temperature T is smaller than the critical temperature $T_c(\alpha)$ given by

$$T_c(\alpha) = \left[C(\alpha) \int \frac{dy}{|y|^{\alpha+1}} \tanh^2(y) \right]^{1/\alpha} . \tag{37}$$

This result coincides with (20) of the cavity approach. To determine the threshold value of β at which the distribution of local fields develops a non-zero second moment it is hence not decisive whether $P(h)$ becomes Gaussian or not. In the limit $\alpha \to 2$ (37) correctly reproduces the value $T_c^{SK} = \sqrt{2}$ of the SK-model [11].

It is interesting to compare the temperature for the spin-glass transition with the temperature at which bonds satisfying $J_{ij} > TN^{1/\alpha}$ start to percolate. From (2) we find for the fraction c of these *strong* bonds per site

$$c = \frac{2C(\alpha)}{\alpha} \, T^{-\alpha} . \tag{38}$$

Since the strong bonds are distributed independently from each other a giant component connected by these bonds appears for $c \geq 1$ [34]. The percolation temperature is hence given by

$$T_p = \left(\frac{2C(\alpha)}{\alpha} \right)^{1/\alpha} . \tag{39}$$

The dependence of T_c and T_p on α is displayed in fig. 1. The percolation temperature is always lower than the spin-glass temperature as expected since a percolating backbone of strong bonds is incompatible with a paramagnetic phase. On the other hand the two temperatures never coincide which means that also the many weak bonds contribute significantly to the spin-glass transition in Levy spin glasses. The transition is therefore not a pure percolation transitions. As can be seen from fig. 1 the difference between T_c and T_p decreases with decreasing α in agreement with the fact that the tails of $P(J)$ comprise a larger and larger part of the probability.

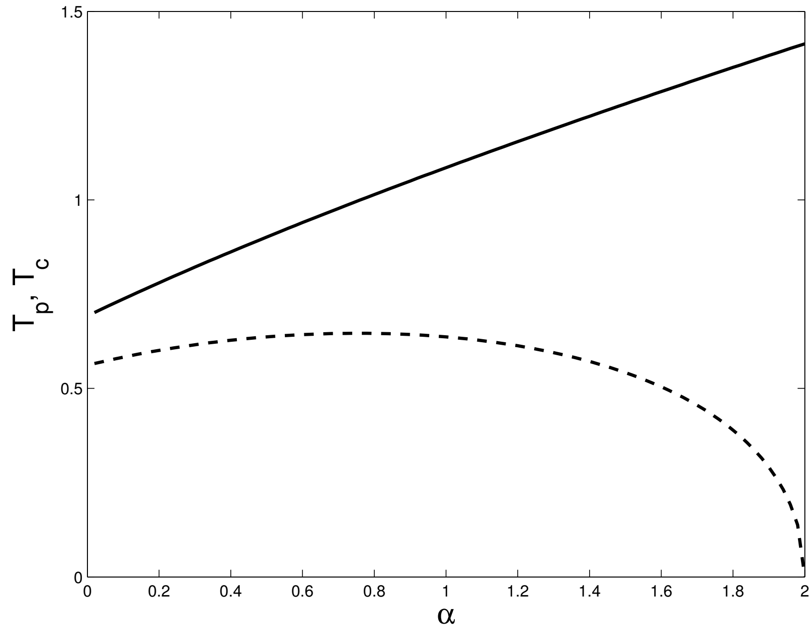

FIG. 1: Spin glass transition temperature T_c (full line) and percolation temperature T_p (dashed line) of an infinite-range spin-glass with Levy-distributed couplings as function of the parameter α of the Levy distribution defined in (2). For the scaling of the coupling strength with N as chosen in (1) there is a finite transition temperature for all values of α.

D. Asymmetric distribution of couplings

The replica calculation described above may be generalized to the case in which the distribution of couplings is not symmetric but shows a ferromagnetic bias, $P_\alpha(J) \neq P_\alpha(-J)$. The couplings are then drawn from a Levy distribution centered at $N^{\frac{1}{\alpha}-1} J_0$ where the N-dependence of the shift guarantees that the replica Hamiltonian remains extensive. Since a shift in the distribution amounts to a phase shift in the characteristic function the exponent of the replicated partition function is supplemented by

$$-\frac{ik}{2N} J_0 \sum_{i,j} \vec{S}_i \cdot \vec{S}_j = -N \frac{ik}{2} J_0 \sum_{\vec{S},\vec{S}'} c(\vec{S}) c(\vec{S}')\, \vec{S} \cdot \vec{S}' \,, \tag{40}$$

where the identity (25) was used. The corresponding saddle-point equation then reads

$$c^{(0)}(\vec{\sigma}) = \Lambda(n) \exp\Big(- k^\alpha \sum_{\vec{S}} c^{(0)}(\vec{S})\, |\vec{S} \cdot \vec{\sigma}|^\alpha - ikJ_0 \sum_{\vec{S}} c^{(0)}(\vec{S})\, \vec{S} \cdot \vec{\sigma} \Big) \,. \tag{41}$$

Using the RS ansatz we find for the new contribution

$$\begin{aligned} -ikJ_0 \int dh P(h) \sum_{\vec{S}} c^{(0)}(\vec{S})\, \vec{S} \cdot \vec{\sigma} &= -kJ_0 \int dh P(h) \int \frac{dr\, d\hat{r}}{2\pi} i r e^{ir\hat{r}} \left[\frac{\cos k(h+\hat{r})}{\cos k(h-\hat{r})} \right]^{\frac{\sigma}{2}} \\ &= -kJ_0 \int dh P(h) \int d\hat{r}\; \delta'(\hat{r}) \left[\frac{\cos k(h+\hat{r})}{\cos k(h-\hat{r})} \right]^{\frac{\sigma}{2}} \\ &= -kJ_0 \sigma \int dh P(h) \tan(kh), \end{aligned} \tag{42}$$

where the limit $n \to 0$ was performed after the single-site trace was completed. Performing the step back to real temperatures $k = i\beta$ we get a self-consistent equation for the replica symmetric field distribution. Since the field distribution is no longer symmetric for biased couplings, it is necessary to distinguish between the cases $0 < \alpha < 1$

and $1 < \alpha < 2$. In the former case we get

$$P(h) = \int \frac{ds}{2\pi} \exp\Bigg(ish + C(\alpha) \int dh' P(h') \int \frac{d\hat{r}}{|\hat{r}|^{\alpha+1}} \left[\exp\Big(-i\frac{s}{\beta} \text{artanh}(\tanh \beta h' \tanh \beta \hat{r}) \Big) - 1 \right] - isJ_0 \int dh' P(h') \tanh \beta h' \Bigg), \tag{43}$$

while for the latter case the equation reads

$$P(h) = \int \frac{ds}{2\pi} \exp\Bigg(ish + C(\alpha) \int dh' P(h') \int \frac{d\hat{r}}{|\hat{r}|^{\alpha+1}} \left[\exp\Big(-i\frac{s}{\beta} \text{artanh}(\tanh \beta h' \tanh \beta \hat{r}) \Big) - 1 + i\hat{r}s \tanh(\beta h') \right] - isJ_0 \int dh' P(h') \tanh \beta h' \Bigg). \tag{44}$$

From the structure of the self-consistent equation we again infer that the paramagnetic field distribution $P(h) = \delta(h)$ is always a solution. To test its stability we use the same procedure as in the previous section, taking into account that also a ferromagnetic instability may occur. To this end we plug into the r.h.s. of the self-consistent equation a distribution $P_0(h)$ with mean $\gamma_0 := \int dh P_0(h) h \ll 1$ and variance $\epsilon_0 := \int dh P_0(h)(h - \gamma_0)^2 \ll 1$. We calculate the l.h.s. (to be denoted by $P_1(h)$) to the leading order in the small parameters, and compare the resulting cumulants of the distribution P_1 with the corresponding quantities of the distribution P_0. The phase transition from a paramagnetic to a ferromagnetic state occurs, if

$$\gamma_0 < \gamma_1 = J_0 \beta \gamma_0 + \mathcal{O}(\gamma_0^2, \gamma_0 \epsilon_0, \epsilon_0^2). \tag{45}$$

We therefore find an instability toward a ferromagnetic state at $T_c^{\mathrm{FM}} = J_0$ which is independent of α. The result for the spin glass transition temperature remains the same as in the unbiased case.

V. NUMERICAL SIMULATIONS

In order to check our analytical results for the spin glass transition temperature we have performed Monte Carlo simulations [35, 36] using the parallel tempering approach [37, 38]. For a given realization $\{J_{ij}\}$ of the disorder, K independent configurations $\{S_i^k\}$ $(k = 1, \ldots, K)$ are simulated at K different temperatures $T_1 < T_2 < \ldots < T_K$, i.e. $\{S_i^1\}$ at T_1, $\{S_i^2\}$ at T_2 etc [41]. One step of the simulation, i.e. one Monte Carlo sweep, consists of the following steps:

- For each of the configurations $k = 1, \ldots, K$, one sweep of *local Metropolis steps* is performed. Each sweep consist of N times selecting a spin $i_0 \in \{1, \ldots, N\}$ randomly (uniformly). For each selected spin, the energy difference ΔE between the current configuration $\{S_i^k\}$ and the configuration where just spin $S_{i_0}^k$ is flipped ($S_{i_0}^k \to -S_{i_0}^k$) is calculated: $\Delta E = H(\{S_i^k\}) - H(\{S_i^k| - S_{i_0}^k\})$. The flip of spin $S_{i_0}^k$ is actually performed with the Metropolis probability $p_{\text{flip}} = \min\{1, \exp(-\Delta E/T_k)\}$, otherwise the current configuration remains unaltered.
- $K-1$ times an *exchange step* is tried: A temperature $k_0 \in \{1, \ldots, K-1\}$ is selected randomly, each temperature with the same probability $1/(K-1)$. The energy difference $\Delta E_{\text{exch}} = H(\{S_i^{k_0}\}) - H(\{S_i^{k_0+1}\})$ between the configurations at neighboring temperatures T_{k_0} and T_{k_0+1} is calculated. The two configurations $\{S_i^{k_0}\}$ and $\{S_i^{k_0+1}\}$ are exchanged with probability $p_{\text{exch}} = \min\{1, \exp(-\Delta E_{\text{exch}}(1/T_{k_0} - 1/T_{k_0+1}))\}$. In this way, the configurations perform a random walk in temperature space and can visit all temperatures T_k.

Furthermore, for each temperature, we simulate two independent sets of configurations $\{S_i^k\}$, $\{\tilde{S}_i^k\}$ which allows for a simple calculation of the overlap

$$q = \frac{1}{N} \sum_i S_i \tilde{S}_i \tag{46}$$

at each temperature T_k. From this overlap we calculate the Binder cumulant

$$B_N(T) = \overline{\frac{1}{2}\left(3 - \frac{\langle q^4 \rangle}{\langle q^2 \rangle^2}\right)} \tag{47}$$

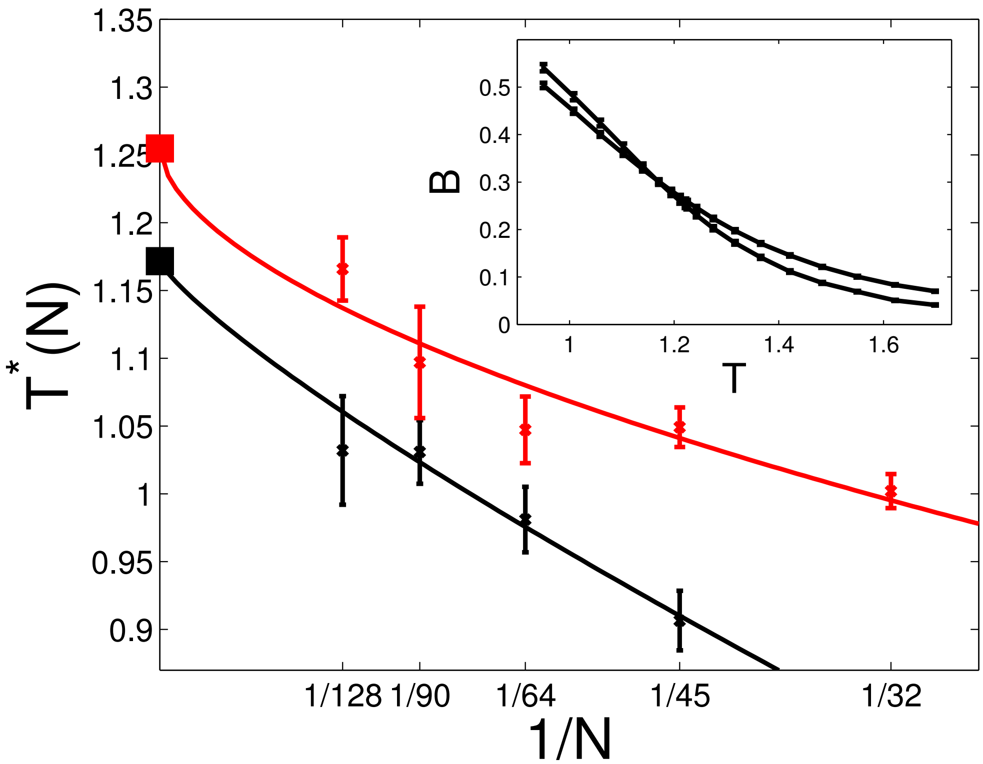

FIG. 2: Transition temperatures $T^\star(N)$ for $\alpha = 1.25$ (black) and $\alpha = 1.5$ (red) as a function of the inverse system size. $T^\star(N)$ is determined from the intersection points of the Binder parameters of the overlap q for system sizes N and $2N$ as shown in the inset for $N = 128$ and $N = 256$ ($\alpha = 1.5$). The lines show fits for the finite-size scaling of the form $T^\star(N) = T_c + aN^{-b}$. Here $T_c(\alpha = 1.25) \simeq 1.172$ and $T_c(\alpha = 1.5) \simeq 1.254$ respectively indicated by the boxes on the left of the figure are the analytical values for the critical temperatures as given by (37) whereas a and b are fit parameters. As can be seen the numerical results and the analytical values are compatible with each other.

for all T_k and various values of N. The critical temperature is then determined from the intersection points of the lines $B_N(T)$ for different values of N.

We have found that the traditional single-spin local update works very well for large values of $\alpha \geq 1.5$. For smaller values, the probability that bonds with a very large magnitude (e.g. $|J_{ij}| > 10$) appear in a realization becomes significant. A spin, which is adjacent to such a bond, will satisfy such a bond on all timescales, for the range of temperatures studied here. Hence, the spin will be frozen under single-spin-flip dynamics. For this reason, we have extended the local update by a *cluster flip*: In advance, all large bonds with $|J_{ij}| \geq J_{\max}$ are determined. Next, we calculate the maximal components of sites connected by these large bonds. The cluster flip consist of an attempt to flip a randomly chosen cluster, i.e. all spins of the cluster simultaneously, with the usual Metropolis p_{flip} probability as stated above, where $\Delta E = H(\{S_i^k\}) - H(\{S_i^k|\text{cluster flipped}\})$. Note that the clusters contain only spin indices, i.e. are indpependent of the actual relative orientations of the spins, since these might change during the simulation by other update steps, e.g. the standard single-spin-flip step.

In case of a single chosen value of $J_{\max}$, if $J_{\max}$ is large, then the clusters will be small, which might lead, in some cases, not to frozen single spins but to some practically frozen clusters. On the other hand, if $J_{\max}$ is small, the clusters will be large, hence the spins inside a cluster are frozen relative to each other. To avoid these problems, we have generated, before the actual simulation starts, several sets C_n of clusters for different values of $J^n_{\max}$. We started at $J^1_{\max} = 2T_K$ to obtain C_1. Then we increment $J_{\max}$ iterativly by 1. A new set C_n is stored, if it differs from the previous set C_{n-1}. This is continued until $C_{n_{\max}}$ consists only of clusters of size 2. During the simulation of the single configurations, each time the local Metropolis step is chosen with probability $p = 0.8$ and a cluster attempt with probability $1 - p = 0.2$. Note that for the Metropolis step still all spins are considered for single-spin flips, independent on how the clusters look like. Hence, large bonds which are not satisfied will become satisfied in this way, and also there is a small probability that bonds with a large magnitude become unsatisfied during the simulation. Hence, ergodicity is guaranteed. For the cluster attempt, one set C_n of clusters is selected randomly (all with the same probability), and from the set one cluster, again with equal probability. Hence, detailed balance is fulfilled.

After checking our code by reproducing the known result $T_c(\alpha = 2) = \sqrt{2}$ for the SK model we have investigated the cases $\alpha = 1.25$ and $\alpha = 1.5$ in more detail. Guided by the analytical result (19) for $T_c(\alpha = 1.25) \simeq 1.172$ and $T_c(\alpha = 1.5) \simeq 1.254$ we chose in both cases 19 temperatures in the range $[0.87 : 2.0]$. The temperatures T_i are determined such that for the largest system size $N = 256$ the average acceptance rate of the exchange steps is at least 0.5 for all pairs of neighboring temperatures. For all system sizes, the same set of temperatures is used, which allows for a better comparison of the results.

At the beginning of the simulation all configurations are random. We equilibrate the system until the squared overlap q^2 as a function of time, averaged over the last half of the simulation, has become independent of the number of Monte Carlo sweeps for all temperatures. Furthermore, we verify that the distribution of overlaps measured during this period is symmetric with respect to $q = 0$. For the case $\alpha = 2.0$, we have additionally employed the equilibration criterion from [39] and verified that the above listed criteria are compatible with it.

After equilibration, spin configurations are stored for later analysis at all temperatures every Δt Monte Carlo sweeps. Δt is chosen such that it corresponds to the typical time one configuration needs to walk in temperature space from the lowest temperature T_1 to the highest T_K and back to T_1. Since at the highest temperature, well above the phase transition temperature, the configurations forget their history at low temperatures, the stored configurations are statistically independent. Typical values for Δt range from $\Delta t = 150$ ($N = 32$) to $\Delta t = 250$ ($N = 256$). For each realization, we sample 1000 configurations and average for each system size over 1000 realizations. The results for T_c obtained in this way are compatible with the theoretical result as shown in fig. 2.

VI. DISCUSSION

Infinite-range spin glasses with Levy-distributed couplings are interesting examples of disordered systems. Due to the long tails in the distribution of coupling strengths they interpolate between systems with many weak couplings per spin as the Sherrington-Kirkpatrick model and systems with few strong couplings per spin as the Viana-Bray model. The broad variations in coupling strengths brought about by the power-law tails in the Levy distribution violate the Lindeberg condition for the application of the central limit theorem and give rise to non-Gaussian cavity field distributions with diverging moments. In the present paper we have shown that it is nevertheless possible to derive the replica symmetric properties of the system in a compact way by using the replica method as developed for the treatment of strongly diluted spin glasses and optimization problems [27]. This approach focuses from the start on the complete distribution of fields rather than on its moments.

The central result of our analysis is the self-consistent equation for the distribution of local fields, $P(h)$, as given by eq. (35). From this equation the expression (37) for the critical temperature of the spin glass transition may be derived. In Levy spin glasses there is for all temperatures a fraction of strong bonds per site which cannot be broken thermally. Comparison of the spin-glass transition temperature with the temperature at which these strong bonds start to percolate through the system reveals that the spin-glass transition in a Levy glass is not a pure percolation transition. The contribution of the many weak couplings cannot be neglected and becomes increasingly important as the parameter α in the Levy distribution approaches the limit $\alpha = 2$ corresponding to the SK model.

Our results show similarities and differences with those of the cavity analysis of Cizeau and Bouchaud [21]. The results for the critical temperature are the same because the expressions for the second moment of the local field distribution coincide. However, we do not find a Gaussian distribution of local fields for $T < T_c$ as assumed by Cizeau and Bouchaud on the basis of the cavity expression (15). From the numerical solution of TAP equations for the SK model it is known that for all $T < T_c$ a certain fraction of local magnetizations $m_i = \tanh \beta h_i$ are extremely near to 1 [40]. As this seems likely to be the case in Levy spin glasses as well it is conceivable that the distribution of h_i in (15) is such that the Lindeberg criterion is again violated and that the central limit theorem may not be applicable.

Several open questions may be addressed in forthcoming work in order to completely characterize the properties of Levy spin glasses. First the self-consistent equation for $P(h)$ should be solved, either numerically or analytically in limiting cases. Building on these results the replica symmetric picture of the low-temperature phase may be completed and compared with the findings from the cavity approach. Since replica symmetry is certainly broken at low temperature a stability analysis of the RS saddle point (28) needs to be performed and it is to be checked whether the deAlmeida-Thouless temperature T_{AT} is indeed smaller than T_c as found within the cavity approach. Finally the structure of the solution with broken replica symmetry is to be elucidated. Within the replica approach adopted in the present paper this is known to be very complicated such that for this task a cavity analysis looks more promising. Finally, improved numerical simulations will contribute to a better understanding of the intricate properties of Levy spin glasses.

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