Geometry of Domain Walls in disordered 2d systems

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Applications of POLYNOMIAL combinatorial optimization methods in Stat-Phys.

(T=0)

- Flux lines with hard core interactions
- Vortex glass with strong screening
- Interfaces, elastic manifolds, periodic media
- Disordered Solid-on-Solid model
- Wetting phenomena in random systems
- Random field Ising systems (any dim.)
- Spin glasses (2d polynomial, d>2 NP complete)
- Random bond Potts model at T_c in the limit $q \to \infty$
- ...

c.f.: A. K. Hartmann, H.R.,
Optimization Algorithms in Physics (Wiley-VCH, 2001);
New optimization algorithms in Physics (Wiley-VCH, 2004)

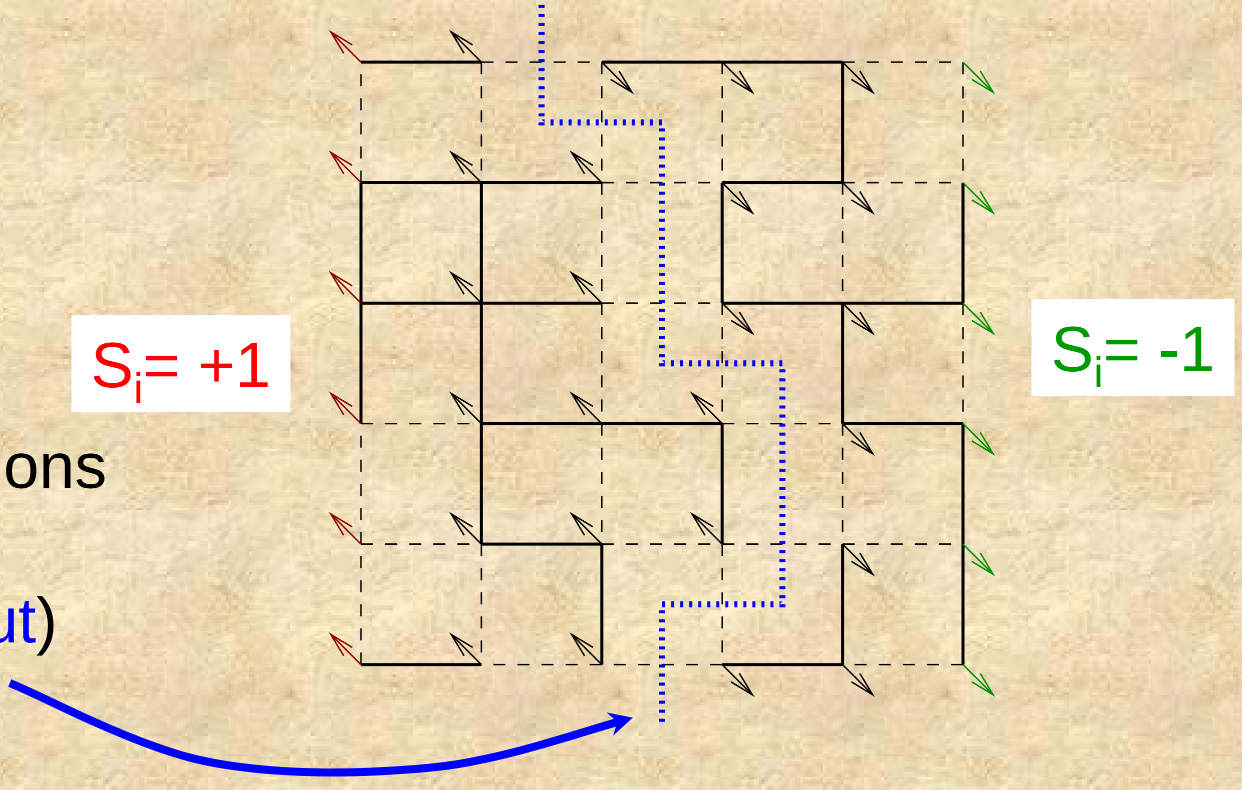
Paradigmatic example of a domain wall: Interfaces in random bond Ising ferromagnets

$$H = -\sum_{i} J_{ij} S_i S_j$$

$$J_{ij} \geq 0, \quad S_i = \pm 1$$

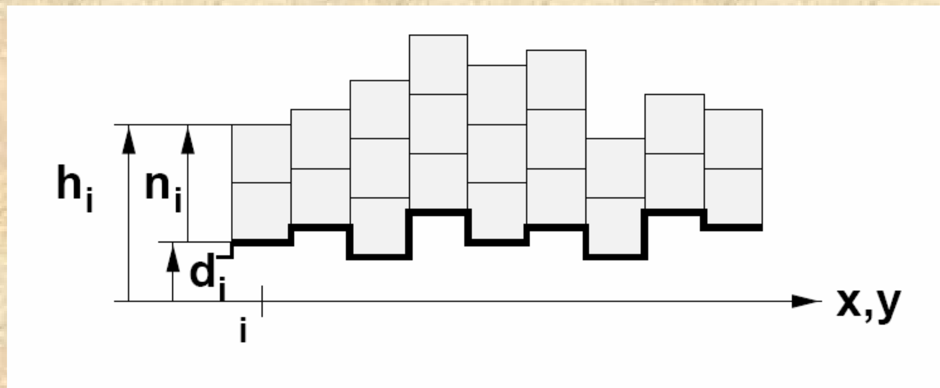
Find for given random bonds J_{ij} the ground state configuration $\{S_i\}$ with fixed +/- boundary conditions

⇔ Find interface (cut) with minimum energy



The SOS model on a random substrate

{n} = height variables (integer)



$$H = \sum_{(ij)} (h_i - h_j)^2 , \quad h_i = n_i + d_i , \quad d_i \in [0,1]$$

Ground state (T=0):

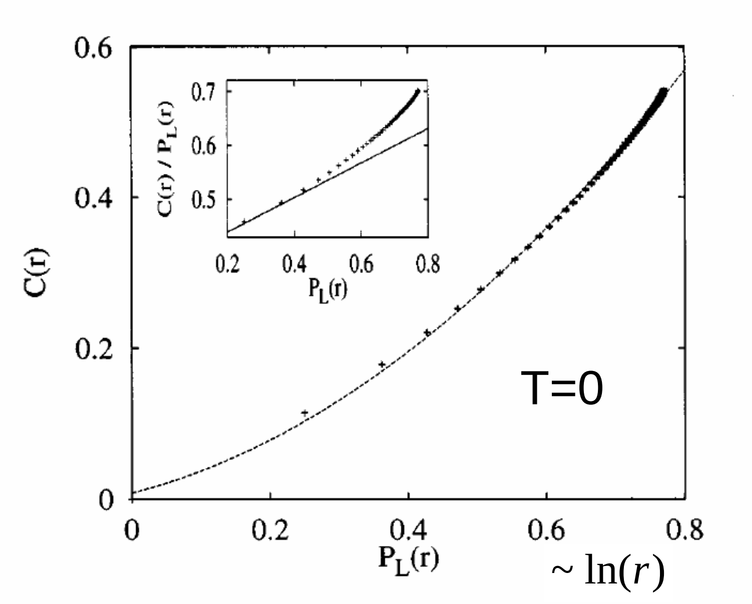
In 1d: $h_i - h_{i+r}$ performs random walk

$C(r) = [(h_i - h_{i+r})^2] \sim r$

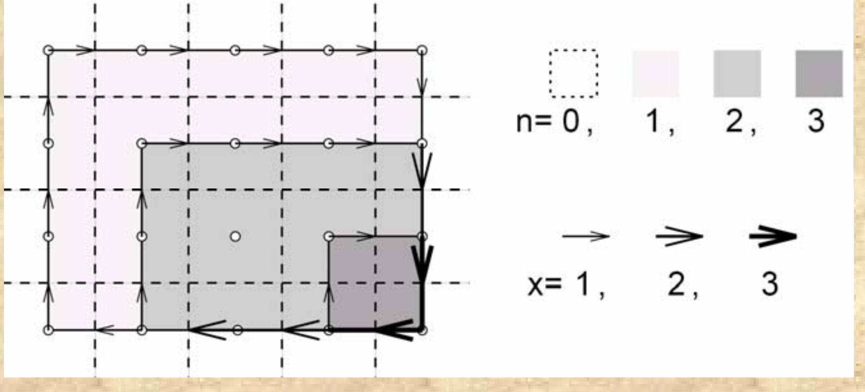
In 2d: Ground state superrough,

$C(r) \sim \log^2(r)$

Stays superrough at temperatures $0<T<T_g$



Mapping on a minimum-cost flow problem



Height profile ↔ Flow configuration

{x}, the height **differences**, is an integer flow in the dual lattice

$$x_{ij} = n_i - n_j$$
$$d_{ij} = d_i - d_j$$

Minimize $$H = \sum_{(ij)} (x_{ij} - d_{ij})^2$$ with the constraint

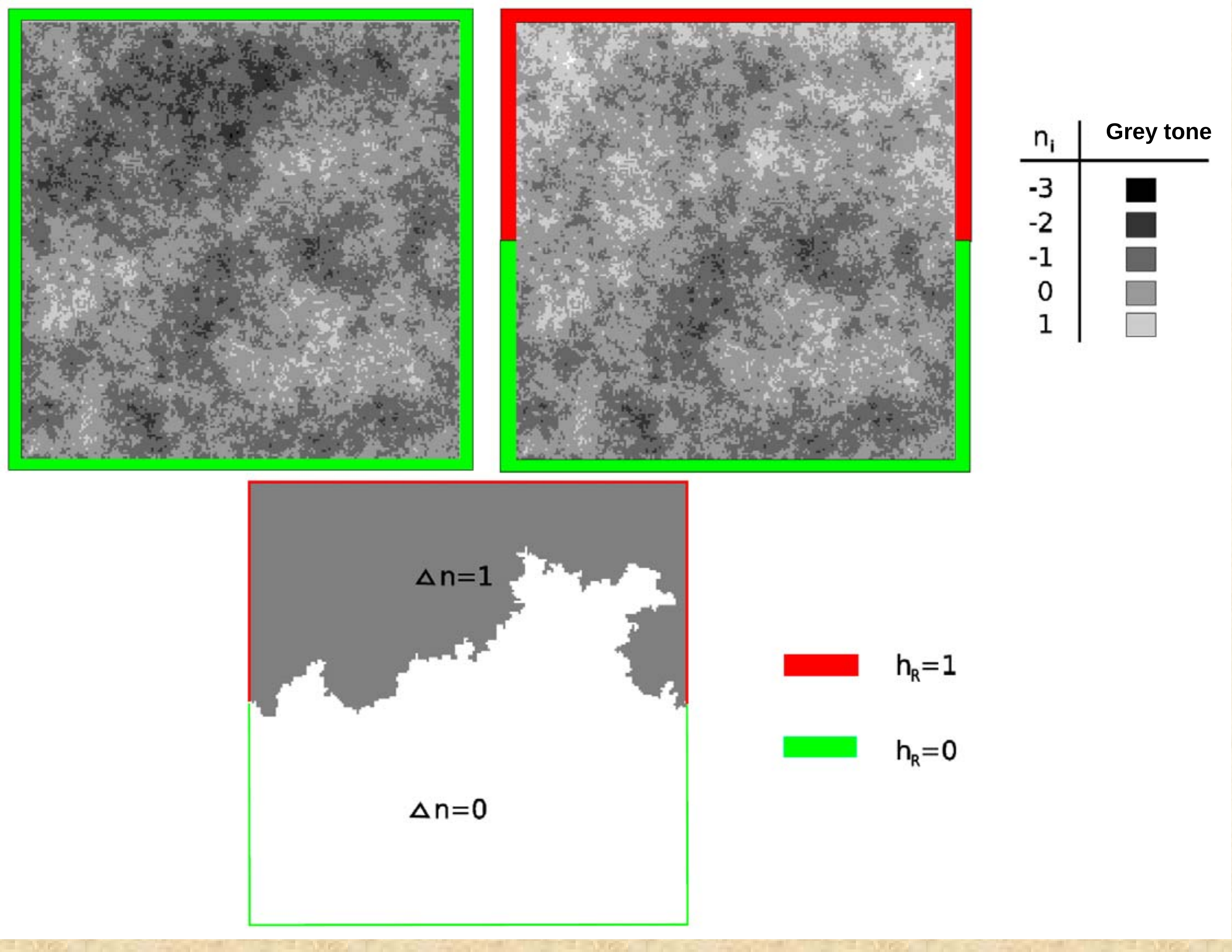
$$(\nabla \cdot \mathbf{n})_i = 0$$

(mass balance on each node of the dual lattice)

→ Minimum cost flow problem

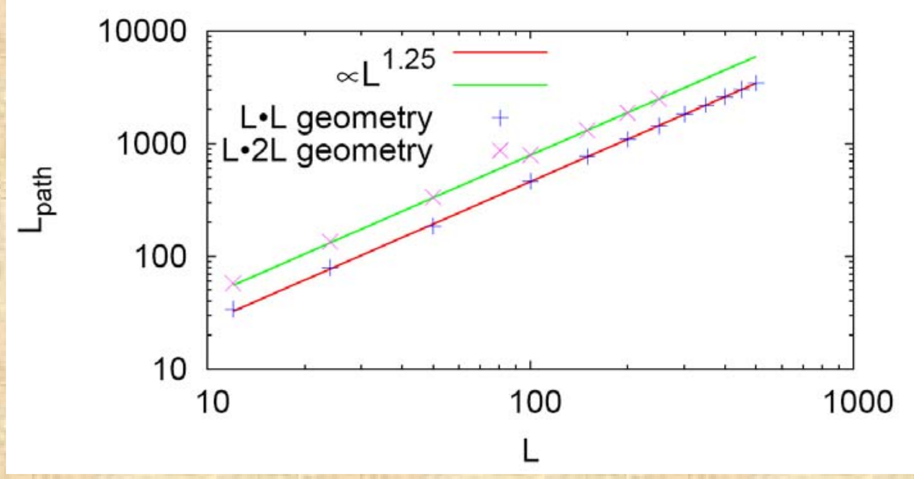
Domain walls in the disordered SOS model

Fixed boundaries



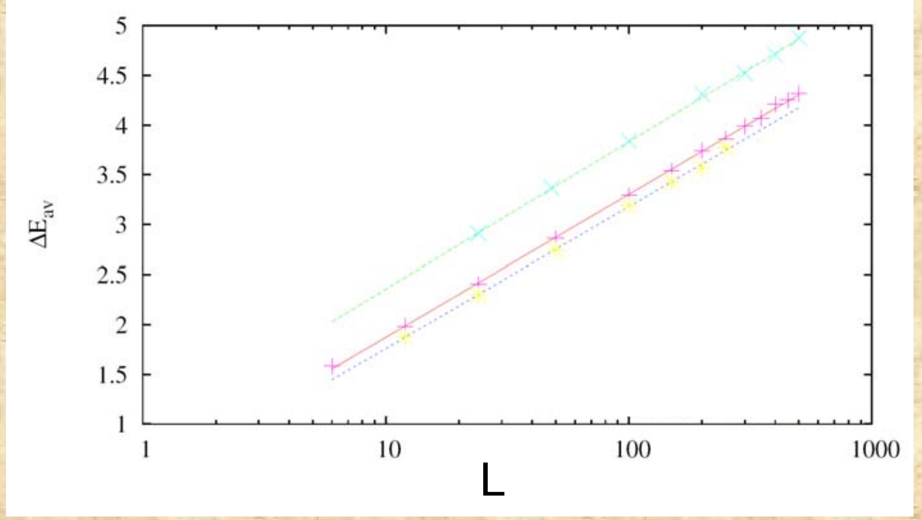
DW scaling in the disordered SOS model

„Length“
→ fractal dimension



⇒ $d_f = 1.25 \pm 0.01$

Energy



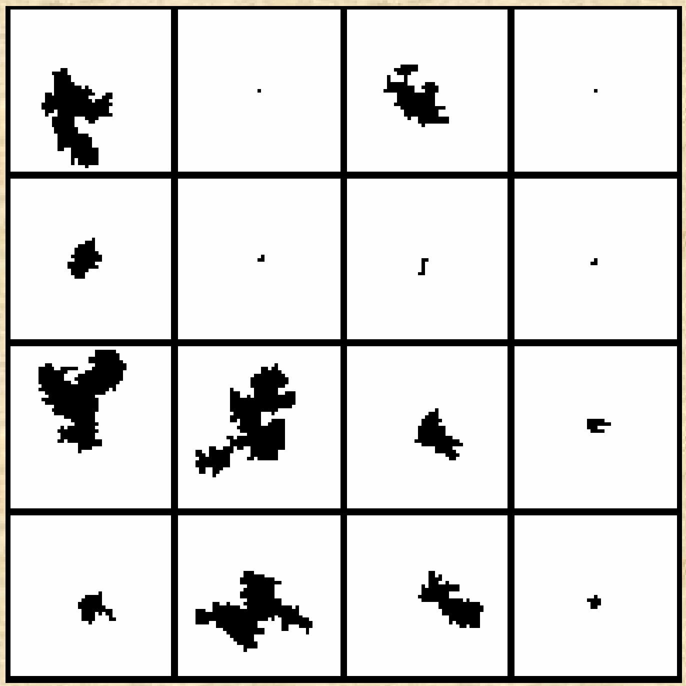
⇒ $\Delta E \sim \log L$

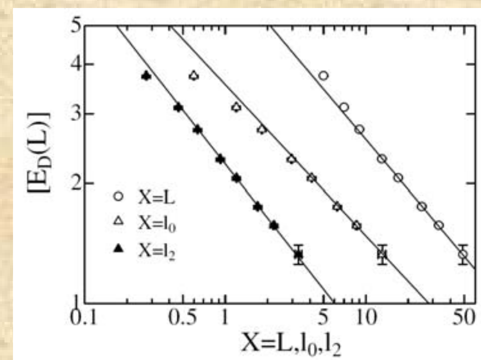
Energy scaling of excitations

Droplets – for instance in spin glasses (ground state $\{S_i^0\}$):

Connected regions $\mathcal{C}$ of lateral size ℓ^d with $S_i = -S_i^0$ for $i \in \mathcal{C}$
with OPTIMAL excess energy over GS energy E^0.

Spin glass:





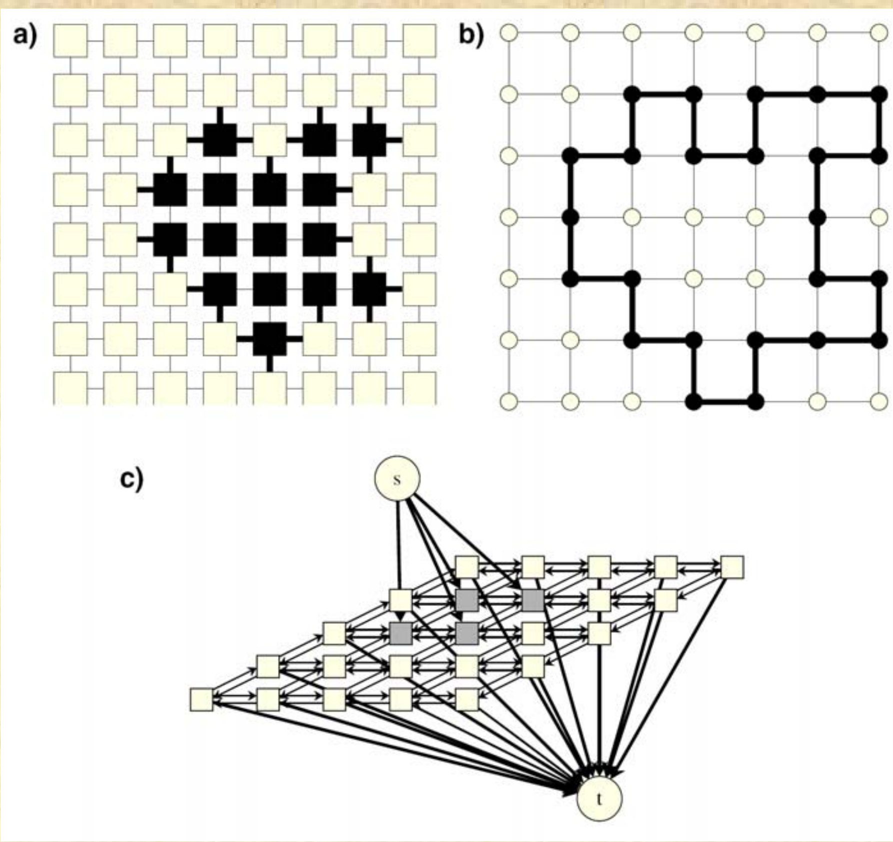
Droplets of ARBITRARY size in 2d spin glasses

[N. Kawashima, 2000]

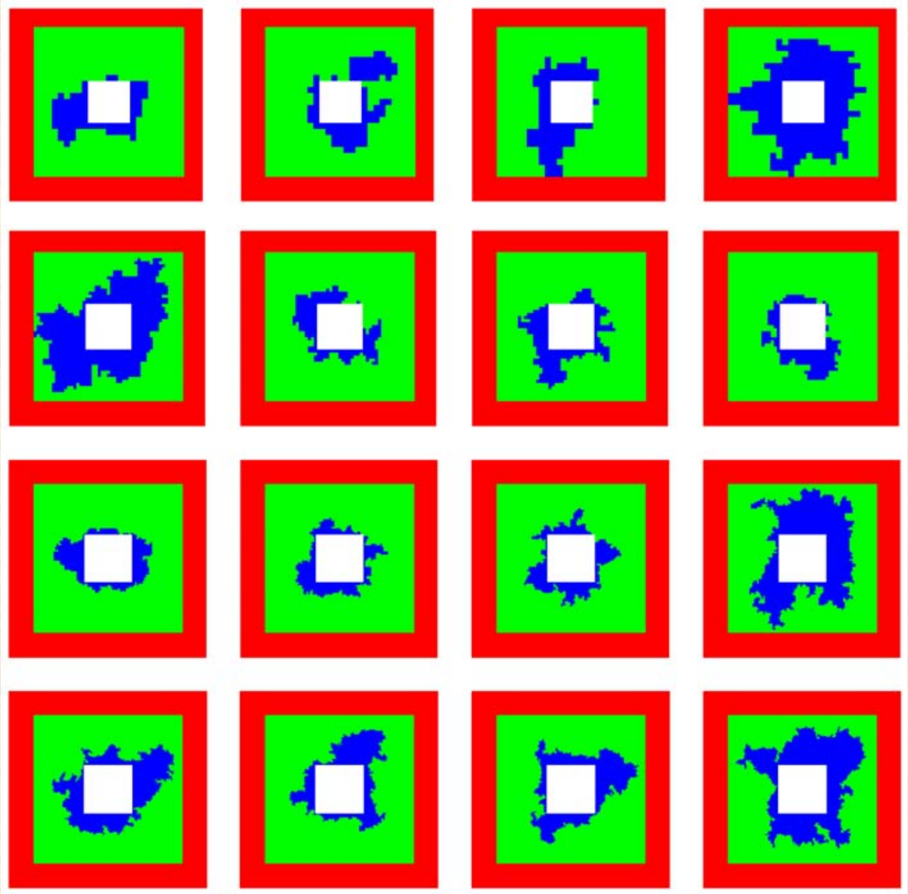
For SOS model c.f. Middleton 2001.

Droplets of FIXED size in the SOS model

Droplets: Connected regions $\mathcal{C}$ of lateral size **L/4 < ℓ < 3L/4** with $h_i = h_i^0 + 1$ for $i \in \mathcal{C}$ with OPTIMAL energy (= excess energy over E^0).

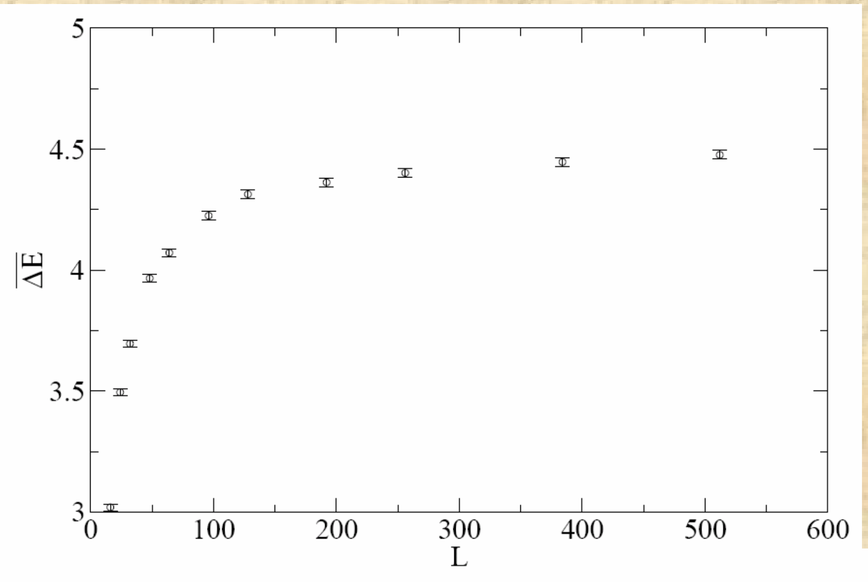


Efficient computation:
Mapping on a minimum s-t-cut.

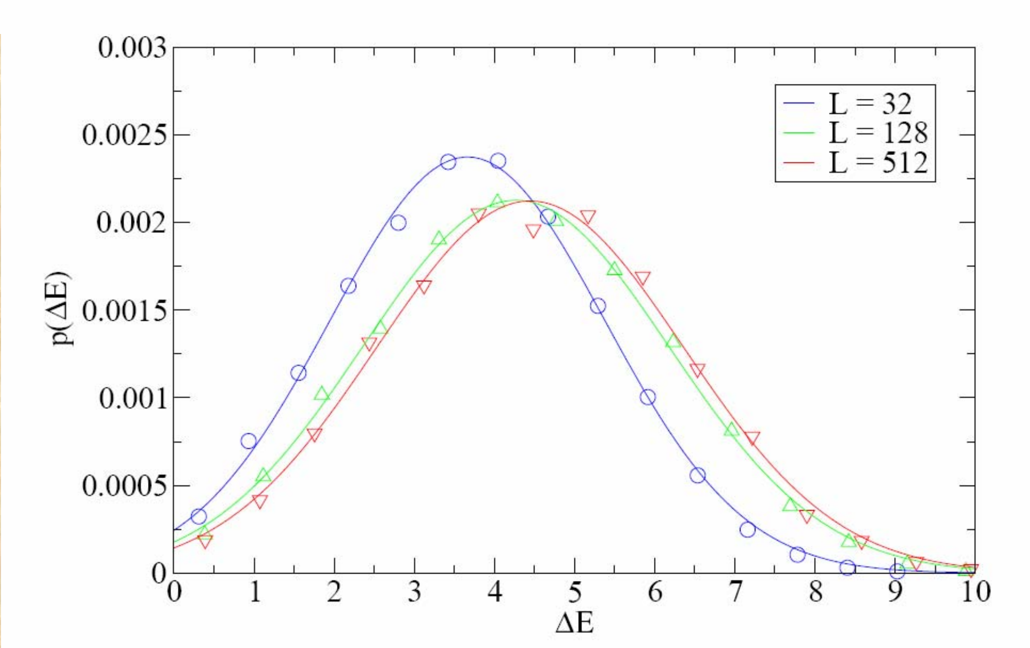


Example configurations
(excluded white square enforces size)

Results: Scaling of droplet energy



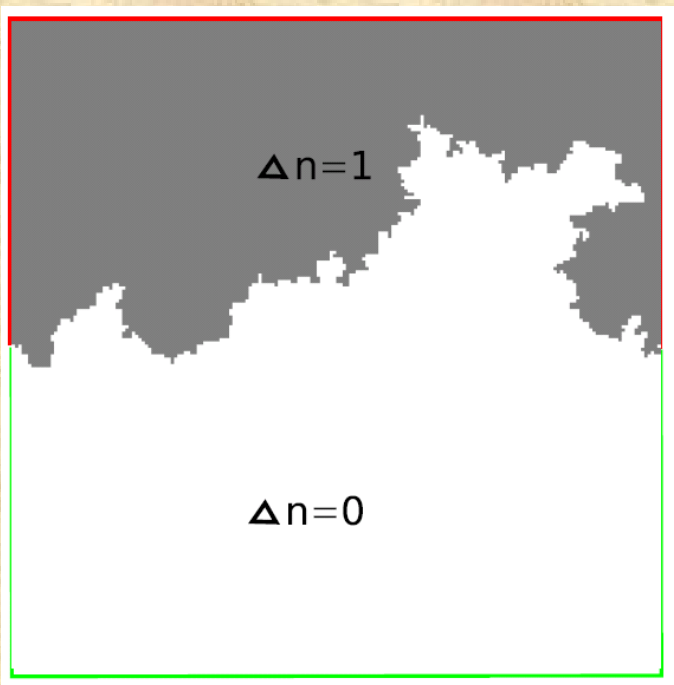
Average energy of droplets of lateral size ~L/2 saturates at **FINITE** value for L→∞



Probability distribution of excitations energies: **L-independent** for L→∞.

n.b.: Droplet boundaries have fractal dimension **d_f=1.25**, too!

Geometry of DWs in disordered 2d models

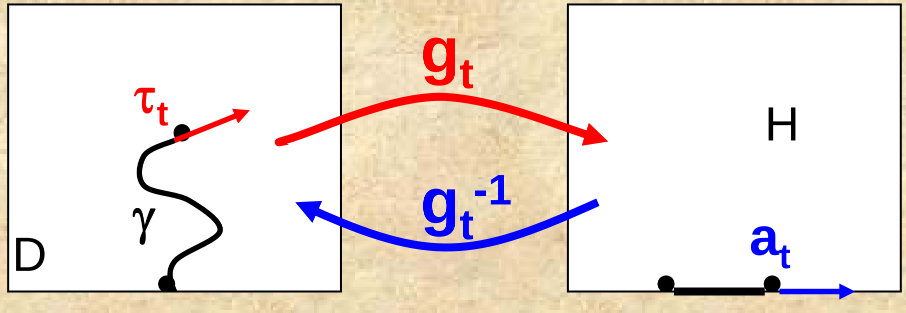


DWs are **fractal curves** in the plane
for spin glasses, disordered SOS model, etc
(not for random ferromagnets)

Do they follow **Schramm-Loewner-Evolution (SLE)?**
Yes for spin glasses (Amoruso, Hartmann, Hastings, Moore,
Middleton, Bernard, LeDoussal)

Schramm-Loewner Evolution (1)

The random curve γ can be grown through a continuous exploration process
Paramterize this growth process by “time” t:



When the tip τ_t moves, a_t moves on the real axis

At any t the domain D/ γ can be mapped onto the standard domain H, such that the image of γ_t lies entirely on the real axis

Loewners equation:
$$\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - a_t}$$

Schramm-Loewner evolution:

If Proposition 1 and 2 hold (see next slide) than a_t is a Brownian motion:

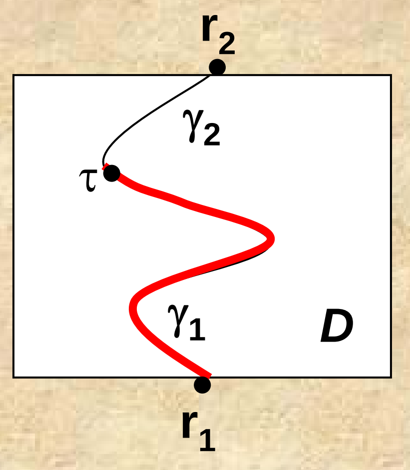
$a_t = \sqrt{\kappa} B_t$ κ determines different universality classes!

Schramm-Loewner Evolution (2)

Define measure μ on random curves γ in domain D from point r_1 to r_2

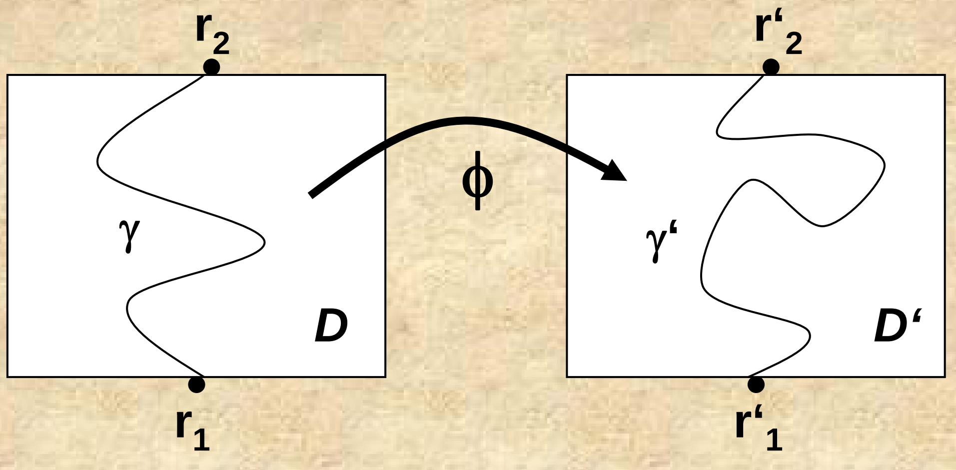
Property 1: Markovian

$$\mu(\gamma_2|\gamma_1; D, r_1, r_2) = \mu(\gamma_2; D/\gamma_1, \tau, r_2)$$

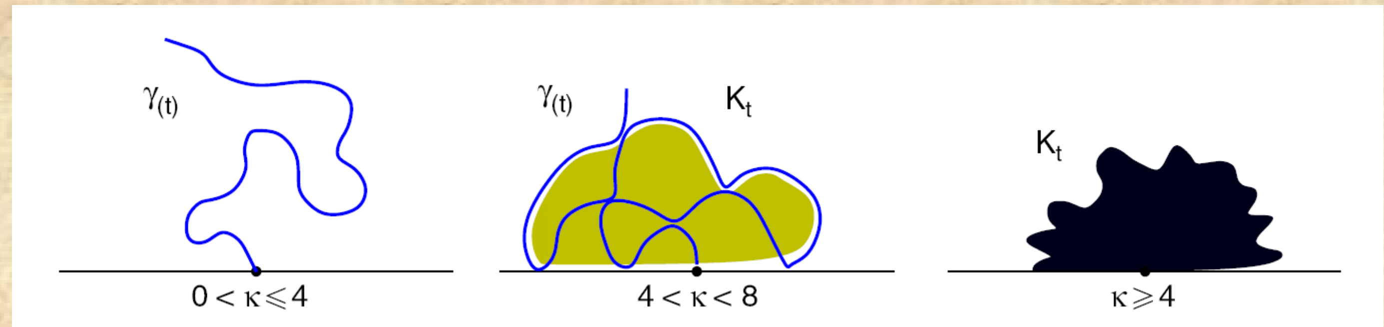


Property 2: Conformal invariance

$$\phi \star \mu(\gamma; D, r_1, r_2) = \mu(\phi(\gamma); D', r_1', r_2')$$



Examples for SLE$_\kappa$

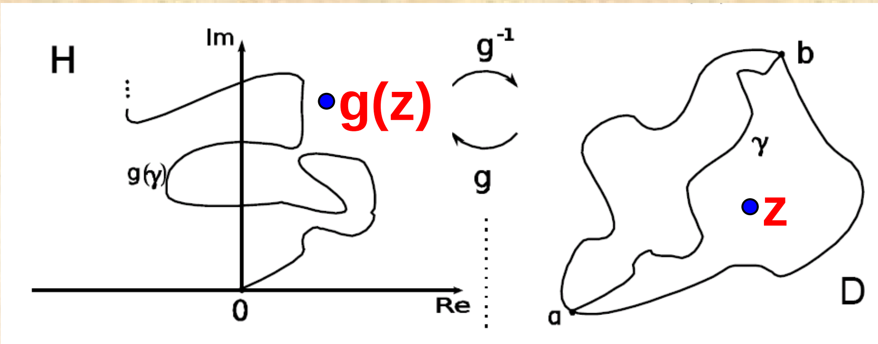


- κ = 2: Loop erased random walks
- κ = 8/3: Self-avoiding walks
- κ = 3: cluster boundaries in the Ising model
- κ = 4: BCSOS model of roughening transition, 4-state Potts model, double dimer models, level lines in gaussian random field, etc.
- κ = 6: cluster boundaries in percolation
- κ = 8: boundaries of uniform spanning trees

Properties of SLE_κ

1) **Fractal dimension** of γ: $d_f = 1+\kappa/8$ for $\kappa \le 8$, $d_f = 2$ for $\kappa > 8$

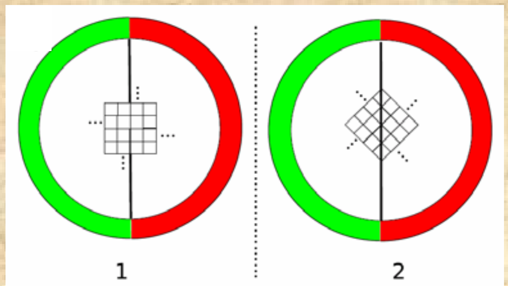
2) **Left passage probability**: (prob. that z in D is to the left of γ)



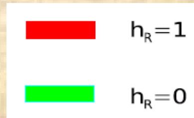
Schramm's formula:

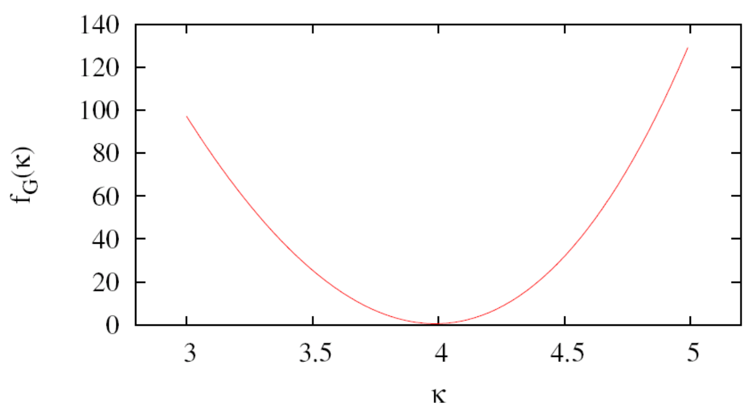
$$P_{\kappa,D,a,b}(z) = P(g(z)) = \frac{1}{2} + \frac{\Gamma\left(\frac{4}{\kappa}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{8-\kappa}{2\kappa}\right)} \cdot {}_2F_1\left(\frac{1}{2},\frac{4}{\kappa};\frac{3}{2};-\left(\frac{\mathrm{Re}(g(z))}{\mathrm{Im}(g(z))}\right)^2\right)\frac{\mathrm{Re}(g(z))}{\mathrm{Im}(g(z))}$$

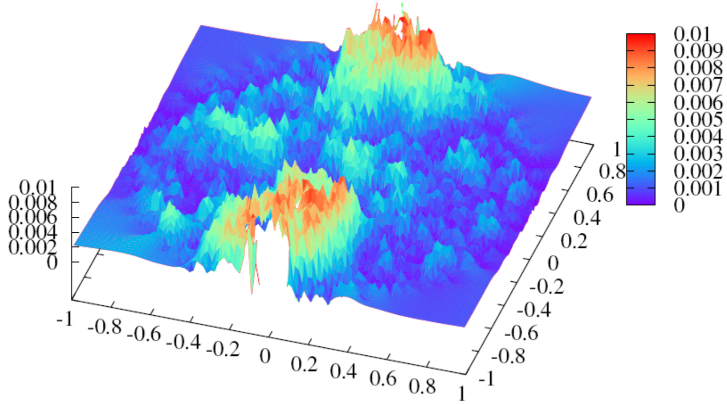
DW in the disordered SOS model: SLEκ?



Let D be a **circle**, a=(0,0), b=(0,L)
Fix boundaries as shown





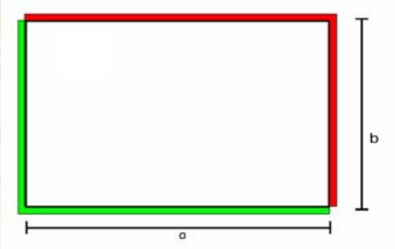


Cumulative deviation of
left passage probability
from Schramm's formula f. κ

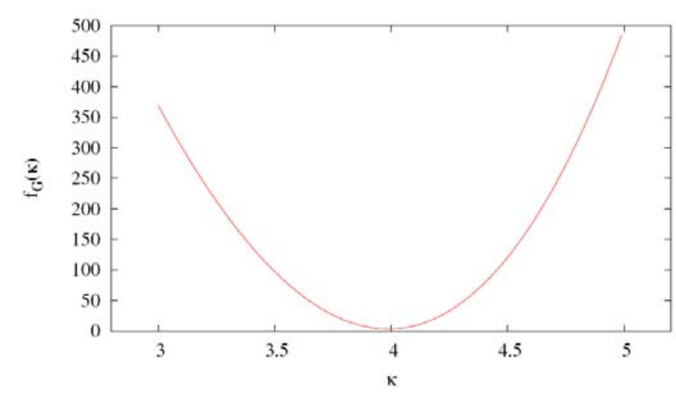
Minimum at **κ=4**!

Local deviation of
left passage probability
from $P_{\kappa=4}$

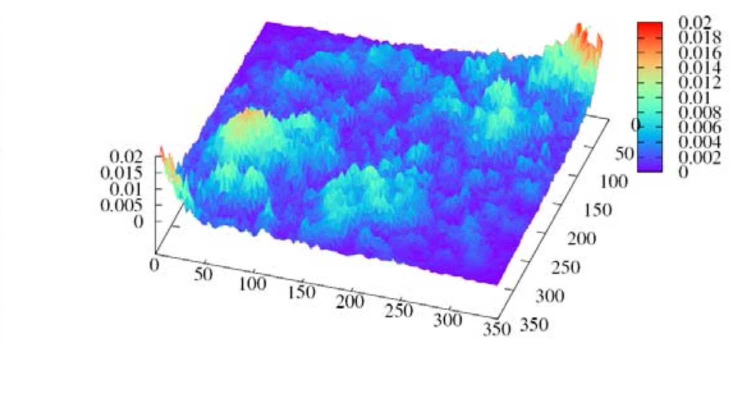
Other domains (→conformal inv.):



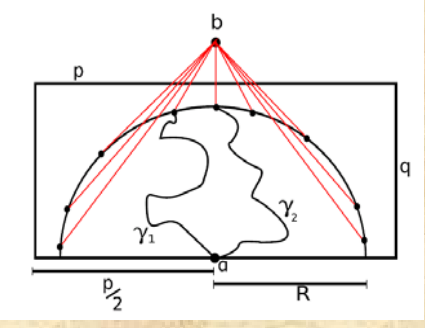
D = square



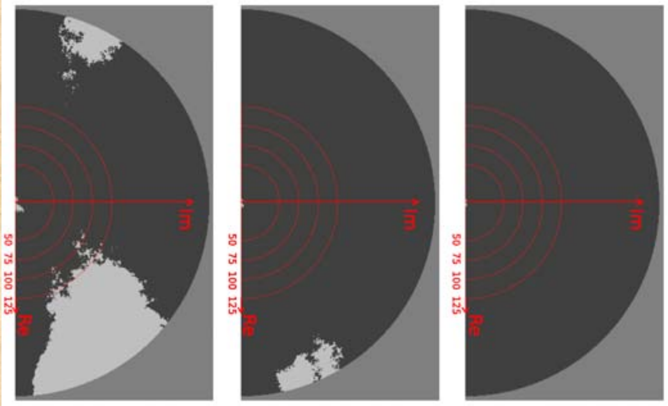
Cum. Deviation: Minimum at κ=4!



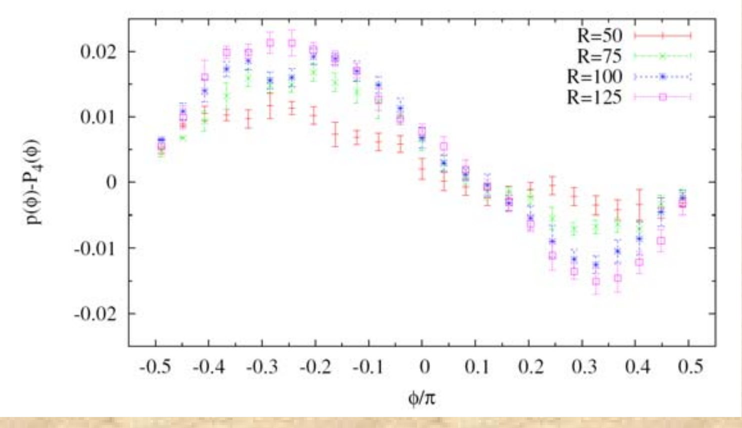
Local dev.



D = half circle



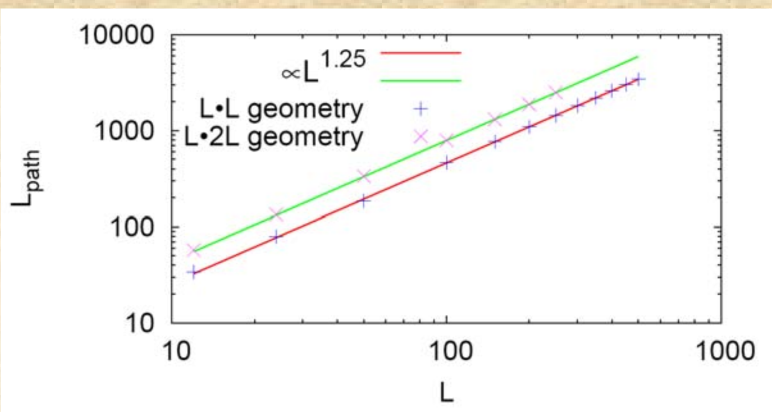
Dev. From $P_{\kappa=4}$ larger than
0.02, 0.03, 0.035



Deviation from $P_{\kappa=4}(\phi)$

DWs in the disordered SOS model are **not** described by chordal SLE

Remember: **$d_f = 1.25 \pm 0.01$**



Schramm's formula with **κ=4** fits well left passage prob.

IF the DWs are described by $SLE_{\kappa=4}$: $d_f = 1+\kappa/8 \Rightarrow$ **$d_f = 1.5$**

But: Indication for conformal invariance!

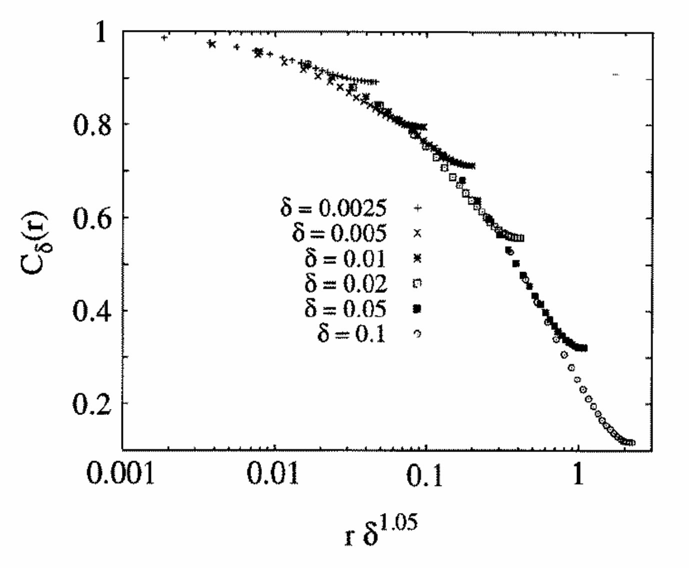
Conclusions / Open Problems

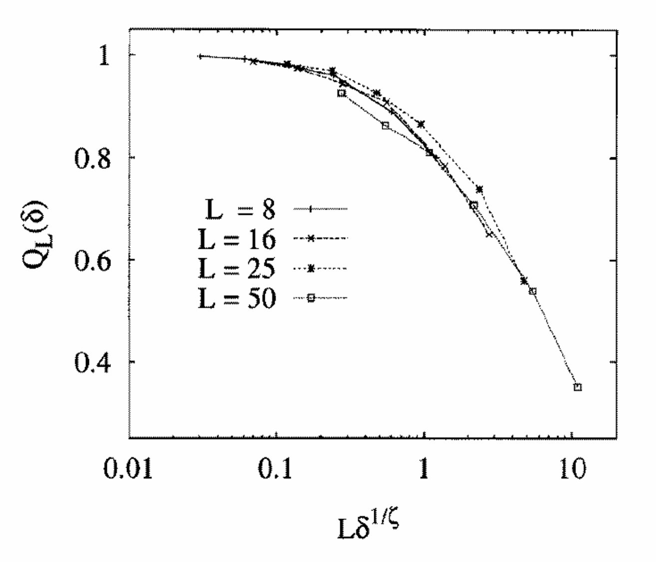
- Droplets for $\ell \to \infty$ have finite average energy,
  and ℓ-independent energy distribution
- Domain walls have fractal dimension d_f=1.25
- Left passage probability obeys Schramm‘s formula with κ=4 [$\neq 8(d_f-1)$]
- … in different geometries → conformal invariance?
- DWs not described by (chordal) SLE – why (not Markovian?)
- Contour lines have d_f=1.5 Middleton et al.): Do they obey $SLE_{\kappa=4}$?
- What about SLE and other disordered 2d systems?

Disorder chaos (T=0) in the 2d Ising spin glass

$$C_\delta(r) = \left[\frac{1}{N}\sum_{i=1}^{N} S_i S_{i+r} S_i'(\delta) S_{i+r}'(\delta)\right]_{\text{av}}$$

$$Q_L(\delta) = N^{-1} \left| \sum_{i=1}^{N} S_i S_i'(\delta) \right|$$





$$C_\delta(r) \sim \tilde{c}(r\delta^{1/\zeta})$$

HR et al, JPA 29, 3939 (1996)

Disorder chaos in the SOS model – 2d

Scaling of $C_{ab}(r) = [(h_i^a - h_{i+r}^a)(h_i^b - h_{i+r}^b)]$:

$C_{ab}(r) = \log^2(r)\, f(r/L_\delta)$ with $L_\delta \sim \delta^{-1/\zeta}$ „Overlap Length“

Analytical predictions for asymptotics $r \to \infty$:

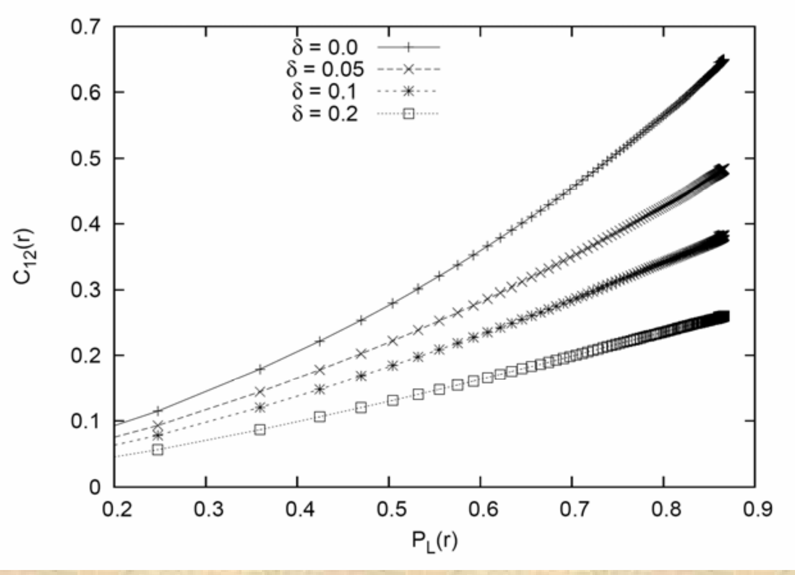
Hwa & Fisher [PRL 72, 2466 (1994)]: $C_{ab}(r) \sim \log(r)$ (RG)

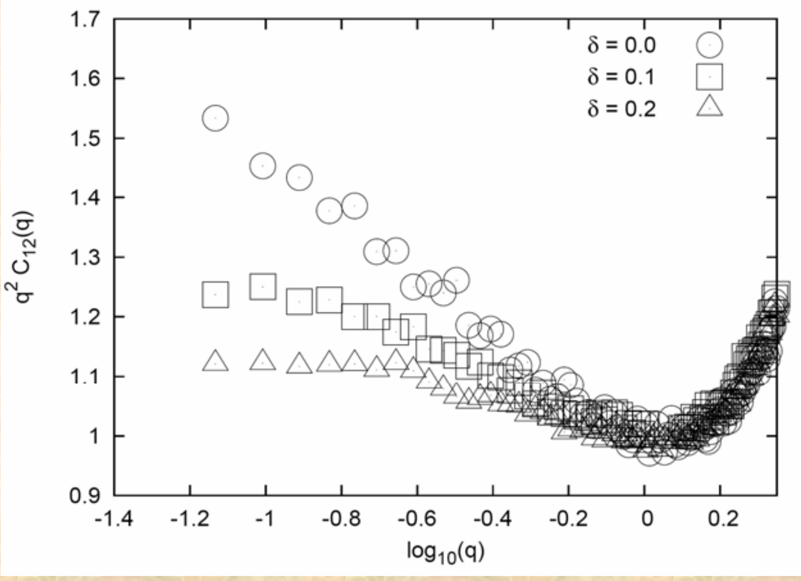
Le Doussal [cond-mat/0505679]: $C_{ab}(r) \sim \log^2(r) / r^\mu$ with $\mu = 0.19$ in 2d (FRG)

Exact GS calculations, Schehr & HR `05:

$q^2 \cdot C_{12}(q) \sim \log(1/q)$ $\Rightarrow$ $C_{12}(r) \sim \log^2(r)$

$q^2 \cdot C_{12}(q) \sim$ const. f. $q \to 0$ $\Rightarrow$ $C_{12}(r) \sim \log(r)$





$\Rightarrow$ Numerical results support RG picture of Hwa & Fisher.