The many faces of percolation

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If you can establish long range connectivity,
then you can ...

... eat your breakfast egg without a spoon, ...

... eat your breakfast egg without a spoon, ...
... watch TV at the same time, ...

... eat your breakfast egg without a spoon, ...
... watch TV at the same time, ...
... win a game of “go”, ...

... eat your breakfast egg without a spoon, ...
... watch TV at the same time, ...
... win a game of "go", ...
... or catch the flu.

In spite of this broad range of phenomena (& much more!), conventional folklore:

percolation is simple & universal:

- continuous ("second order") phase transition
- for spatial systems (lattices, Voronoi tessalations, ...) : anomalous critical exponents depending on dimension d
- for random networks (locally loop-less, e.g. Erdös-Renyi): mean field exponents
- for directed systems (SIS epidemics): directed percolation has different exponents
- finite systems: Finite Size Scaling (FSS)

That's it !

NO, THERE IS MORE!

Growing random networks (Callaway, Hopcroft, Kleinberg, Newman & Strogatz, PRE 64, 041902 (2001):

- begin with empty graph (no nodes, no links)
- add one node
- with probability δ add a link between two randomly chosen nodes, if there are not yet linked node pairs
- repeat

NB: disregard connectivity, no preferential attachment

- For $\delta < \delta_c = 1/4$, there will be no ∞ cluster, i.e. $S_{max}/N \to 0$
- For $\delta > 1/4$, $S_{max}/N \to$ finite value $\rho(\delta) > 0$.
- $\rho(\delta)$ has no power law singularity $\rho(\delta) \sim \delta^\beta$ as in OP, but an essential singularity:
  all derivatives $d^k\rho(\delta)/d^k = 0$: "infinite order"
- for all $\delta < 1/4$, $S_{max} \sim N^{-\alpha(\delta)}$: "critical phase"
  (cf. Kosterlitz-Thouless-Berezinskii)

Hmm!

Boettcher, Singh, & Ziff, arXiv 1109.6567:
bond percolation on a hierarchical lattice with finite ramification
$\rightarrow$ discontinuous transition: S_{max}/N jumps (for $N \rightarrow \infty$) at $p = 1/2$.

Hmmm!

Are there more models with first order percolation transitions?

Large Percolation on interdependent networks
[Buldyrev et al., 2010, Parshani et al., 2010, S.-W. Son et al. 2011, ...]

Ordinary SIR epidemic on sparse random graph (locally loop-less);
infection probability = 1:

$u =$ prob{ end point of random link is not infected}
$\quad = u^{k-1} \quad (k = \text{degree})$

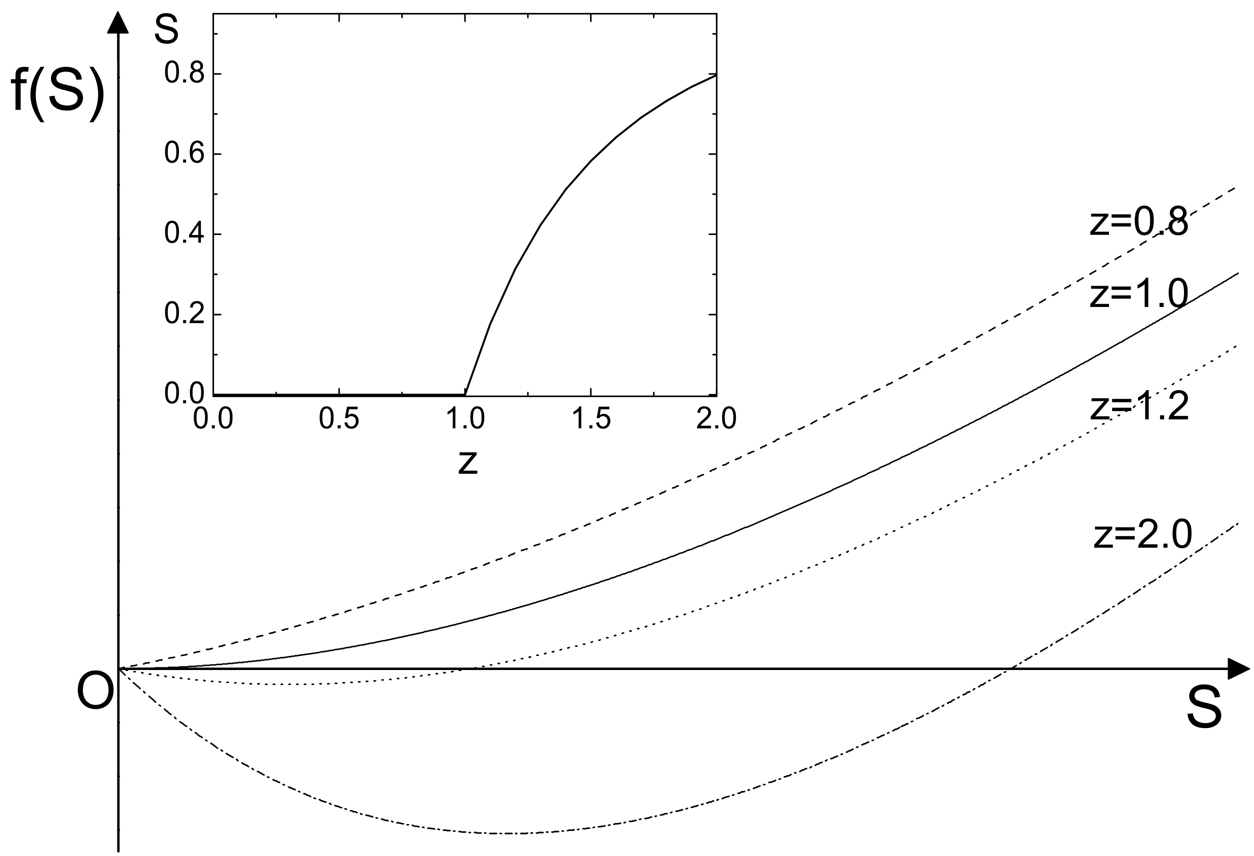
Let $p'(k) = kp(k)/\langle k \rangle$ = degree distribution of link endpoint;

$S = 1 - u$ = prob{ point is infected} = relative size of infected cluster

$S = 1 - \sum_k p'(k)(1-S)^{k-1} = 1 - G(1-S)$

$G(x) = -\sum_k p'(k)x^k = 1 + \langle k \rangle x + \frac{1}{2}\frac{\langle k(k-1) \rangle}{\langle k \rangle} x^2 + \ldots$

Def.: $f(S) = S - 1 + G(1-S)$
$\rightarrow f(S) = 0 :$ $\rightarrow$ self-consistency condition for relative cluster size



Erdös-Rényi:

Large Interdependent networks:

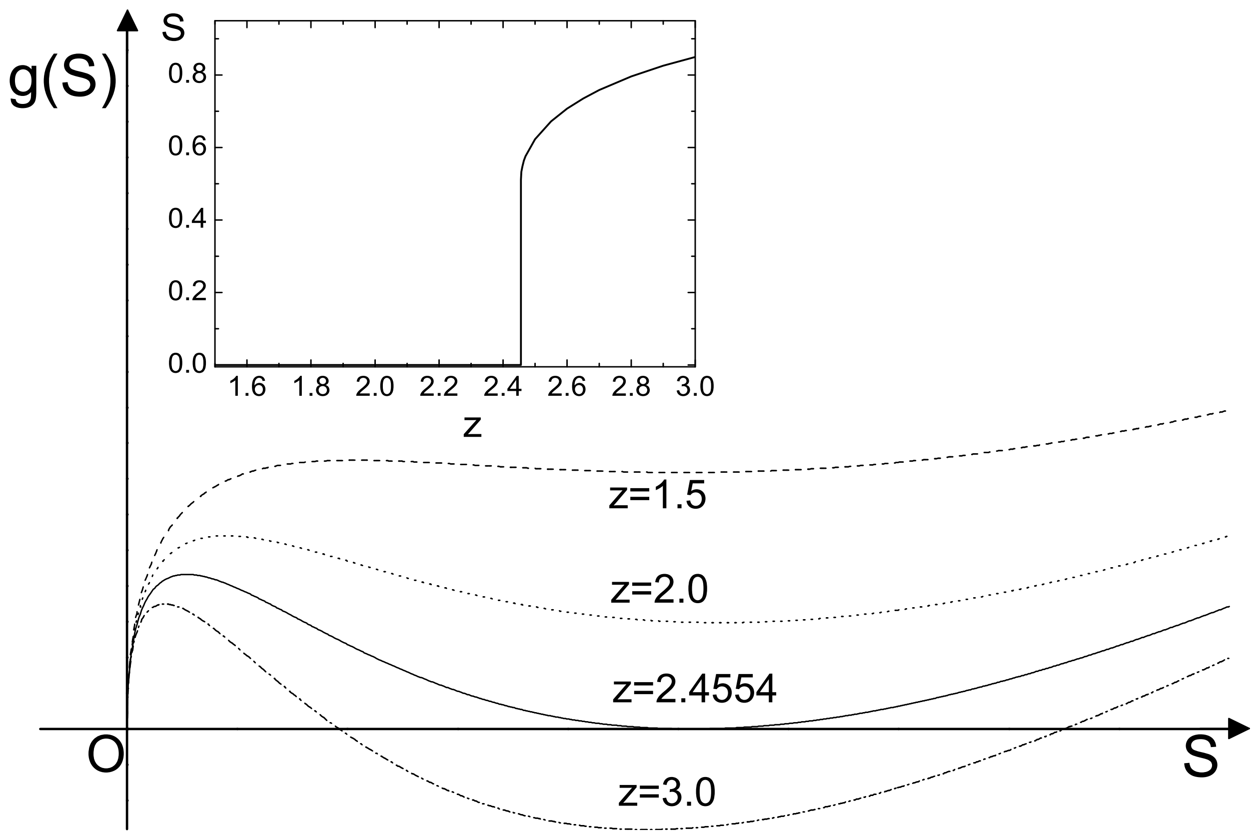
- same set of nodes
- two different sets of links: $\{\mathcal{A}, \mathcal{B}\}$
- cluster $C_{\mathcal{AB}}$ of sites is $(\mathcal{AB})$-connected, if any two sites $i, j \in C$ are connected both by $\mathcal{A}$-path and by $\mathcal{B}$-path, both entirely within C

node $i \in C_{\mathcal{AB}} \leftrightarrow$

- i is connected to at least one other node $j \in C_{\mathcal{AB}}$ through an $\mathcal{A}$-link
- & at least one node $\in C_{\mathcal{AB}}$ through a $\mathcal{B}$-link

For Erdös-Rényi with $\langle k \rangle_{\mathcal{A}} = \langle k \rangle_{\mathcal{B}}$:

$g(S) \equiv S - (1 - G(S))^2 = 0$



g(S)
S
0.8
0.6
0.4
0.2
0.0
1.6
1.8
2.0
2.2
2.4
2.6
2.8
3.0
z
z=1.5
z=2.0
z=2.4554
z=3.0
O
S

First order (discontinuous) transition!

Same simple analytic treatment also for other types of dependencies (Son et al., 2011):

- > 2 interdependent sets of links,
- mixture of “connectivity” & “dependency” links,
- some nodes need only single connection
- ...

Intuitive picture (Buldyrev et al., 2010):

Italy, September 28, 2003: Nation-wide power black out
$\mathcal{A}$ = network of power lines
$\mathcal{B}$ = information network

Failure of first node
to all nodes connected to it via $\mathcal{A}$ fail
to all nodes connected to these via $\mathcal{B}$ fail
to all nodes connected to these via $\mathcal{A}$ fail
...

Cascades of mutually induced failures → more abrupt onset of giant cluster

NO!

For $\mathcal{A}, \mathcal{B}$ subsets of links on 2-d lattices:

- $S \sim (\langle k \rangle - \langle k \rangle_c)^\beta$: continuous transition
- $\beta_{\mathrm{depend}} > \beta_{\mathrm{OP}}$: transition is *less* abrupt

Large Cooperative contagion

- Barbara tells Thilo: “movie X is good!”
  Thilo: ”Hm”
- Stefan tells Thilo: “movie X is good!”
  Thilo: ”Hm”
- Ginestra tells Thilo: “movie X is good!”
  Thilo: ”Hm”

- Barbara & Stefan & Ginestra tell Thilo: “movie X is good!”

  Thilo: ”Oh – Where can I see it!?”

More generally:

p_1 = prob{site is infected at first attack}
p_2 = prob{site that was not infected at first attack is infected at 2^{nd}}
p_3 = prob{site that is still not infected is infected at 3^{rd} attack}
etc

Ordinary bond percolation: $p_1 = p_2 = p_3 = \ldots$
Ordinary site percolation: $p_2 = p_3 = \ldots = 0$

p_k decrease with k: immunity strengthened by successive attacks: dull
p_k increases with k: cooperative contagion – interesting!

Alternative notation:
q_k = prob{site is infected after k attacks}

$$= p_1 + (1 - p_1)p_2 + \ldots (1 - q_{k-1})p_k$$

On sparse (i.e., locally loop-less) random graphs:
Similar treatment as before

Ordinary (bond) percolation with infection probability p:

$$1 - S = \sum_{k}^{\infty} kp'(k)(1 - pS)^{k-1}$$

$$F(S) \equiv \sum_{k=1}^{\infty} kp(k)\{(1 - pS)^{k-1} + (S - 1)\} = 0$$

Complex contagion:

$$(1 - pS)^{k-1} \to \sum_{n=0}^{k-1} \binom{k-1}{n} (1 - q_n) S^n (1 - S)^{k-n-1}$$

(each term corresponds to exactly n infective neighbors)

Transition continuous → discontinuous percolation transition:
$F(0) = F'(0) = F''(0) = 0$

$$q_1 = p_1 = \frac{\langle k \rangle}{\langle k(k-1) \rangle}$$

$$q_2 = 2q_1$$

(Dodds & Watts 2004).

Spatially embedded (regular lattice, ...):

cooperativity
→ surface of clusters become smoother
→ holes in cluster become smaller
→ cluster density increases

Tricritical point:
cluster becomes compact,
with rough but non-fractal surface

Beyond tricritical point:

“First-order percolation” ≡ rough pinned surfaces

Why did "explosive percolation" make such a splash?
Why did we, like Molière's Bourgeois Gentilhomme, not know that we speak 1^{st} order percolation, whenever we spoke rough pinned surfaces?

- Density of ∞ cluster has always finite density
- Nucleation: growth from point seed is hindered & goes through bottleneck

BUT:

- No jump in any plot of "order parameter" S against "control parameter" = #(bonds)

Is this a bona fide "first order" phase transition in the usual sense?

A short course on first & second order phase transitions:

Thermodynamics:

- order parameter is density of extensive quantity (magnetization, particle density, ...) or inverse density (specific volume)
- control parameter is conjugate variable (magnetic field, chemical potential, pressure, ...)
- e.g. van der Waals gas ($\sim$ water / vapor): if pressure is kept fixed below critical point, then energy jumps at $T = T_{boil}$ $\rightarrow$ forst order transition. If $p > p_c$, no such jump, all is continuous. If $p = p_c$, then no jump, but dE/dT is not continuous.
- similar, if $T = const$, p is used as control parameter

! if volume is used at fixed T, then NO jump, even if always $p < p_c$!

(water in pot with movable tight lid: as lid is moved, vapor $\leftrightarrow$ coexisting vapor/water $\leftrightarrow$ water)

Percolation:

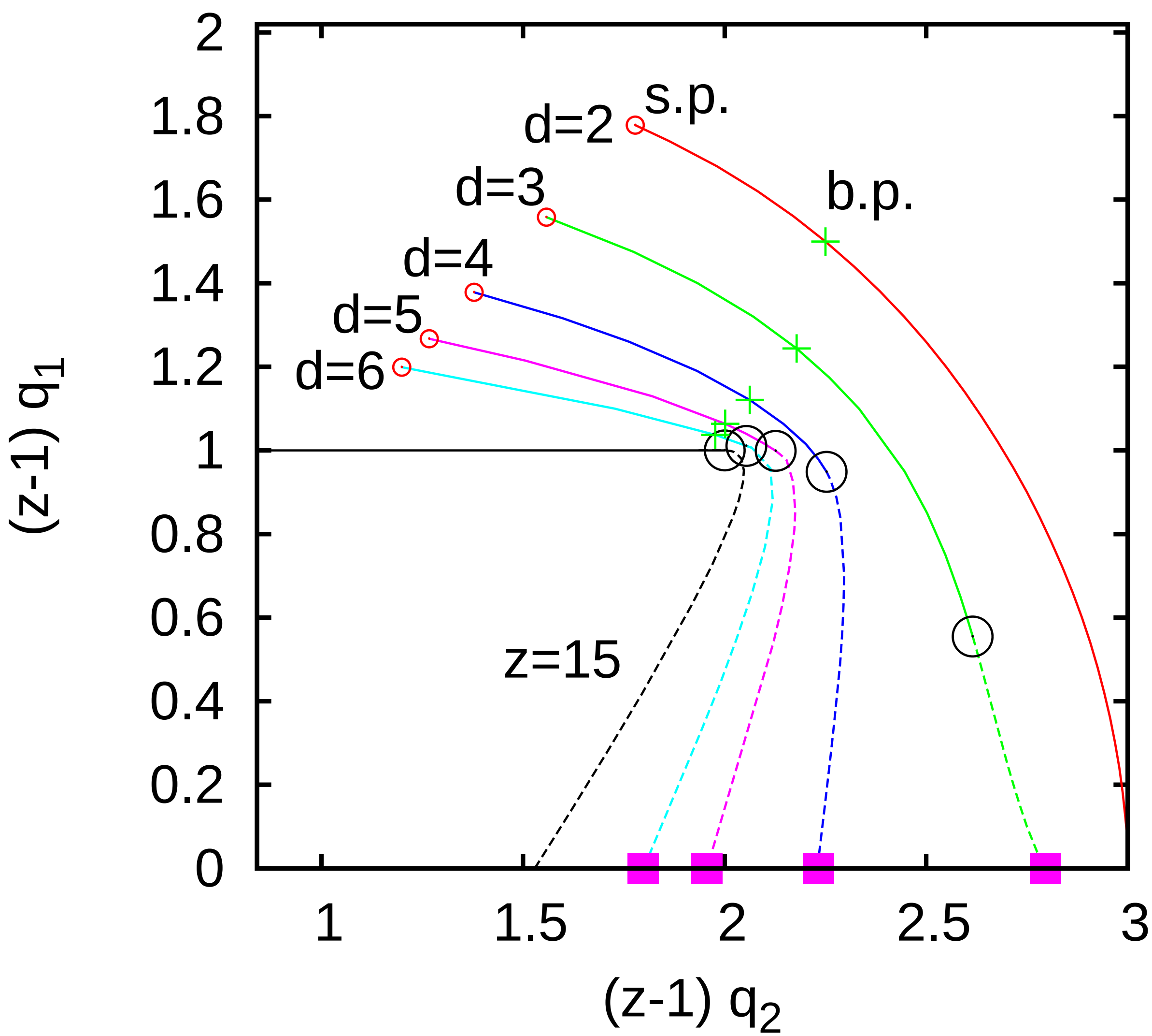
p=fraction of est'd bonds (bond percolation)
p=fraction of est'd sites (site percolation)

neither is conjugate to a density or to an inverse density, but both are densities themselves!

?!?! S_{max}/N is not expected to jump at a bona fide first order transition !?!?!?

Cooperative contagion: q_1, q_2 are NOT directly related to bond densities

Phase diagram for regular graphs:


s.p.
d=2
b.p.
d=3
d=4
d=5
d=6
z=15
(z-1) q_1
(z-1) q_2
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
1
1.5
2
2.5
3

$d = 2$: no tricritical point, pinned surfaces are always fractal & in percolation universality class

$d = 5$: Upper critical dimension for tricritical point, also upper critical dimemnsion for rough surfaces

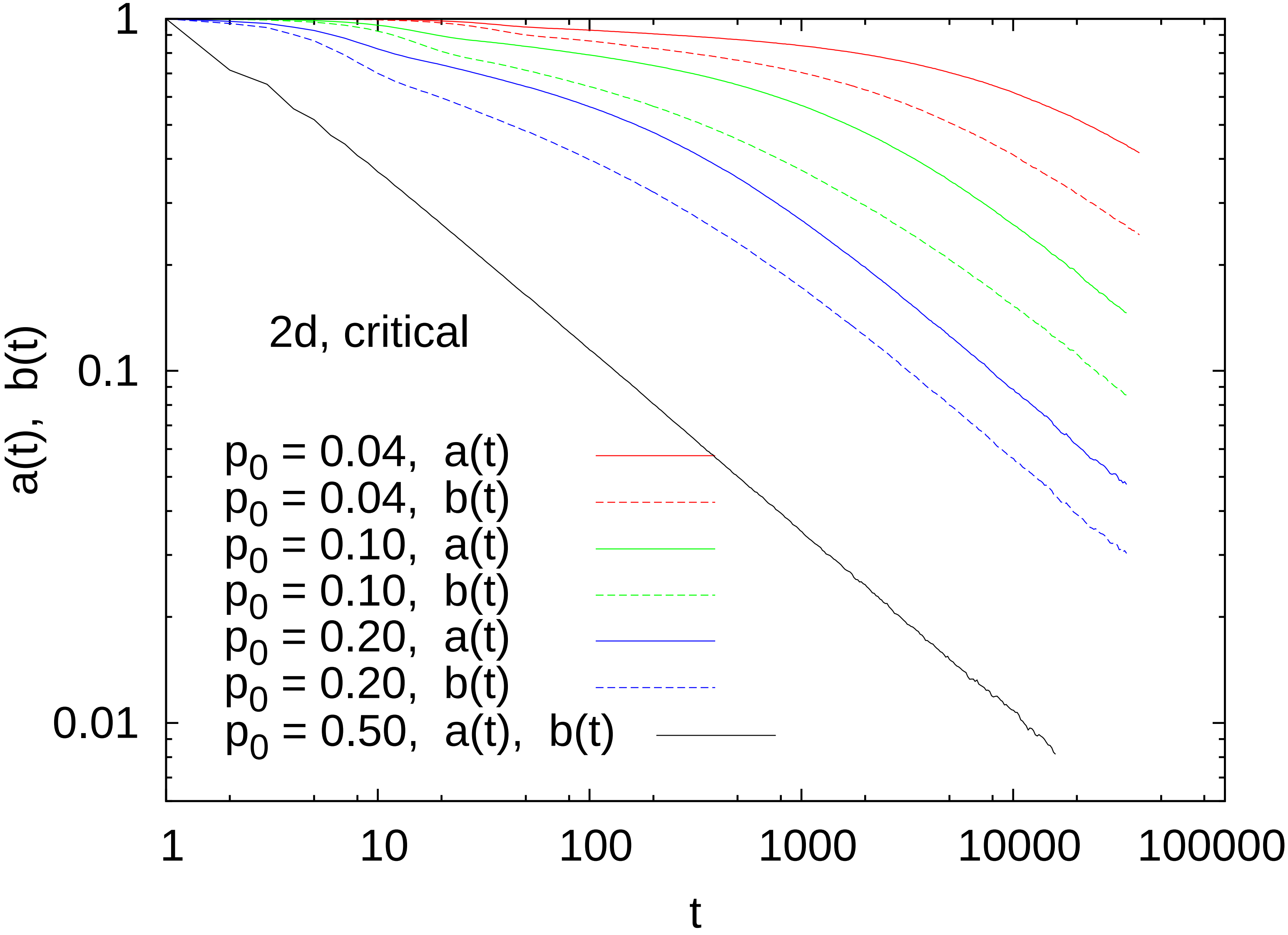
$d \leq 5$: ϵ-expansion (field theoretic RG)
H.K. Janssen, M. Müller, & O. Stenull, PRE **70**,026114 (2004):
– agrees with simulations for $d = 4, 5$
– disagrees for $d = 3$

Points on x-axis ($p_1 = 0$): "Bond bootstrap" percolation
Ordinary bootstrap percolation: all bonds are present, but sites are only present when $> k$ neighbors.
Here: Bonds are present with probability < 1, sites are only present when ≥ 2 neighbors
– $d > 3$: "Edwards-Wilkinson-type" theory (no overhangs) give correct scaling – $d = 3$: Numerically observed scaling disagrees with EW-predictions

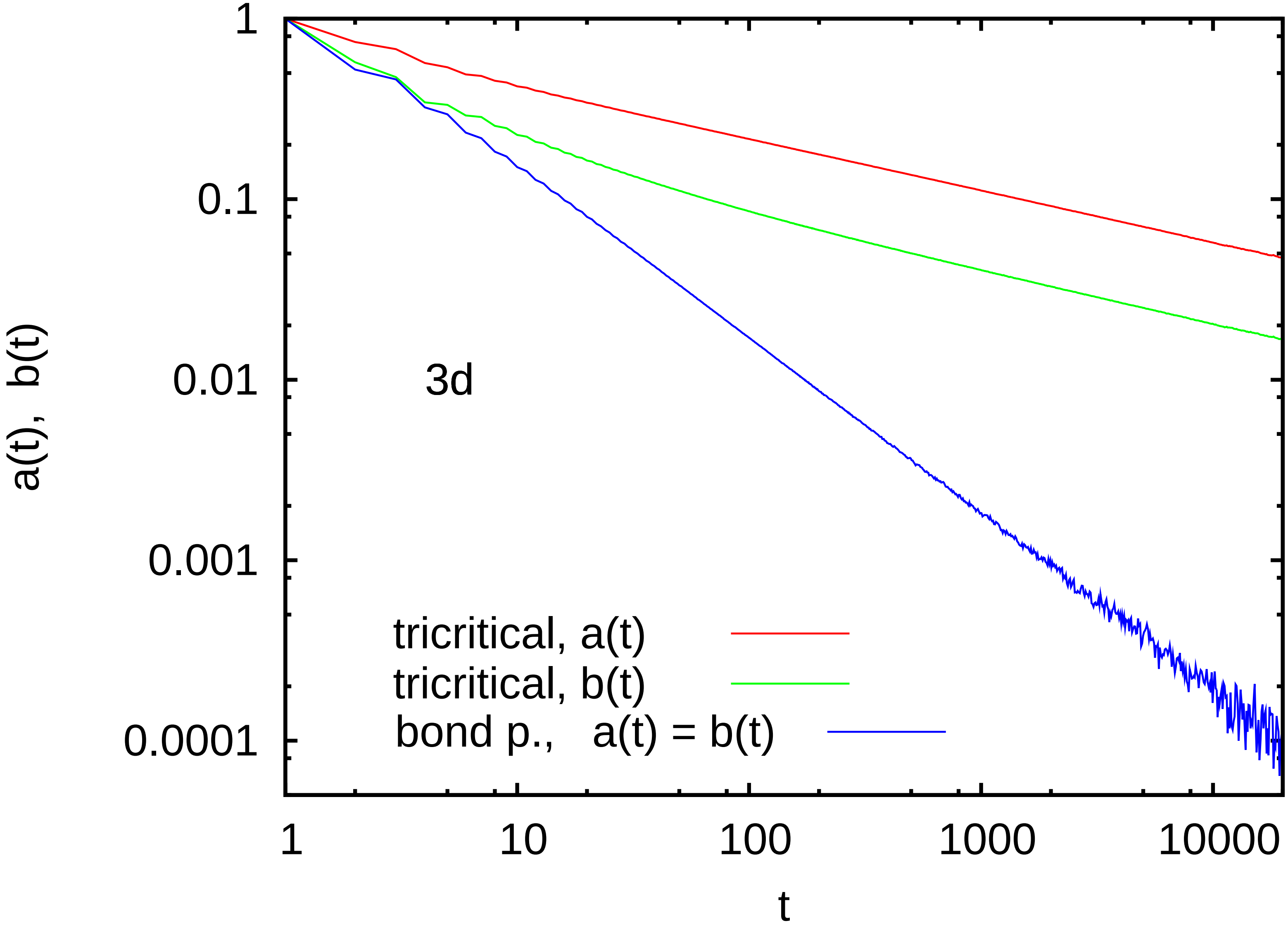
All dimensions:

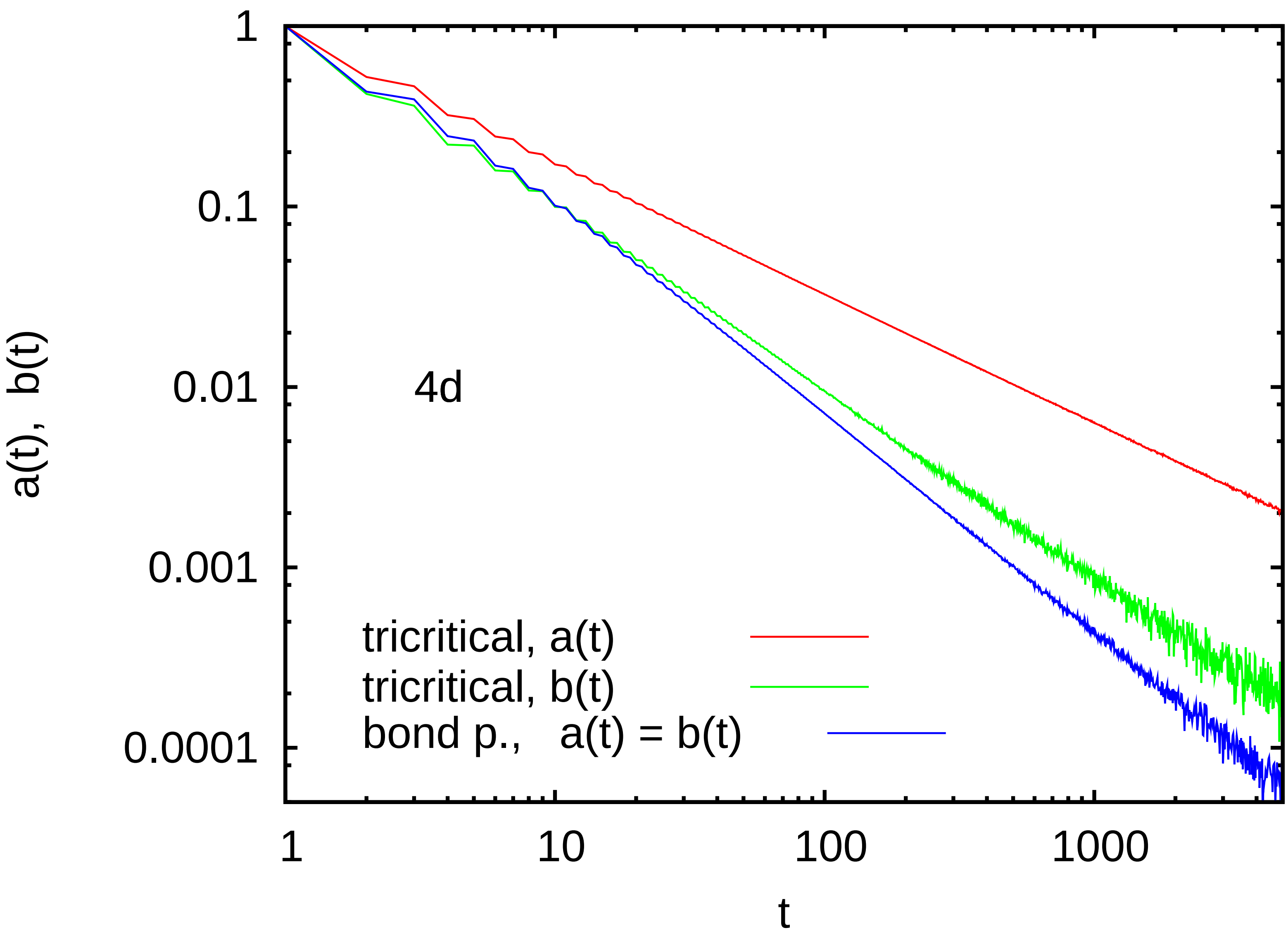
Cluster surfaces become ($t \to \infty$) locally isotropic, when in ordinary percolation universality class,
stay anisotropic when in first-order regime

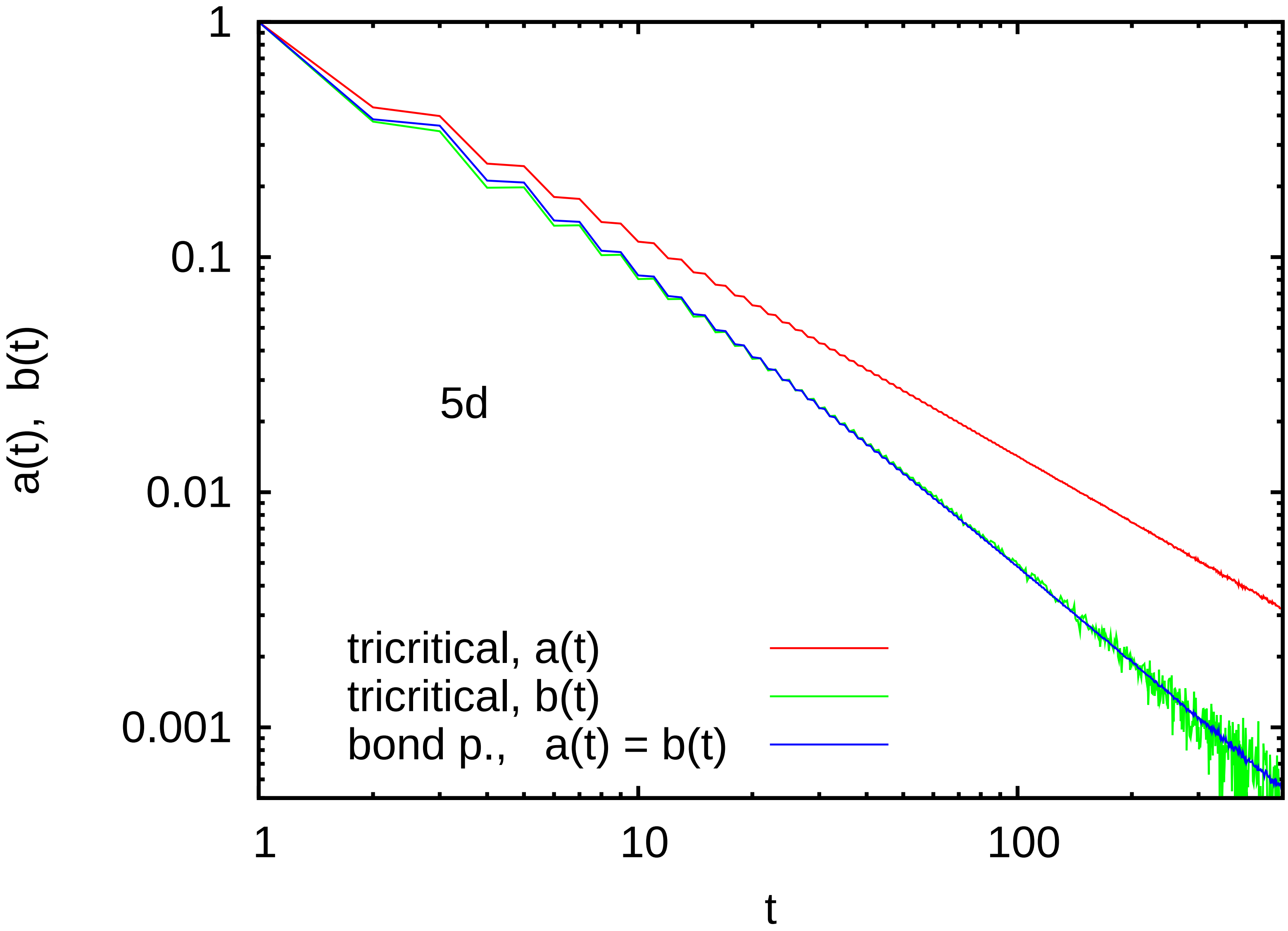
Ordinary & tricritical percolation: decay of local surface anisotropy $\sim t^a$ with
– one new exponent a for $d < 5$; – two new exponents for $d \geq 5$
(one exponent for directionalities of new contacts, other for anisotropy of new infections)

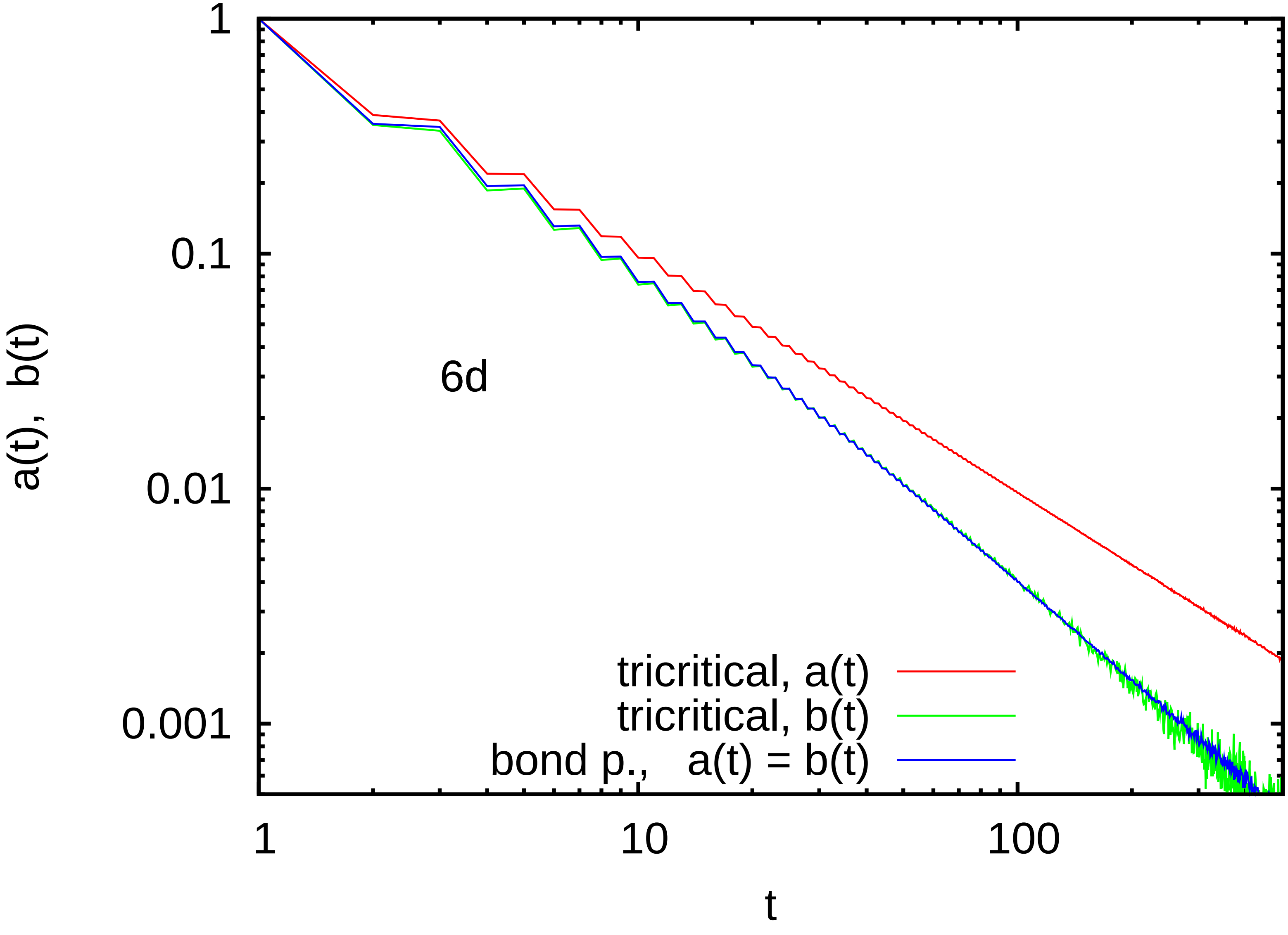


1
0.1
0.01
a(t), b(t)
2d, critical
p0 = 0.04, a(t)
p0 = 0.04, b(t)
p0 = 0.10, a(t)
p0 = 0.10, b(t)
p0 = 0.20, a(t)
p0 = 0.20, b(t)
p0 = 0.50, a(t), b(t)
1
10
100
1000
10000
100000
t

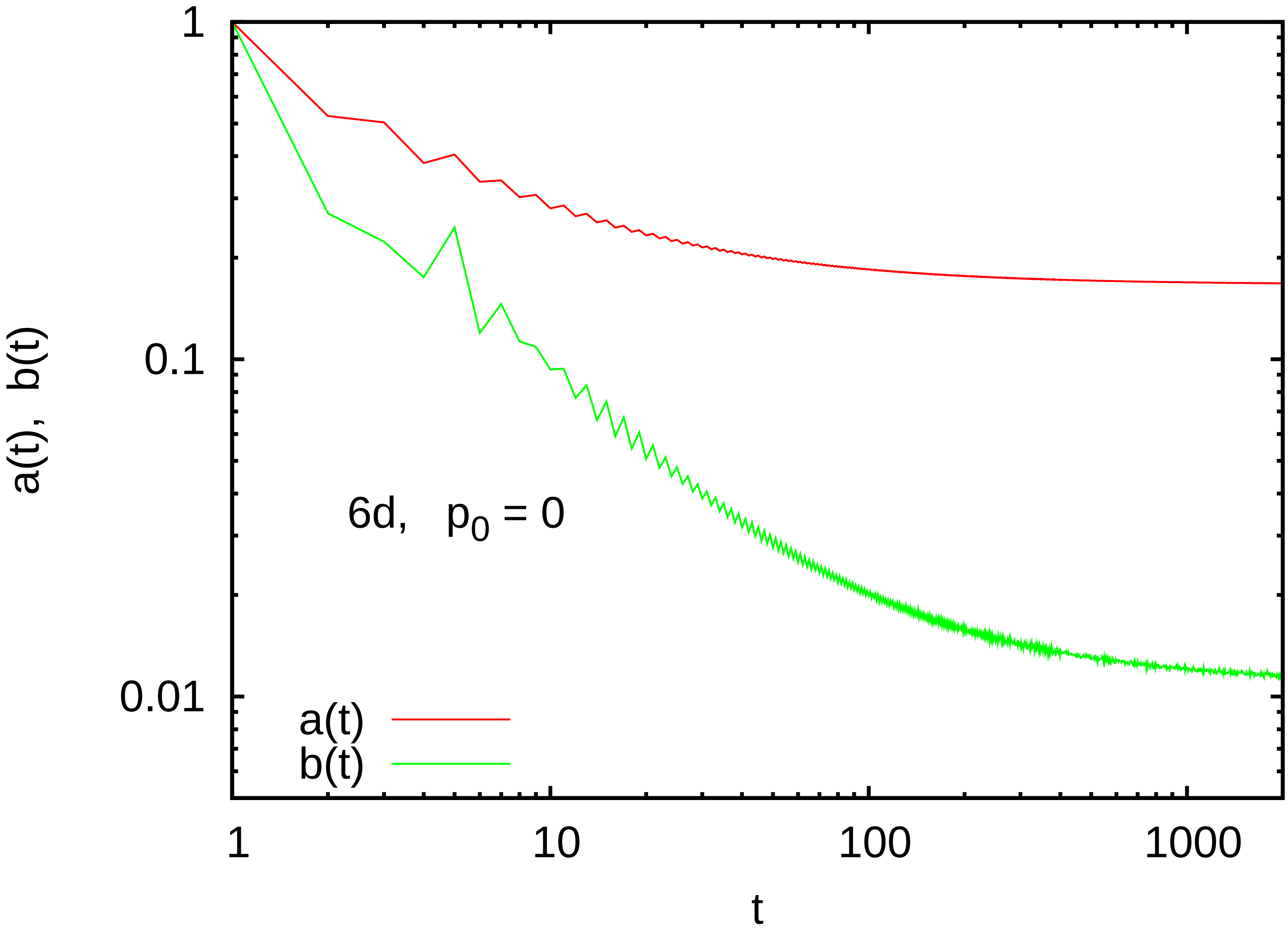

1
0.1
0.01
0.001
0.0001
a(t), b(t)
3d
tricritical, a(t)
tricritical, b(t)
bond p., a(t) = b(t)
1
10
100
1000
10000
t


1
0.1
0.01
0.001
0.0001
a(t), b(t)
4d
tricritical, a(t)
tricritical, b(t)
bond p., a(t) = b(t)
1
10
100
1000
t


1
0.1
0.01
0.001
a(t), b(t)
5d
tricritical, a(t)
tricritical, b(t)
bond p., a(t) = b(t)
1
10
100
t



a(t), b(t)
1
0.1
0.01
0.001
1
10
100
t
6d
tricritical, a(t)
tricritical, b(t)
bond p., a(t) = b(t)



a(t), b(t)
1
0.1
0.01
6d, p_0 = 0
a(t)
b(t)
1
10
100
1000
t

$d \geq 4$:

- Finite size scaling as in second-order regime ok.
- Cleanest power laws for $p_0 = 0$ ("bond bootstrap percolation"), powers agree with Edwards-Wilkinson type models (no overhangs)

Large Other way to implement cooperativity:

Hamiltonian ("exponential") graph models

$P(G) = e^{-H(G|\theta_1,\theta_2,\ldots)}$
(Gibbs-Boltzmann)

- ER-model, fluctuating bond number L:
  $H(G|\theta) = \theta L$
- Strauss model:
  $H(G|\theta) = \theta L + B/N n_\Delta$,
  n_Δ = number of triangles ...
- 2-star model:
  $H(G|\theta) = \theta L + B/N n_v$,
  n_v = number of "2-stars"

- ...

Percolation thresholds: $\theta = O(\ln N)$ $(gives L \sim N)$

For $\theta/B = O(1), \theta \geq 1$:
first order "clustering" transition with huge jump in L. (Park & Newman)

For $\theta/B = O(1), \theta = O(\ln N)$:
second order percolation transition is "overrun" by clustering transition,
becomes also first order,
but with "schizophrenic" hysteresis loop

Agglomerative percolation (AP)

Ordinary bond percolation (OP): in each step, one random pair of nodes is joined.
AP: in each step, one cluster is chosen randomly and joined with ALL of its neighbors

[original motivation: claim by Song, Havlin, & Makse (Nature **433**, 392 (2005), that similar procedure shows finite fractal dimensions for small world graphs]

Random trees
Erdös-Renyi networks
1-d chains
$\rightarrow$ different results from OP.

2-d lattices:

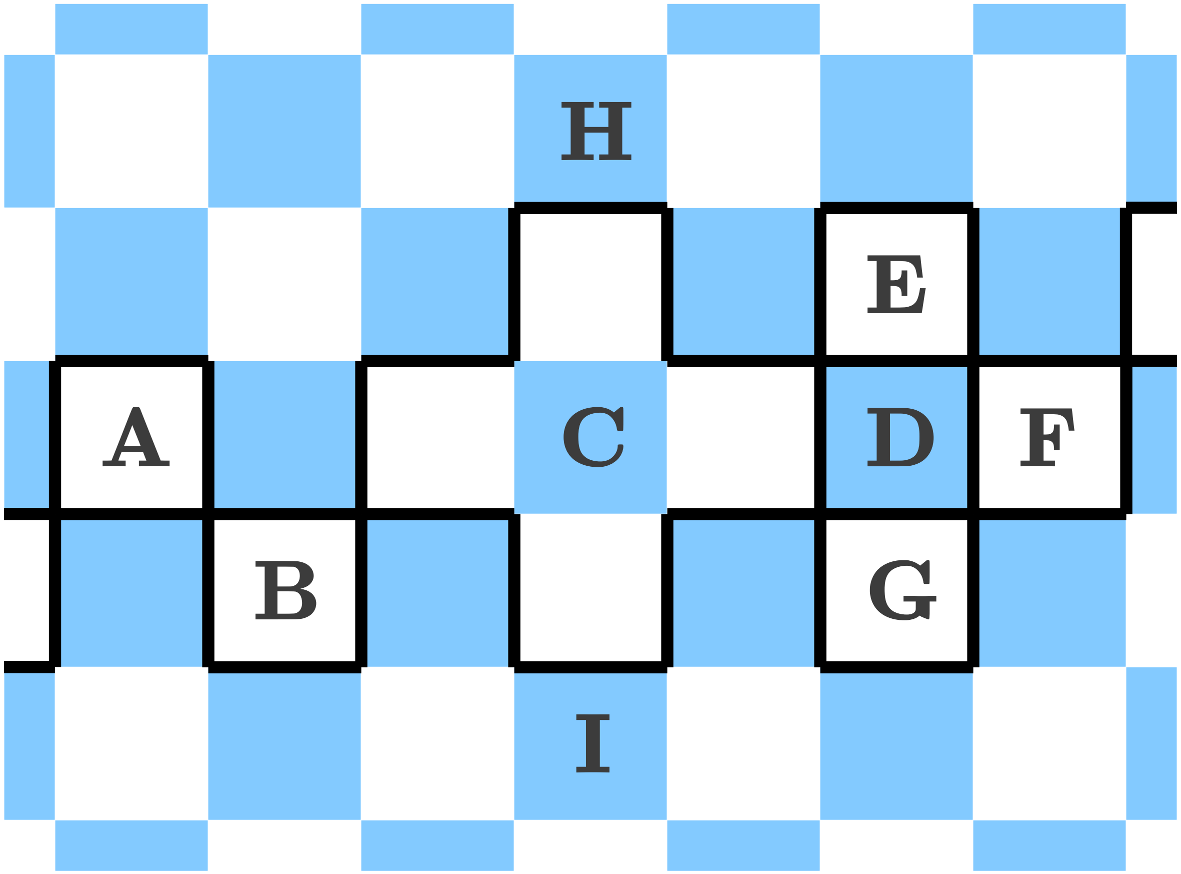
triangular lattice: same universality class as OP ($\tau = 2.055, D = 1.89, \nu = 4/3$) square lattice: completely different: $\tau = D = 2, \nu = \infty$

??

Hon Wai Lau, PG, Maya Paczuski:

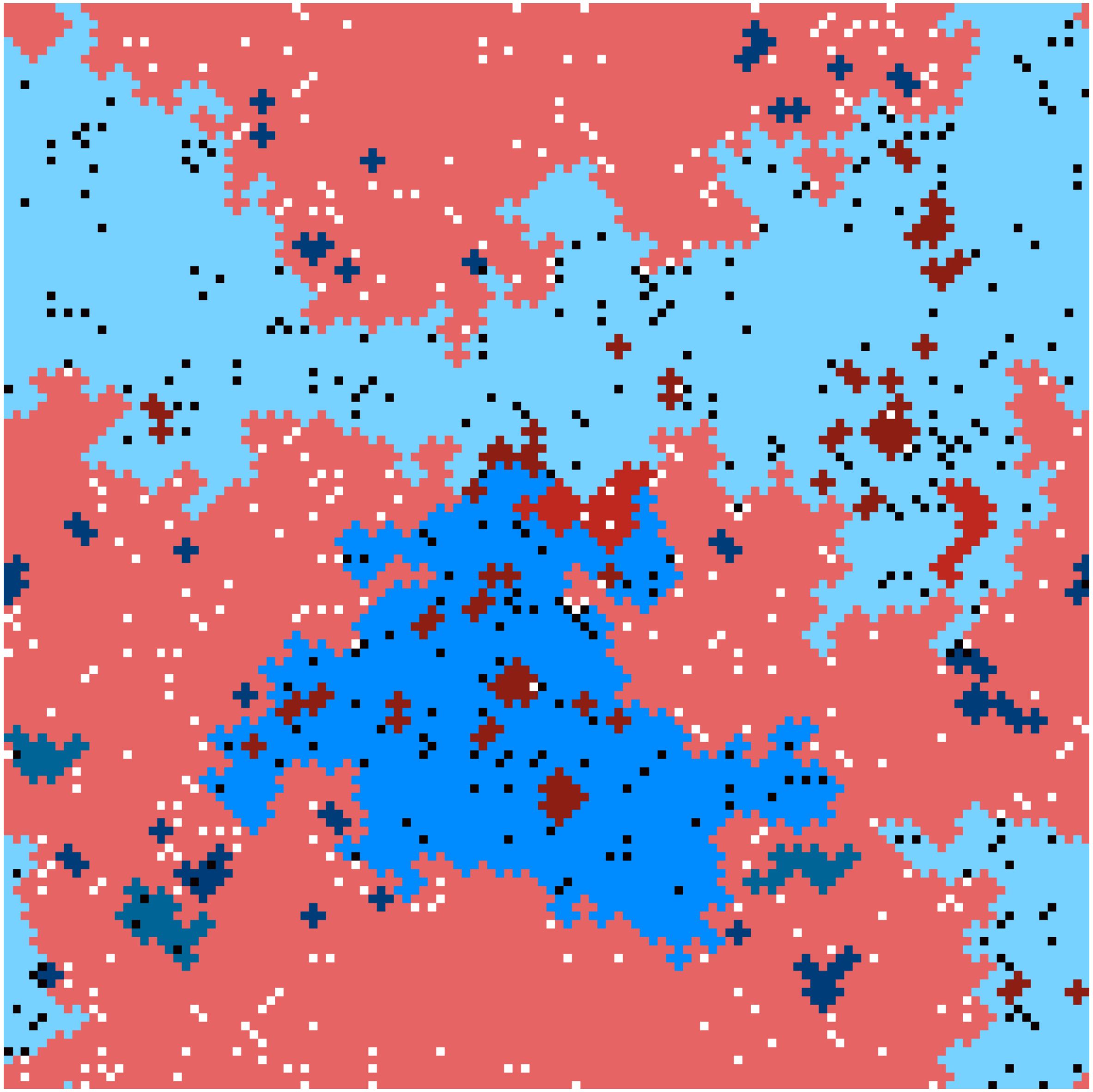
Square lattice is *bipartite*,

"color" of site/node
to unique color of cluster starting at this site/node
AP transition on bipartite graph coincides with spontaneous symmetry breaking



Conclusions:

- Depending on topology, percolation on graphs can have orders ranging from 1 to ∞



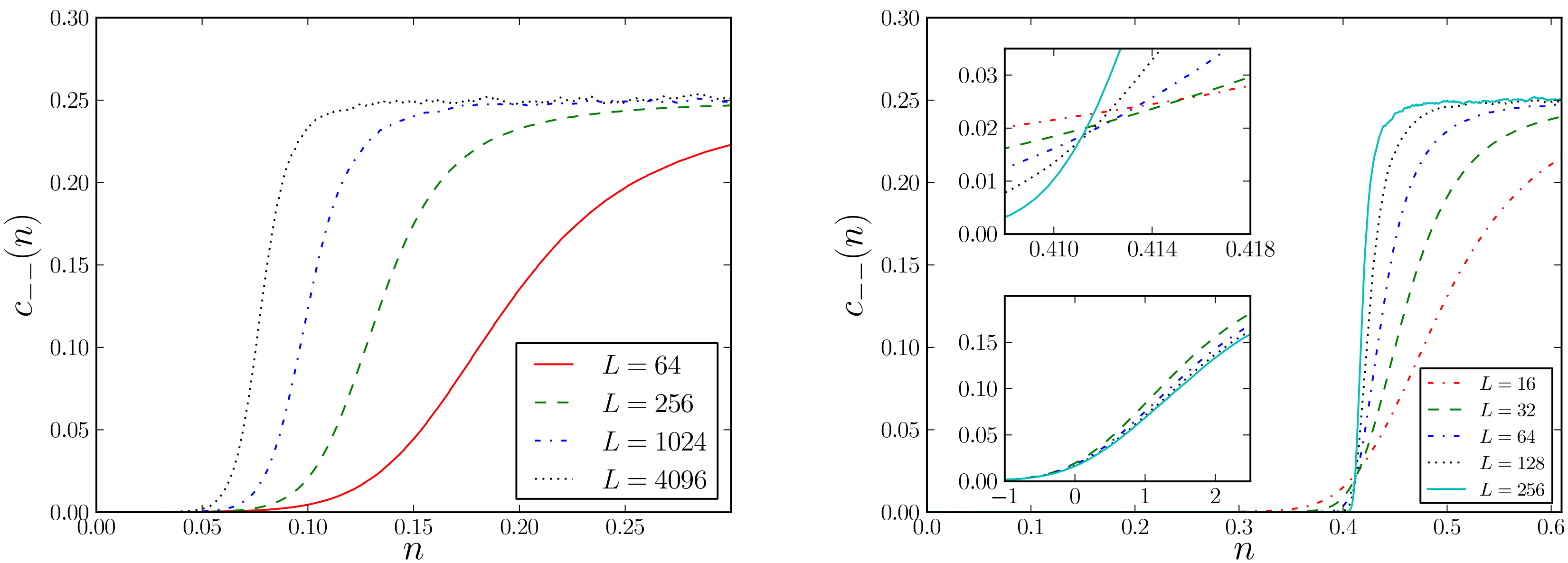


Figure 1: (Color online) Probabilities that the two largest clusters have the same color. These probabilities should vanish in the supercritical phase, if $L \to \infty$. Panel (a) is for the square lattice, panel (b) for the cubic. The upper inset in panel (b) shows the region close to the critical point. The lower inset shows a data collapse plot, $c_{--}(n)$ against $(n - n_c)L^{1/\nu}$ with $n_c = 0.4109$ and $\nu = 1.01$.

- Definition of "order" of percolation transition requires care
- In a well defined way, pinned rough surfaces are just first order percolation transitions.
- Their off-lattice equivalents are models for cooperative infectious spreading studied in the sociological literature
- Although Achlioptas processes are coninuous, they completely different finite size behavior
- Interdependencies in random sparce (loop-less) networks can be treated theoretically much easier, if cascades are not followed explicitly –
- – but this might be not so interesting, because geometric networks show opposite effects
- Agglomerative percolation shows dramatic violations of universality ;
  – it can do so because it is non-local;

– it does so, because the percolation transition on bipartite lattices involves spontaneous symmetry breaking