Bootstrap percolation, the role of anisotropy. Questions, some answers and applications.

Bootstrap Percolation (also called: diffusion percolation, jamming percolation, k-core percolation, neuropercolation, quorum percolation...). Models for metastable behaviour; growth, fracture, nucleation, glasses, the brain, contagion...
Contributions by computational phyicists, mathematical physicists, neurophyicists, probabilists, combinatorialists...

We consider variables $\sigma_i^t = 0, 1$, living on a lattice, so $i \in Z^d$, at discrete times t. (Sometimes more general graphs)
Dynamics deterministic in discrete time.
Cellular Automaton rule:
$\sigma_i^{t+1} = f(\{\sigma_j^t, j \in N(i)\}$.
Depends on choice of function f and choice of neighborhood N.
We choose f increasing (the more 1's at time t, the more $1's$ at the next step, and allow only the change from 0 to 1 (empty to occupied, or healthy to sick).

Standard choice in d=2: transition only if an empty (= healthy) site has at least 2 occupied (=infected) nearest neighbours.
Very similar model: modified BP, at least 2 occupied neighbours in two orthogonal directions.
(Site) percolation occurs if there is a top-down connection (an infinite connected cluster).

QUESTIONS: Start with i.i.d initial conditions, probability of being sick is p, of being healthy is $1-p$
(Primordial Soup, only probability is in the beginning).
Q1) What happens in the long run?
Q2) Is there a threshold (critical) value of p, above which everyone gets infected?
Q3) How big is it?
Q4) What happens if you have a large but finite box?
Q5) What happens in higher dimensions?
Q6) What happens with different, anisotropic, neighbourhoods (or rules)?
Q7) What happens with anisotropic neighborhoods in higher dimensions?
Q8) What are these models good for?
Known results on Q1) to Q5), new on Q6) and Q7).

Examples and notation:

$d = 2$, (a, b)-models;

Neighborhood $\mathcal{N}$ with a neighbours in x-direction, b neighbours in y-direction.

$d = 3$, (a, b, c)-models;

Neighborhood $\mathcal{N}$ with a neighbours in x-direction, b neighbours in y-direction, c neighbours in z-direction.

Growth if at least half of the neighborhood sites are occupied.

Example: (1,2)model

$$\mathcal{N} \quad = \quad \begin{array}{ccccc} & & \bullet & & \\ \bullet & \bullet & 0 & \bullet & \bullet \\ & & \bullet & & \end{array}$$

Standard-BP answers to Questions 1-4)
For every p everyone will get sick, so $p_c = 0$.
Proof: Let large (N by N) square be occupied. Large here means $N \geq C \times \frac{1}{p}$, with C large. This happens with density p^{N^2}, which is positive, so somewehere there is one.
The probability that it grows to an $N+2$-by-$N+2$ square is
$[1-(1-p)^N]^4 \simeq [1-e^{-Np}]^4$
which is close to 1.
The probability that it keeps growing is the product $\Pi_{j=N,\ldots\infty}$ of the above.
The log of this probability will then be a sum of terms approximately of the form
$4\Sigma_{j=N,\ldots\infty} - e^{-pj} = -O(\frac{1}{p}e^{-Np})$ which is small when $N \geq C \times \frac{1}{p}$. So then the probability to keep growing forever is an infinite product which still will be close to one, if C is large.
Such a box is example of "Critical Droplet".
(In fact an occupied diagonal is enough already.) You cannot do much better than above. (meaning choose N smaller).

Proper scale? Some results:
The minimal size of a volume V to show the " infinite-volume" behaviour (to contain a critical droplet -size $N = \frac{C}{p}$- with large probability) is given by $|V_c| = e^{O(N)} = e^{O(\frac{1}{p})}$.
Inversely
$p_c = O(\frac{1}{ln|V|})$,
large finite-size correction. Computational confusion.

The hard direction of the proof is based on an Iteration Lemma of Aizenman and Lebowitz. Call a rectangular box " internally spanned " if it fills up from the inside.
The Lemma says that for a large box (size N) to be internally spanned, there has to be a moderately large box, that is with size of longest side at least half as big (to be precise, $\frac{N}{2} - 2$), which also is internally spanned.

Then establish for which size the probability for a box to be internally spanned is minimal, and for which sizes it grows to 1.

More answers:
Sharp constants.
$V = e^{Cst\frac{1}{p}+o(\frac{1}{p})}$
with Cst often computable (Holroyd, leads to integral which is computable, but not by Mathematica). Bounds on second-order correction term $o(\frac{1}{p})$.

"Sharp threshold property": Statistical error much smaller than systematic error. Hence numerical values of Cst are way off from the true ones, and the literature is full of wrong claims by numerical (computational) physicists.
(Theorems by Balogh-Bollobas-statistical error-, and Holroyd-Gravner and Morris-systematic error). For your info, for BP and modified BP $Cst_{BP} = \frac{\pi^2}{18}$, and $Cst_{mBP} = \frac{\pi^2}{6}$.
Remark: Large finite-size effects translate in slow dynamics. Long wait till origin is occupied. Finite speed of growing critical droplet.

Answer to Q5.
Higher dimensions.
In dimension d take threshold for infection $\frac{1}{2}d$ neighbours. Still on the infinite lattice everyone gets sick for any initial p.
Proof : Induction on dimension (Schonmann). If a **HUGE** cube is occupied, on the sides we have a two-dimensional BP model which will grow, as we saw, once it is large enough ("reduced" model). By space ergodicity, somewhere such a cube exists. The size of the system to behave like an infinite lattice now becomes, for $d = 3$, $V = e^{e^{O(\frac{1}{p})}}$.
(Or, equivalently, $p_c = O(\frac{1}{lnlnN})$).
Every dimension an extra exponent in the expression for N, cq an extra logarithm in the expression for p_c. Numerics even more hopeless.

Asymptotic bounds Cerf-Cirillo and Cerf-Manzo. Generalisations of Aizenman-Lebowitz lemma. Constants again exactly computable.
(Holroyd,
Balogh-Bollobas-Morris-(Duminil-Copin))
Idea: For a connection in a fixed direction consider d-1-dimensional slices orthogonal to that direction. For a connection it is necessary that one of two things happens in each slice.
a) Either a d-1-dimensional critical droplet occurs in the slice, which is improbable,
b) or there exists no critical droplet but there is a percolation connection to which the slice helps between the filled slices, which typically are quite far apart. This is improbable because one is in a subcritical regime of a kind of percolation model with enough independence.

Answer to Q6:

Anisotropic neighborhoods.

2 neighbours out of NSW (Duarte model).

3 neighbours out of EENWWS, the $(1,2)$-model.

Critical droplet is large NS interval (long double NS interval) of occupied sites. Definition of "large" slightly different (logarithmic corrections). Take length

$N = C'\frac{1}{p}ln\frac{1}{p}$

Probability of growing distance N, N^2, orthogonally to the interval, E, or EW, is:
$(1-(1-p)^N)^{N^{(2)}} \simeq (1-e^{-pN})^{N^{(2)}}$
$= (1-p^{cst})^{N^{(2)}}$
Once it is that big, it starts growing with large probability in both directions again.
(Proof: Take again logarithms and sum. Sum is small for large C').
Density of critical droplets becomes
$p^N = e^{O(\frac{1}{p} ln^2 \frac{1}{p})}$.
(v.E.-Hulshof, using Gravner-Griffeath tools)
Exact constants known, e.g.$\frac{1}{12}$ for the $(1,2)$-model, and similarly known for $(1,b)$-models, or conjectured (for Duarte model). (v.E.-Duminil-Copin). (Optimal growth, faster than NS interval, starts fast in NS, then mainly EW, then all directions). Order of magnitude for general (a,b) models. Critical droplets rectangles of length $p^{-a} \ln \frac{1}{p}$ and width b provide
$V = \exp O(\frac{1}{p^a} \ln^2 \frac{1}{p})$
(v.E, A. Fey).

But more a precise analysis, in $d = 2$, gives changing shape of critical droplet. Related to anisotropic Ising model (Kotecký-Olivieri)? Difference is that Ising model always has "standard" scaling. Standard BP closely related to $T = 0$ Ising model. Critical droplets are squares (Wulff), in both situations. Shape-shifting growth of the critical droplet in anisotropic bootstrap percolation. Estimate via sequence of rectangles $R_{n,p}$ with sides

$$p^{-1-\frac{3n}{(\ln \frac{1}{p})}} \text{ by } \frac{n}{p},$$

with n growing from some initial value to $\frac{1}{3} \ln \frac{1}{p}$.

More general rules

Conjecture: the most relevant distinction is "balanced", speed of growth of a droplet the same in different directions. versus " unbalanced", like the $(1,2)$-model, with slow- and fast-growing directions. Partial results:

Duminil-Copin and Holroyd (general isotropic and balanced),

Bringmann-Mahlburg-Mellit (not isotropic but balanced example) ,

Bollobas-Smith-Uzzell (weaker results for very general class of models).

Anisotropy in high dimensions,(a, b, c)-models. Neighborhood with a neighbours in x-direction, b neighbours in y-direction, c neighbours in z-direction, $a \leq b \leq c$ (v.E.-A. Fey).
Answer: double exponent. One side via Schonmann's induction-on-dimension argument. Other side via generalisation of Aizenman-Lebowitz Iteration Lemma.
$(1, 1, 2)$-model: upper and lower bounds on V, of the form $e^{e^{O(\frac{1}{p})}}$, also for $(1, 1, c)$-models, (just as Cerf-Cirillo proved for $(1, 1, 1)$ models).
$(1, 2, 2)$-model: upper and lower bounds on V $e^{e^{O(\frac{1}{p} ln^2 p)}}$, also for $(1, b, c)$-models.

$(a,a,c)-model$: upper and lower bounds on V

$e^{e^{O(\frac{1}{p^a})}}$, also for

$(a,b,c)-model$: upper and lower bounds on V if a is less than b

$e^{e^{O(\frac{1}{p^a} ln^2 p)}}$.

Second-level exponent is the first-level exponent of "reduced" ((a,b)-model.

"Reason":

If e.g. $P = e^{-C\frac{1}{p}}$, it holds

$e^{\frac{1}{CP^n \ln^m P}} = e^{e^{O(\frac{1}{p})}}$ for all m,n.

Note $P^n = e^{-C\frac{n}{p}} = e^{O(-\frac{1}{p})}$.

Answer to last Question (What is it good for?)
Metastability, caricature.
Slow creation, or arrival of ”critical droplet- of stable phase in metastable environment. (Steam bubble in overheated water, icecube in undercooled water, water droplet in supersaturated steam, spontaneous creation by thermal fluctuations, if not caused by mechanical, external disturbance long wait).
Facilitated (kinetically constrained) models for glasses (Fredrickson-Anderson, Kob-Anderson, Martinelli-C.Toninelli-Cancrini-Roberto) you jump, or move, if you have at least k neighbors. Random, not deterministic dynamics, but still similar mechanism. Droplets become ”cages”. Glass, jamming. Still no infinite network of cages, by same argument, hence slow movement. No glass transition at finite density.

Rigidity, neural networks (neuron fires after enough inputs), magnetic models, economic models (buy Ipad if at least two of your neighbours bought one).

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