“Glassy” properties of the Many-Body Localization transition

Marco Tarzia
LPTMC, Sorbonne Université, Paris

In collaboration with
Giulio Biroli (@ENS Paris) and *Alexander Hartmann*

Oldenburg, July 14th, 2022

Many-Body localization

Isolated interacting quantum disordered system fail to reach thermal equilibrium if the disorder is strong enough even at finite energy density [Basko, Aleiner & Altshuler ’06; Anderson ’58]

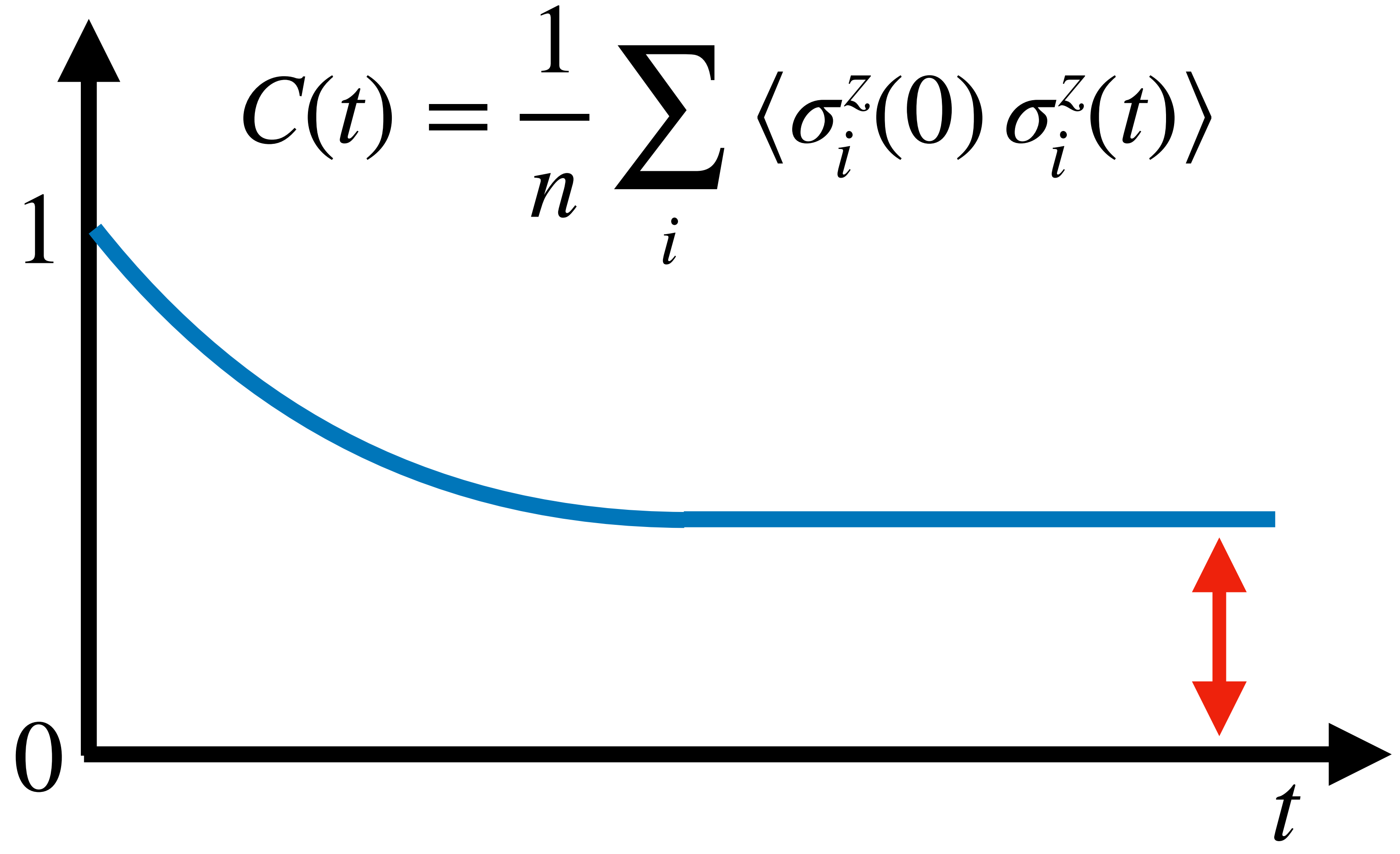
n interacting spins, particles, cold atoms, qbits, …

$$|\psi(t)\rangle = e^{-iHt/\hbar}|\psi_0\rangle$$

Random initial state ($T = \infty$)

$$|\psi_0\rangle = |\uparrow\downarrow\uparrow\uparrow\cdots\rangle$$

$$C(t) = \frac{1}{n}\sum_i \langle \sigma_i^z(0)\,\sigma_i^z(t)\rangle$$



The system keeps memory of the initial condition after infinite time!

Realized in experiments (cold atoms, superconducting qbits, …)
[Schreiber & al ’15; Bordia & al ’16; Choi & al ’16; Smith & al ’16; Xu & al ’18;…]

Novel quantum out-of-equilibrium dynamical phase transition due to the interplay of disorder, quantum fluctuations, and interactions

Emergent integrability in the MBL phase

$$H = \underbrace{\sum_{i=1}^{n} \left(J_i \sigma_i^z \sigma_{i+1}^z + h_i \sigma_i^z \right)}_{H_0} + \underbrace{\Gamma \sum_{i=1}^{n} \sigma_i^x}_{H_{\text{int}}} \qquad h_i \in [-W, W] \qquad J_i \in [0.8, 1.2]$$

Trivial MBL limit $\Gamma = 0 \rightarrow$ The $\{\sigma_i^z\}$ constitute a complete set of conserved quantities

$$[H, \sigma_i^z] = 0 \qquad [\sigma_i^z, \sigma_j^z] = 0$$

This structure is preserved when interactions are turned on [Imbrie '14]

$\Gamma > 0 \rightarrow$ "Dress" the $\{\sigma_i^z\}$ $\qquad \tau_i = \sigma_i^z + \{\text{multispin terms}\}$ $\qquad$ only involve linear combinations of local operators within a finite distance

$$[H, \tau_i] = 0 \qquad [\tau_i, \tau_j] = 0$$

$$H = \sum_{i=1}^{n} \epsilon_i \tau_i + \sum_{i,j} J_{ij} \tau_i \tau_j + \sum_{i,j,k} J_{ijk} \tau_i \tau_j \tau_k + \ldots$$

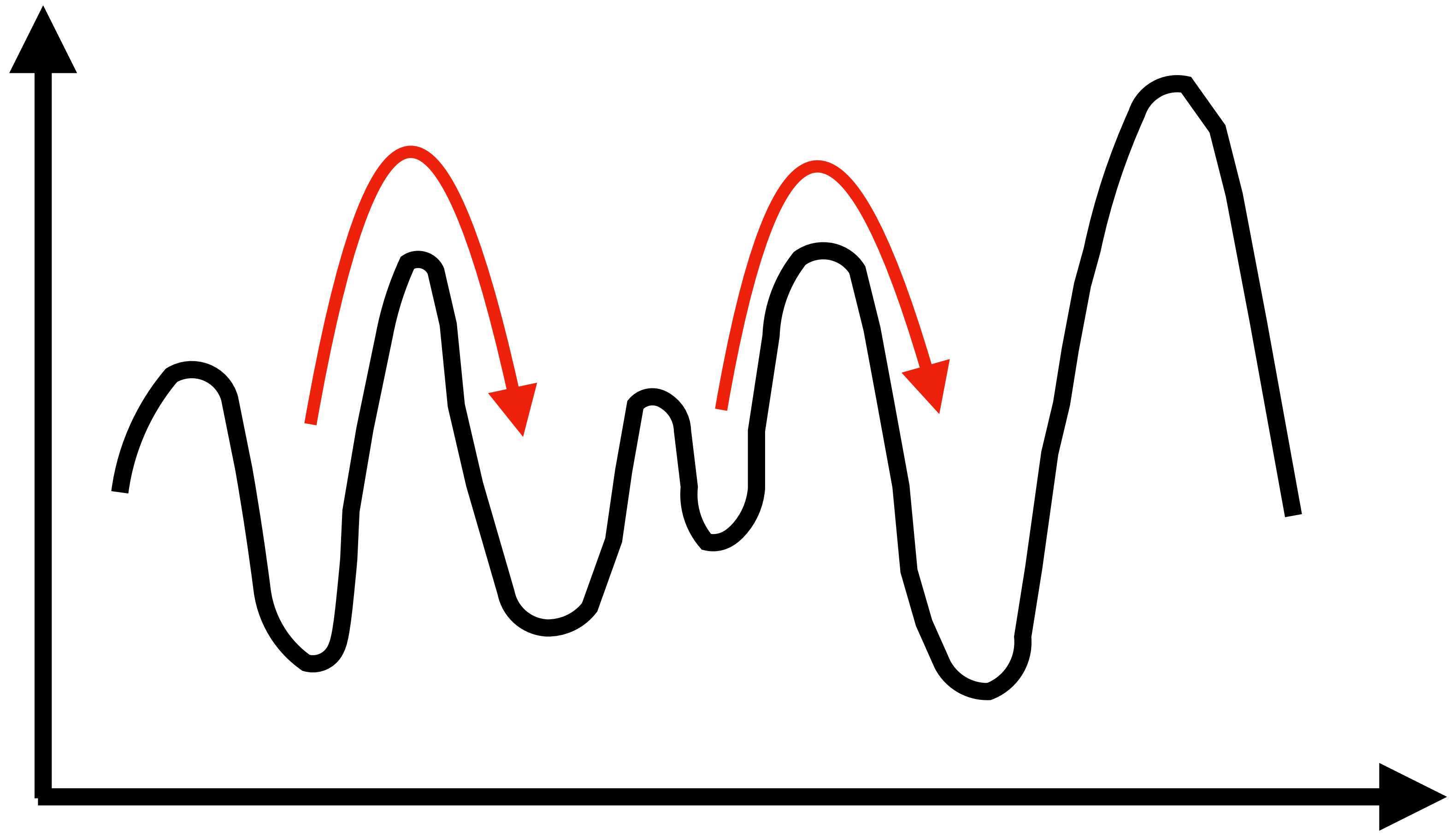
[Serbyn & al '13; Huse & al '14; Ros & al '15]
[Chandran & al '15; O'Brien & al '16; Pekker & al '17; Thomson & Schiro' '18, …]

Complete set of LIOMs: The system is completely localized in the basis of the $\{\tau_i\}$
Equivalent to n independent local degrees of freedom

MBL is stable in a broad range of disorder and interaction strength (no fine tuning of the coupling constants)

MBL *vs* Glasses

Ergodicity breaking in glasses is due to complex free-energy landscapes while in MBL barriers are small (the system breaks down in local independent degrees of freedom)



Free-energy barriers

- Only collective modes freeze (the thermal conductivity is non zero)
- Glasses are stable with respect to the environment (coupling to the bath)
- Glasses supposedly exist in $d > 2$, while the existence of MBL in $d > 1$ is still debated **[De Roeck and Huveneers '17]**
- Yet complex landscapes can induce MBL
  (e.g. QREM **[Faoro & al '19; Smelyanskiy & al '19; Biroli & al '20; Parolini & Mossi '21]**)

A pictorial view of MBL

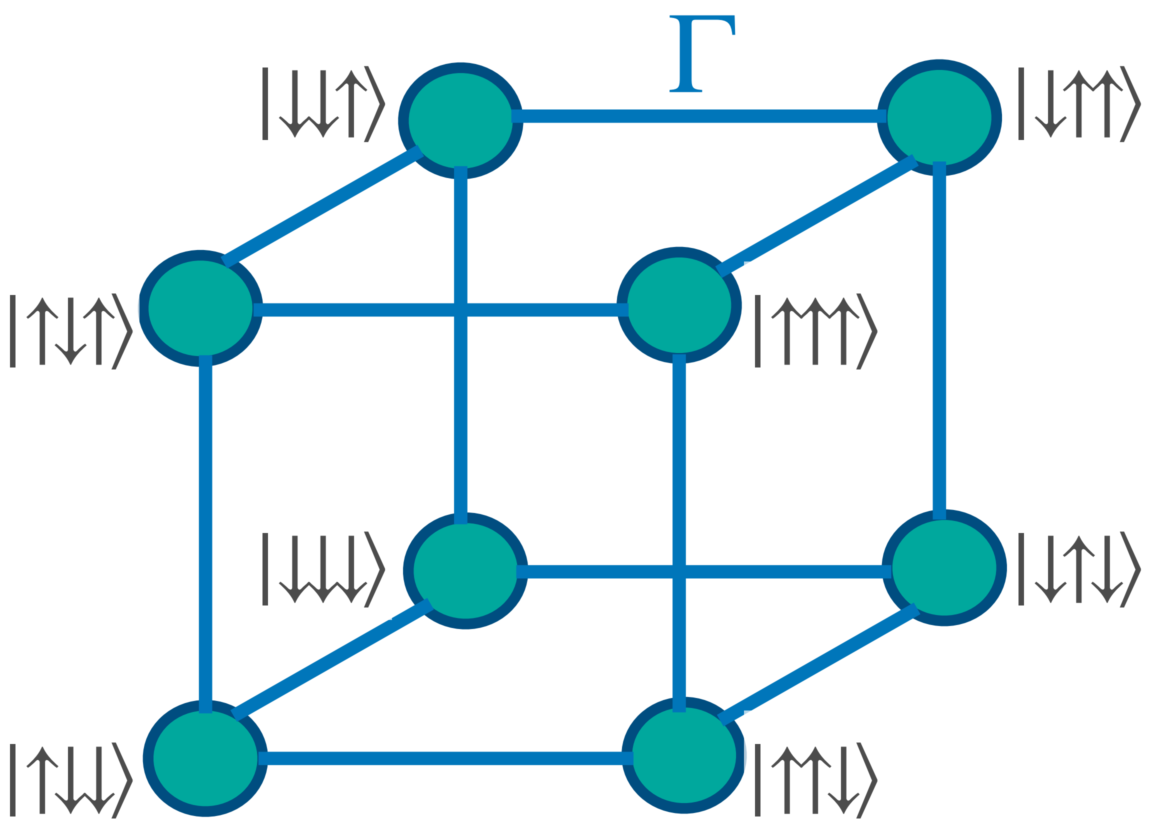
$$H = \underbrace{\sum_{i=1}^{n} \left(J_i \sigma_i^z \sigma_{i+1}^z + h_i \sigma_i^z \right)}_{H_0} + \underbrace{\Gamma \sum_{i=1}^{n} \sigma_i^x}_{H_{\text{int}}}$$

Recast the many-body quantum dynamics as single-particle diffusion in the n-dimensional configuration space (in a basis in which the system is localized in absence of interactions) **[Altshuler, Gefen, Kamenev & Levitov '97]**

Many-body configurations can be taught as "site orbitals" in the Hilbert space $|a\rangle = |\{\sigma_i^z\}\rangle$

$\langle \uparrow\downarrow\downarrow | H_0 | \uparrow\downarrow\downarrow \rangle \rightarrow$ **On-site (correlated) random energies**

$\Gamma \sigma_1^x | \uparrow\downarrow\downarrow \rangle = \Gamma | \downarrow\downarrow\downarrow \rangle \rightarrow$ **Hopping**



n-dimensional hypercube of $N = 2^n$ nodes and degree $n = \log_2 N$

$$H = \sum_{a=1}^{2^n} E_a |a\rangle\langle a| + \Gamma \sum_{\langle a,b \rangle} \left(|a\rangle\langle b| + |b\rangle\langle a| \right)$$

Single particle Anderson localization in a high dimensional disordered lattice (e.g. on the Bethe lattice) provides a pictorial representation of MBL **[De Luca & Scardicchio '13; Roy & Logan '20; Tikhonov & Mirlin '21]**

A schematic phase diagram

[Basko, Aleiner & Altshuler '06; Altshuler & al '97]

W, Γ^{-1}

Metal

- Thermal
- ETH - RMT
- Volume-law entanglement of eigenstates

"Bad metal"

- Slow dynamics
- Power-law decay of correlations
- Sub-diffusive transport

[Bar Lev & Reichman '14; Bar Lev & al '15; Agarwal & al '15; Luitz & Bar Lev '16; Doggen & al '18; Bera & al '17]

W_c

MBL

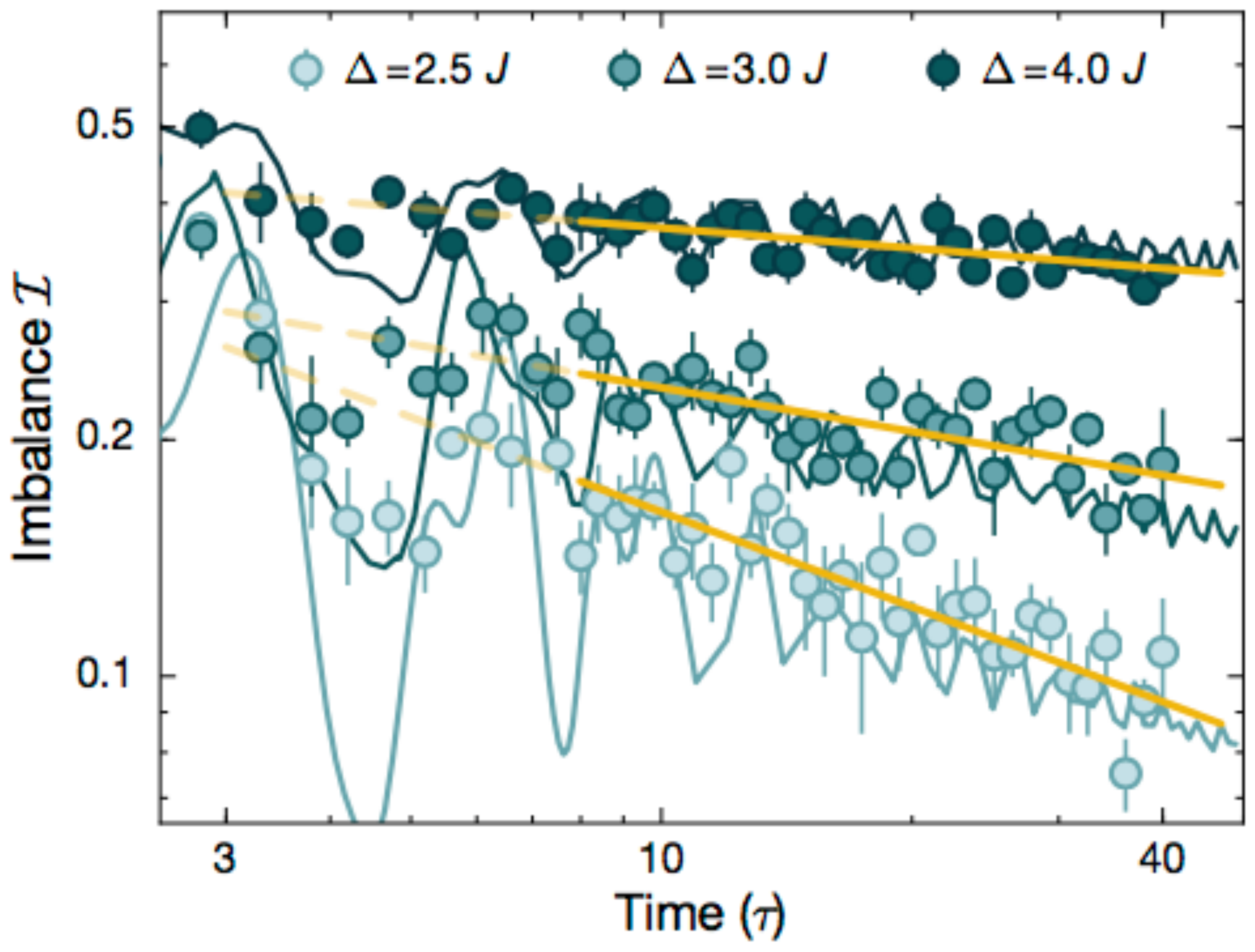
- No thermalization / ETH
- Complete set of LIOMs
- Poisson statistics
- No transport
- Area-law entanglement of eigenstates

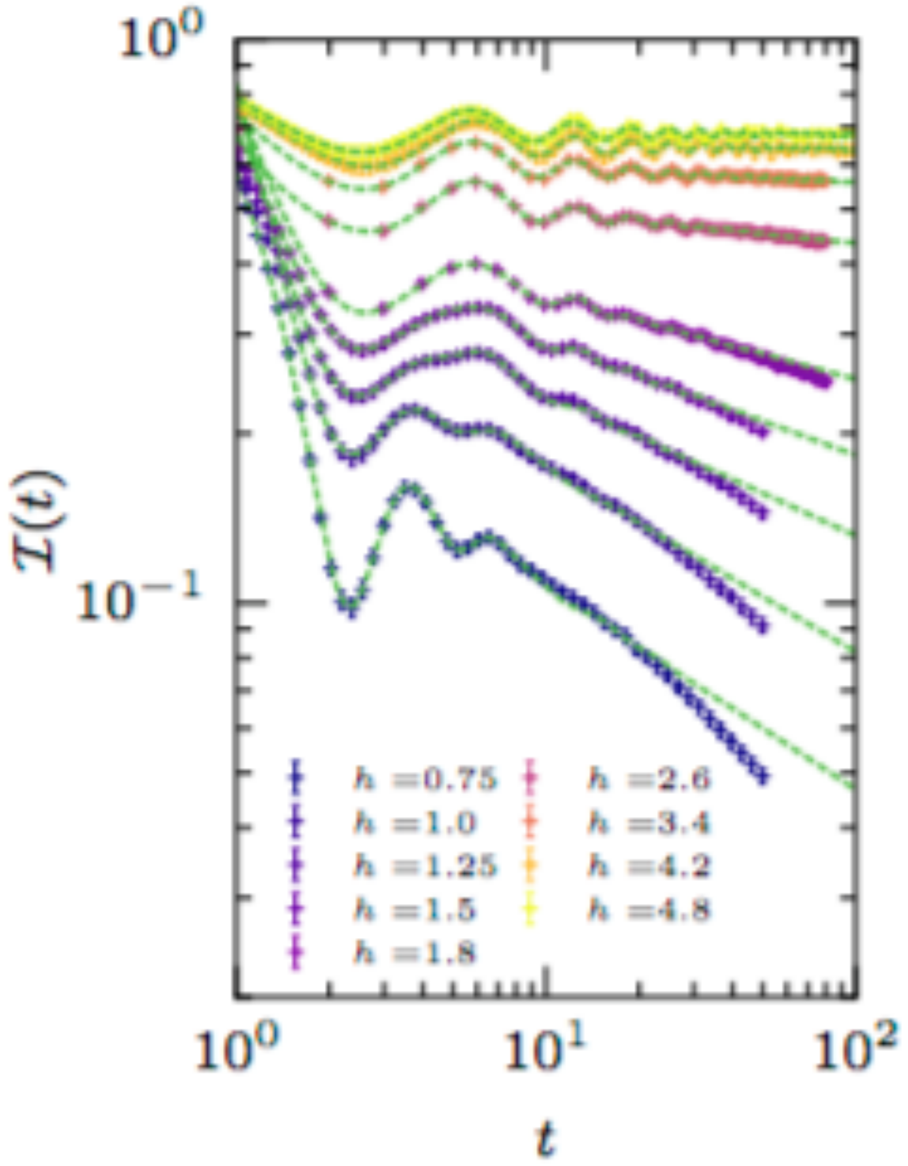
Many important question remain open:

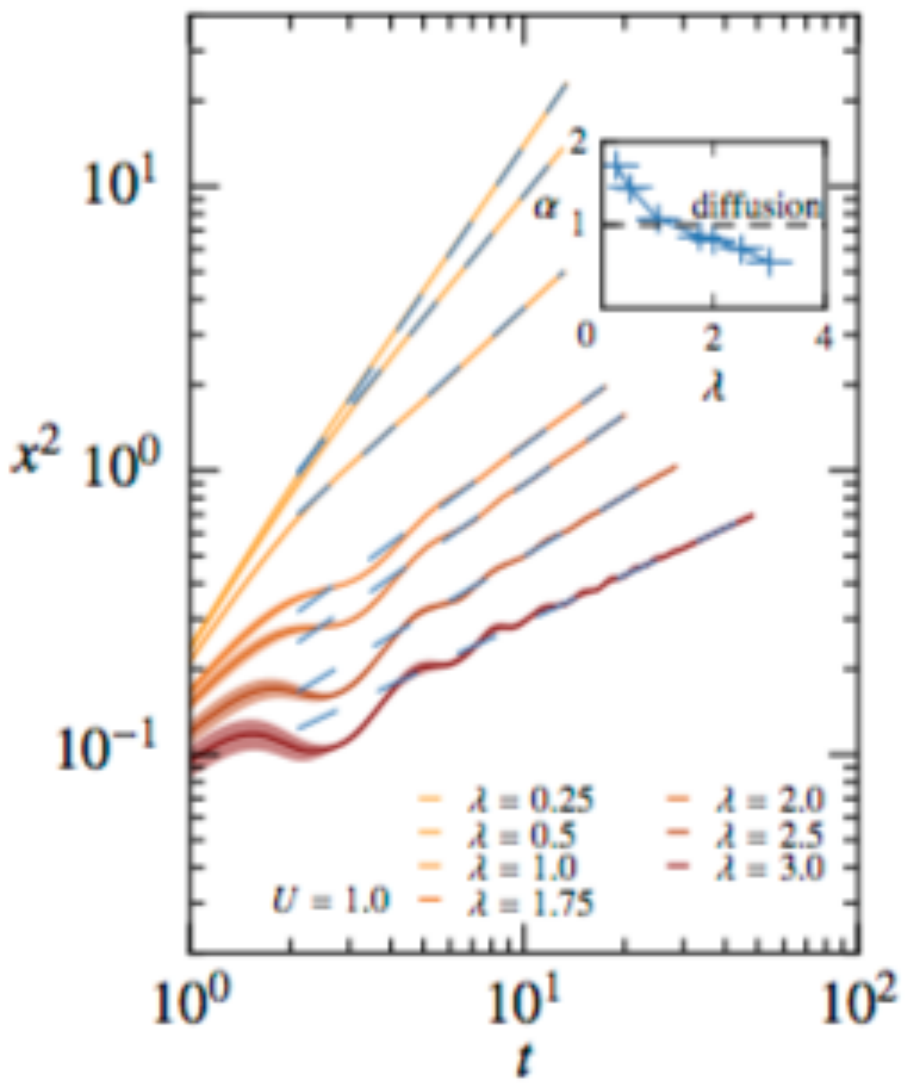
- The very existence of MBL in the infinite size and infinite time limit has been recently questioned. Avalanche instability to thermal inclusions? [Šuntajs & al '20; Sierant & al '20; Sels & Plokovnokov '21; Sels '21]
- MBL in quasiperiodic systems?
- MBL in $d > 1$? [De Roeck and Huveneers '17]
- Critical properties of the MBL transition? KT-like universality class? [Goremykina & al '19; Morningstar & Huse '19; Dumitrescu & al '19; Thiery & Müller '17; Morningstar & Huse '20]
- Properties of the "bad metal" regime? [Luitz & Bar Lev '17; Agarwal & al '17]

The “bad metal” regime

Anomalously slow out-of-equilibrium power-law relaxation and sub-diffusive transport in a broad region of the phase diagram preceding the MBL transition, observed both in the experiments and in numerical simulations **[Bar Lev & Reichman ’14; Bar Lev & al ’15; Agarwal & al ’15; Luitz & Bar Lev ’16; Doggen & al ’18; Bera & al ’17; Schreiber & al ’15; Bordia & al ’16; Lüschen & al ’17; Smith & al ’16; Xu & al ’18]**





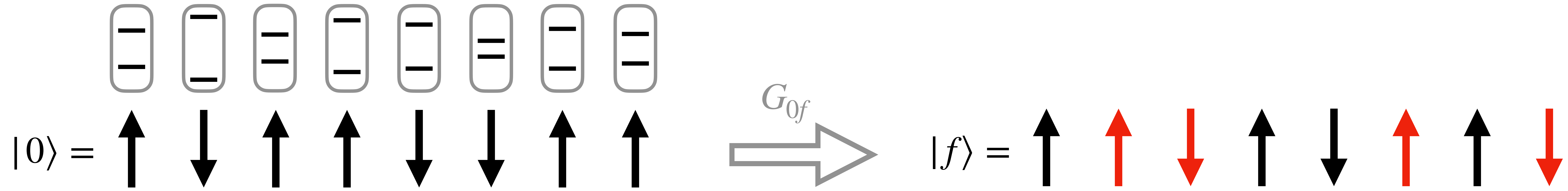


This behavior have been explained in therms of Griffiths regions: Rare inclusions of the localized phases with anomalously large escape times **[Agarwal & al ’15; Vosk & al ’15; Potter & al ’15; Luitz & Bar Lev ’17; Agarwal & al ’17]**

Yet the same behavior is also observed in quasiperiodic **[Bar Lev & al ’17; Li & al ’20; You & al ’20]** and $2d$ **[Bordia & al ’16; Lüschen & al ’17; Bar Lev & Reichman ’16]** systems in which Griffiths effects should be absent or subdominant

Complementary explanation directly based on quantum dynamics in the Hilbert space

Rarefaction of resonances



$W \gg \Gamma \rightarrow$ Suitable specific sequence of spin flips such that $\Delta E \ll \Gamma \rightarrow$ Resonance far away in the Hilbert space

Locator expansion [Anderson '58]

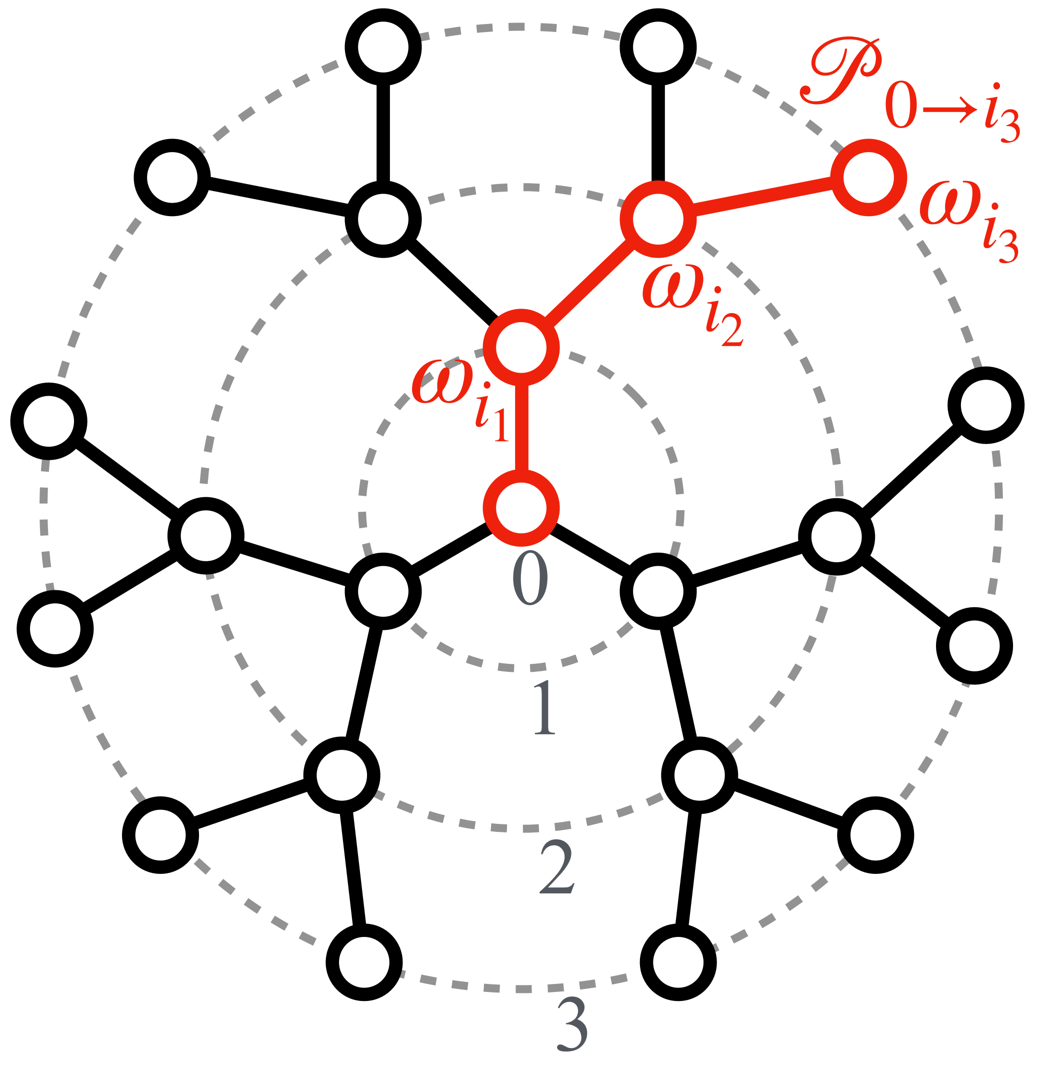
$$G_{0f} = \langle 0 | \frac{1}{E-H} | f \rangle = \sum_{paths\ \mathscr{P}_{0\to f}} \prod_{a \in \mathscr{P}_{0\to f}} \frac{\Gamma}{E - E_a} = \sum_{S.A.\ paths\ \mathscr{P}^\star_{0\to f}} \prod_{a \in \mathscr{P}^\star_{0\to f}} \frac{\Gamma}{E - E_a - \Sigma_a(\mathscr{P}^\star_{0\to f})}$$

$|G_{0f}|^2 \propto$ probability that the system starts in $|0\rangle$ at $t=0$ and is found in $|f\rangle$ at $t=\infty$

Computing $\Sigma_a(\mathscr{P}^\star_{0\to f})$ is in general a formidable task: $W \gg \Gamma \rightarrow$ *Forward-scattering approximation* (leading order term of the perturbative expansion: retaining only the contribution of the shortest paths connecting $|0\rangle$ to $|f\rangle$

Akin to the partition function of a directed polymer in a (complex and correlated) random potential $e^{-\beta \omega_a} = \Gamma/(E - E_a - \Sigma_a)$ [Monthus & Garel '09 & '11; Lemarié '19], a stat mech problem that has been deeply studied

Glass transition of DPRM on the Bethe lattice



$$Z_{DP}(\beta) = \sum_{paths\ \mathscr{P}_{0\to i_n}} e^{-\beta \sum_{i_m \in \mathscr{P}_{0\to i_n}} \omega_{i_m}}$$

$\omega_i \to$ iid random variables

Glass transition at $T_\star$ similar to that of the Random Energy Model [Derrida & Spohn '88]

- High $T > T_\star$ $\to$ An exponential number of paths contribute to the sum
- Low $T < T_\star$ $\to$ Few O(1) specific disorder-dependent paths dominate the sum

"Quenched" and "annealed" free-energy: $\to$

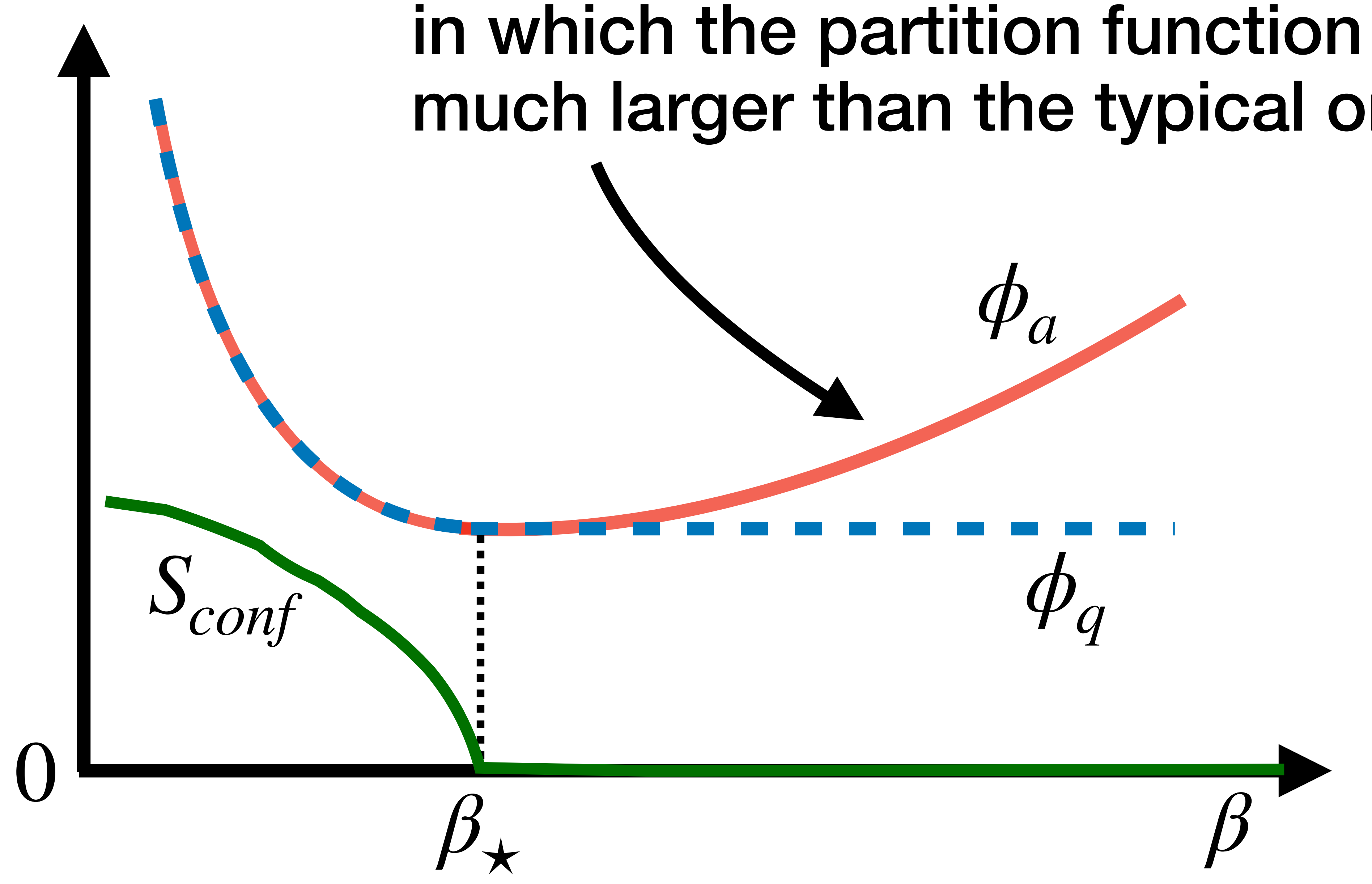
$$\begin{cases} \phi_q(\beta) = \dfrac{1}{\beta n} \overline{\log Z(\beta)} \\ \phi_a(\beta) = \dfrac{1}{\beta n} \log \overline{Z(\beta)} \end{cases}$$

Configurational entropy: Assume that $e^{nS_{conf}(e)}$ paths have energy ne $\to$

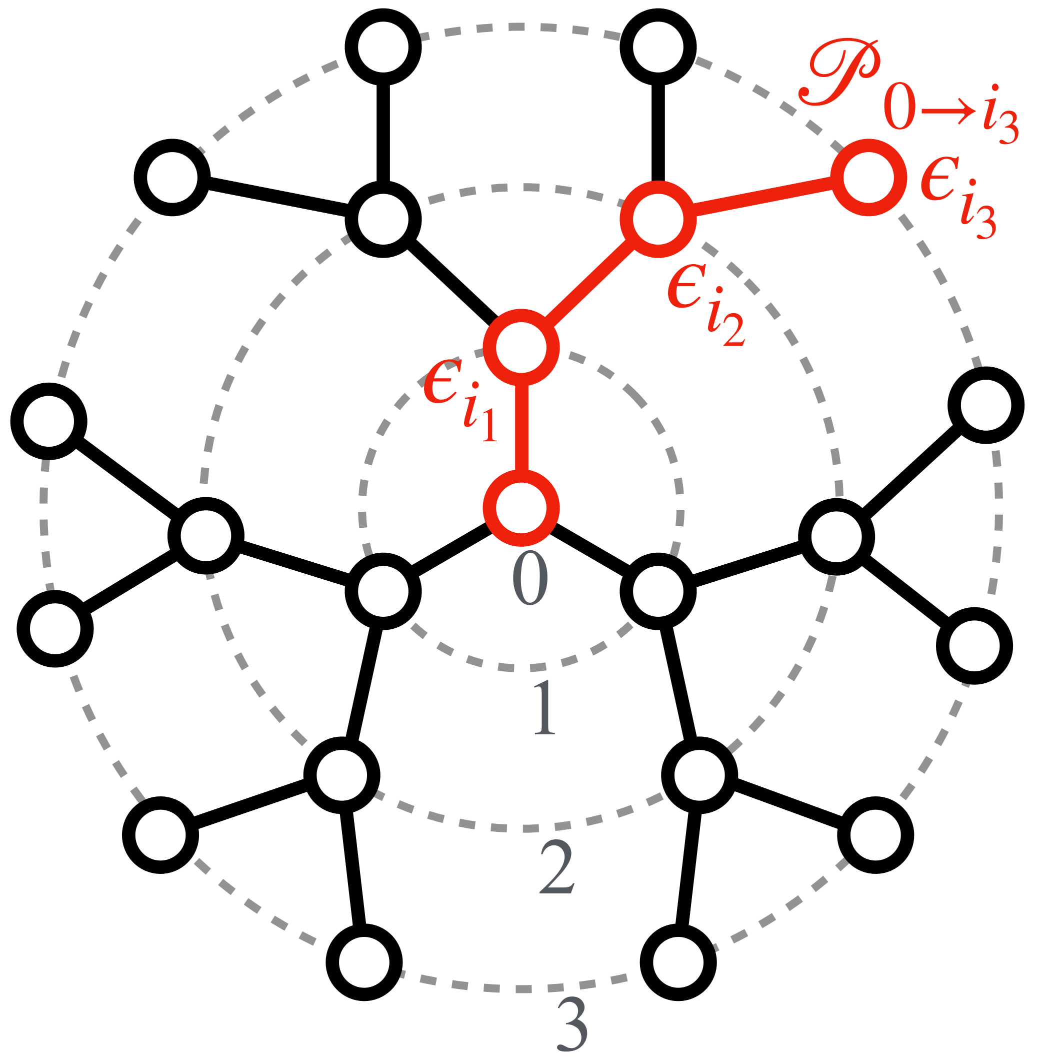
$$Z_{DP}(\beta) = \int de\, e^{n(S_{conf}(e) - \beta e)}$$

$$S_{conf} = -\beta^2 \frac{d\phi_q}{d\beta}$$

Some realizations of the disorder in which the partition function is much larger than the typical one



Benchmark case: AL on the loop-less Cayley tree



$$H = \sum_i \epsilon_i |i\rangle\langle i| - t \sum_{\langle i,j\rangle} \left(|i\rangle\langle j| + |j\rangle\langle i|\right) \qquad \epsilon_i \in [-W/2, W/2] \qquad k+1=3$$

$$G_{0,i_n} = \frac{-t}{\epsilon_{i_1} + \Sigma_{i_1\to i_0}} \frac{-t}{\epsilon_{i_2} + \Sigma_{i_2\to i_1}} \times \cdots \times \frac{-t}{\epsilon_{i_n} + \Sigma_{i_n\to i_{n-1}}}$$

Exact recursion relation for the self-energies $\longrightarrow$ $\Sigma_{i_m\to i_{m-1}} = -\sum_{i_{m+1}\in\partial i_m} \frac{t^2}{\epsilon_{i_{m+1}} + \Sigma_{i_{m+1}\to i_m}}$ [Abou-Chacra & al '73]

Initial conditions $\longrightarrow$ $\Sigma_{i_n\to i_{n-1}} = 0$

$$Z(\beta) = \sum_{i_n=1}^{(k+1)k^{n-1}} \left|G_{0,i_n}\right|^\beta = \sum_{paths\ \mathscr{P}_{0\to i_n}} \prod_{i_m\in\mathscr{P}_{0\to i_n}} \underbrace{\left|\frac{t}{\epsilon_{i_m} + \Sigma_{i_m}}\right|^\beta}_{\equiv\, e^{-\beta\omega_{i_m}}}$$

- Partition function of a DP in presence of correlated and broadly distributed random energies [Monthus & Garel '09 & '11; Biroli & Tarzia '20; Kravtsov & al '18]
- Effective inverse "temperature" $\beta \to$ statistics of the moments of the sum of the propagators
- $Z(2) \propto$ probability that a particle starting in 0 reaches the leaves after infinite time (Fisher-Lee conductivity from the root to the boundaries [Fisher & Lee '81])

Quenched and annealed free-energy

- Position of the minimum $\beta_\star$ → freezing of the paths contributing to transport ($\beta_\star \gtrsim 2$?)

- Height of the minimum → $\overline{\log\left(\sum_{i_n} |G_{0,i_n}|^2\right)} < 0$ → The probability that the particle reaches the boundaries decreases exponentially with n (hallmark of localization)

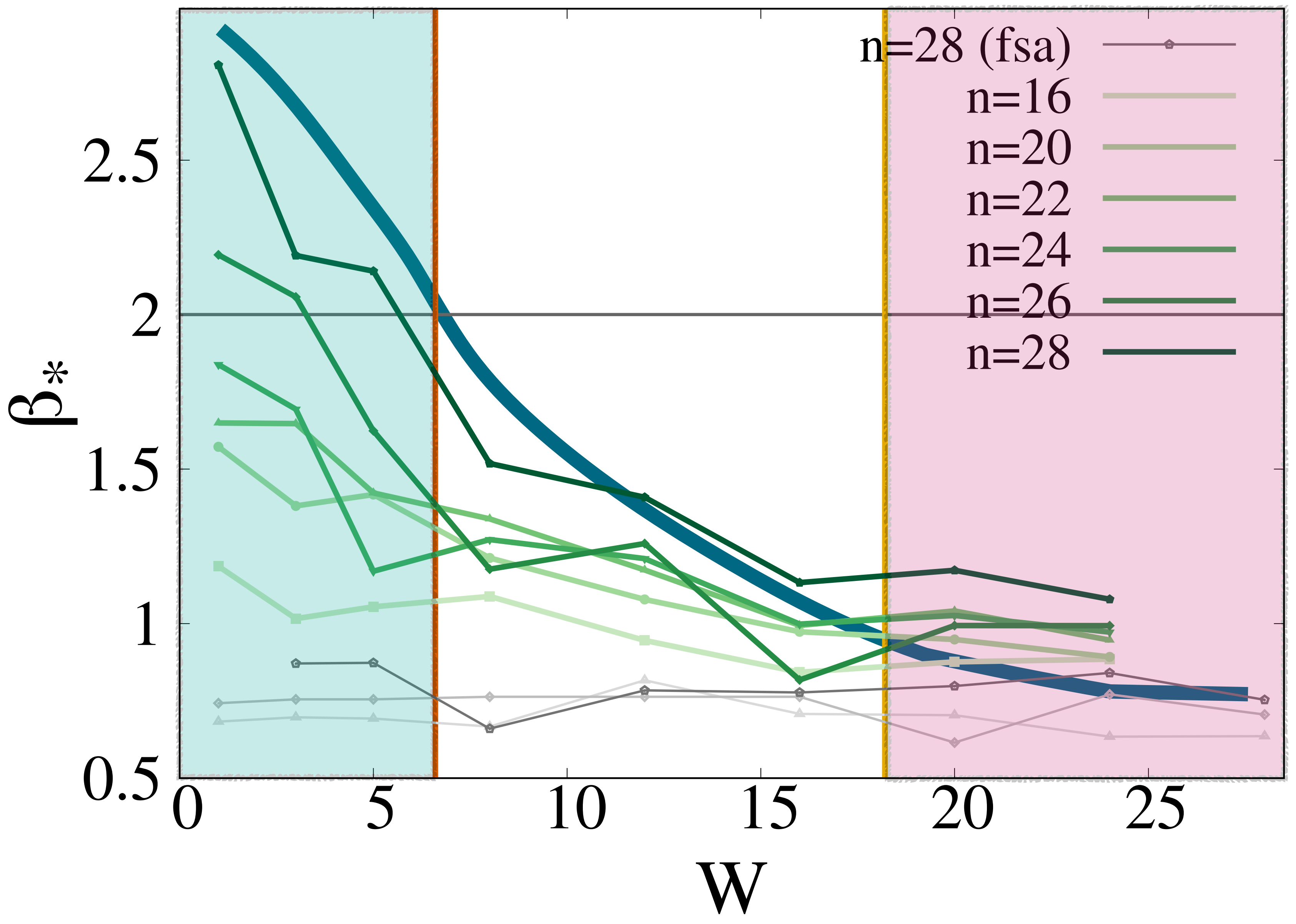
  $\phi_a(\beta_\star) = 0,\ \phi_q(2) = 0$ → Upper and lower bound for AL

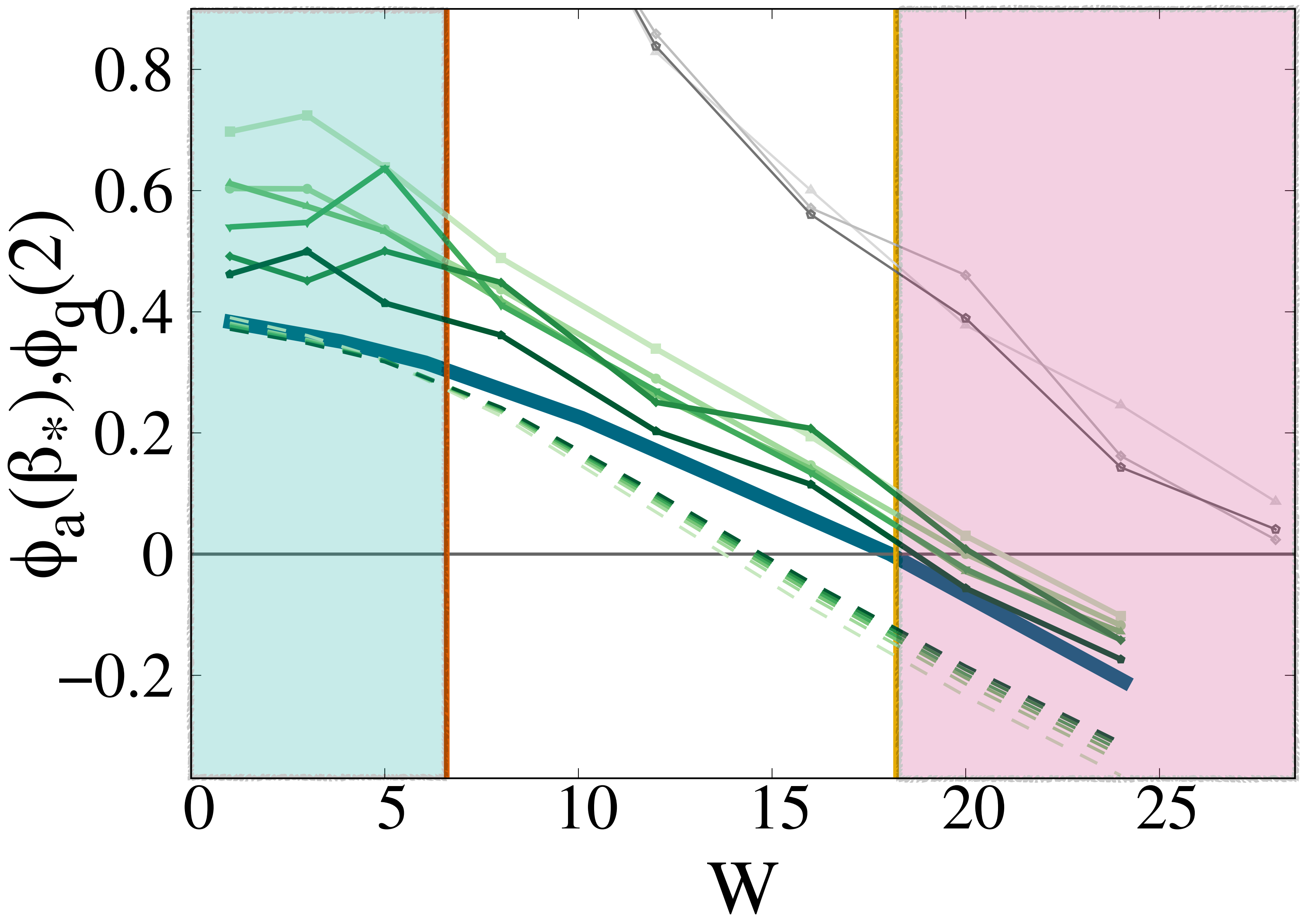
- Configurational entropy $S_{conf} = -\beta^2\phi'$ at $\beta = 2$ → Number of paths contributing to transport (if $\beta_\star > 2$)

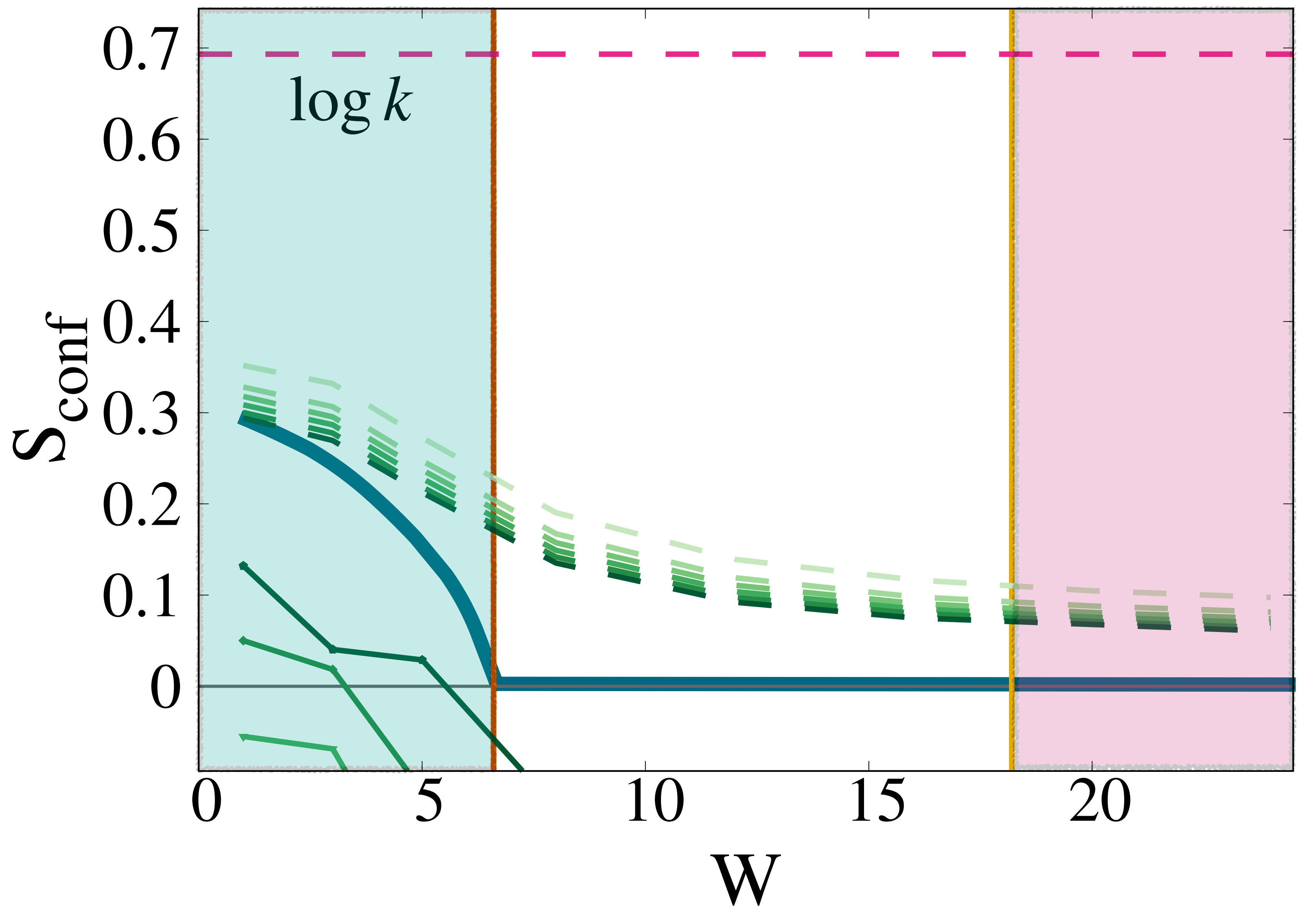
Phase diagram

Phase diagram

Phase diagram





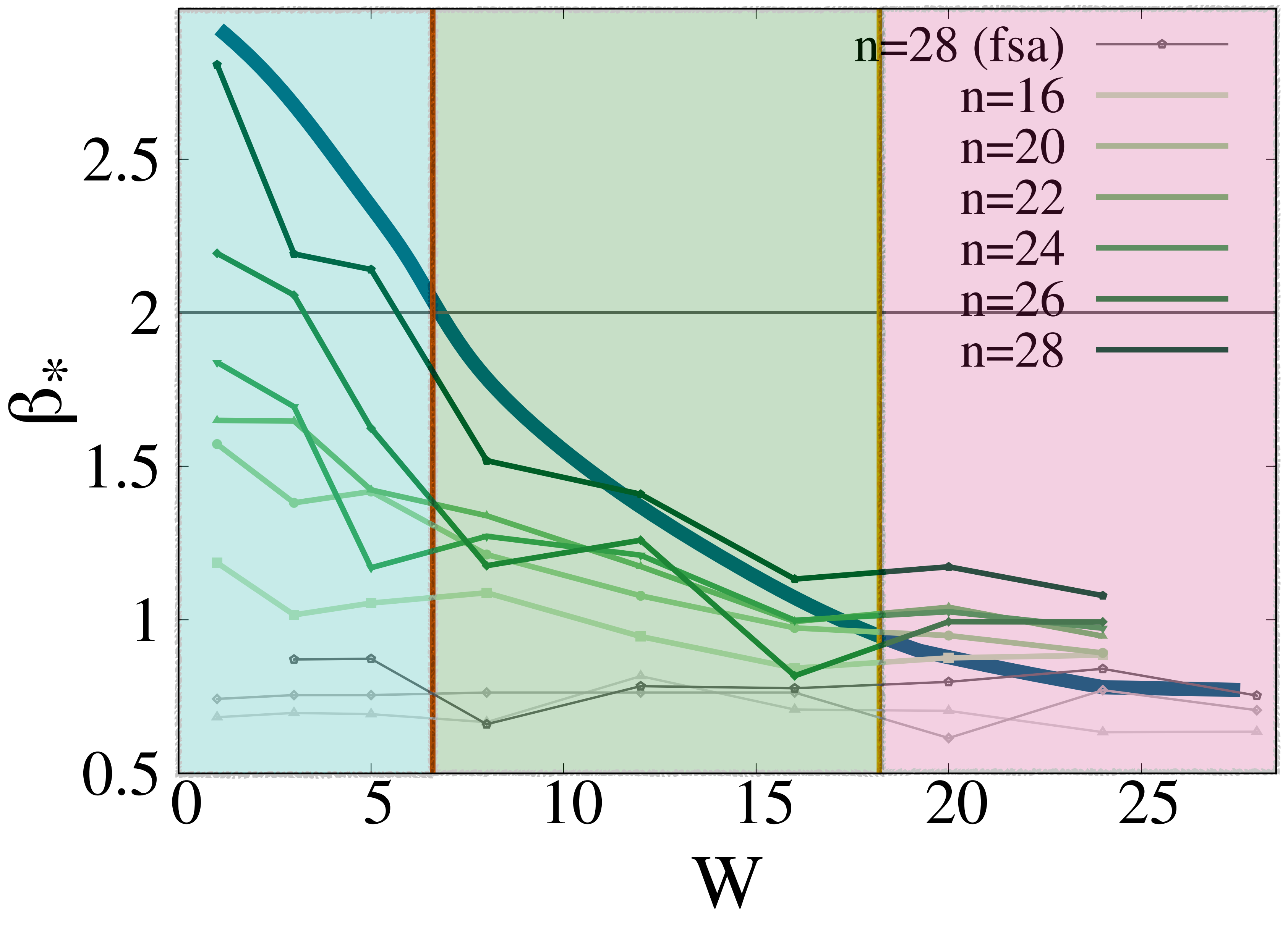


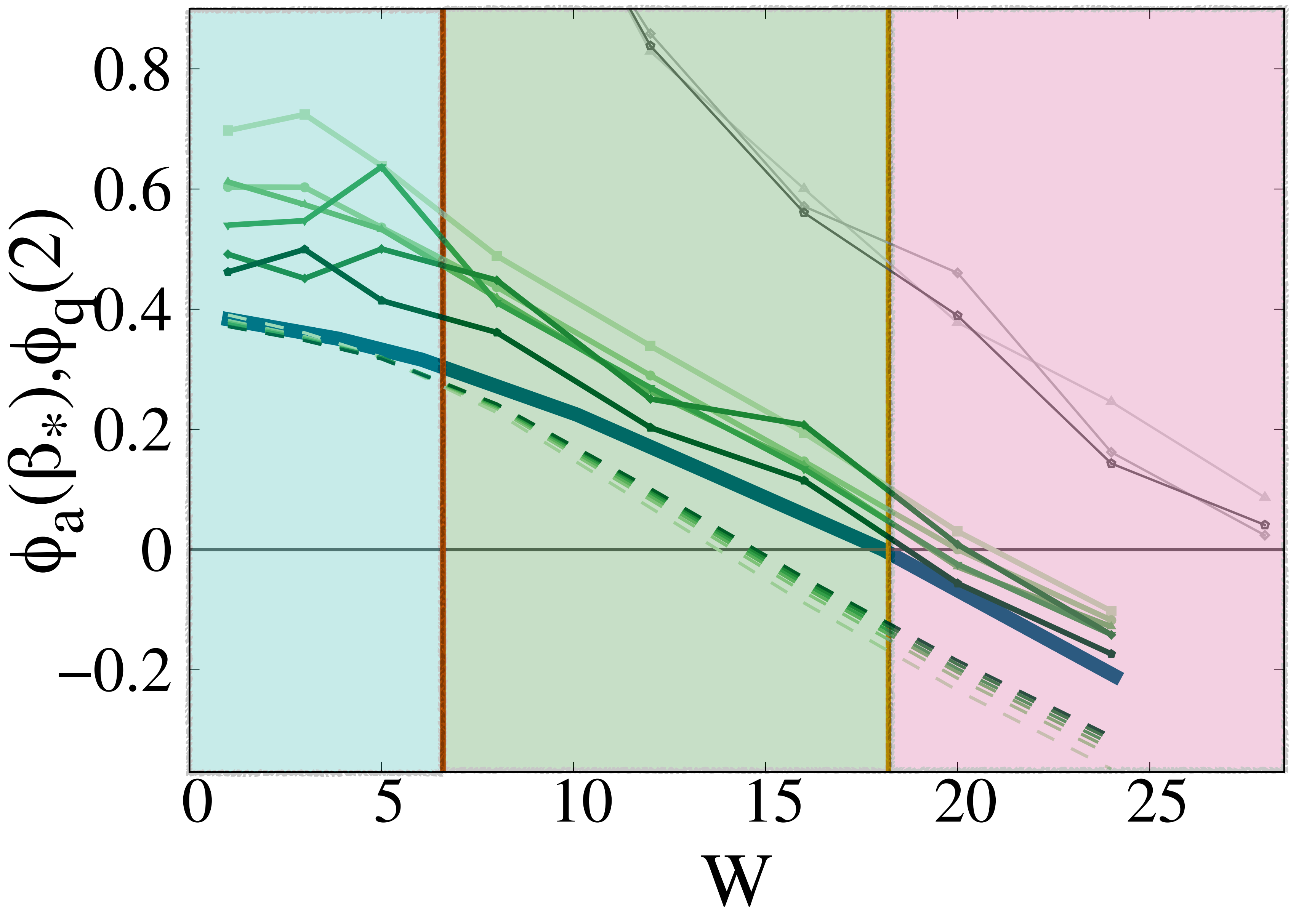
$0 < S_{conf} < \log k$

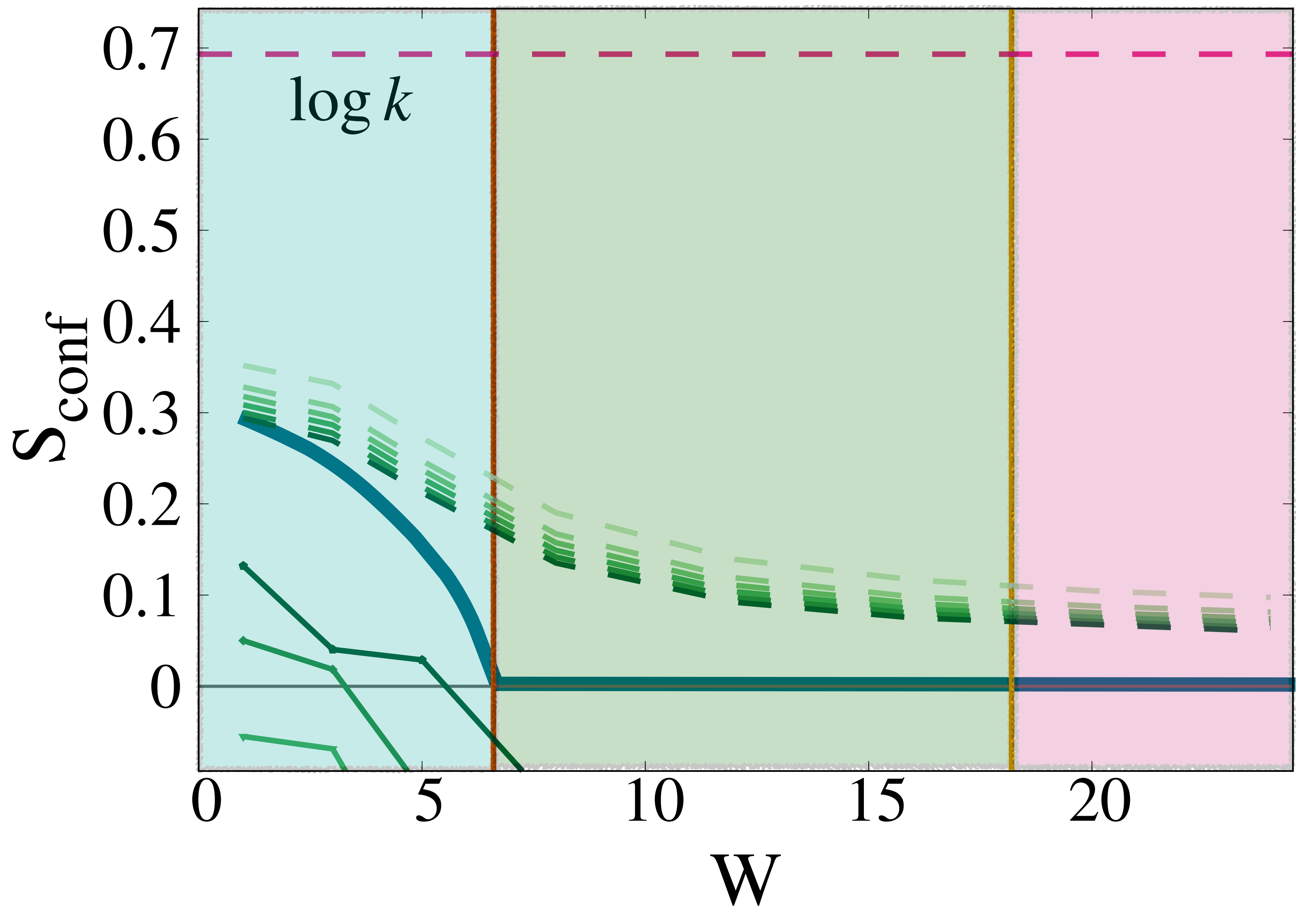
$\beta_\star > 2$

$\phi(2) < 0$

Phase diagram







$0 < S_{conf} < \log k$
$\beta_\star > 2$

$\beta_\star < 2$
$S_{conf} = 0$
$\phi(2) > 0$

$\phi(2) < 0$

$W_g \approx 6.6$ $W_c \approx 18.17$ W

An exponential number of paths (but less than k^n) contribute to transport and dissipation

Transport and dissipation only occur through few specific disorder-dependent paths

Anderson localization

Phase diagram

Typical samples are localized but there exist rare realizations of the disorder for which the conductivity is much larger than the typical one (Griffiths region)

$W_g \approx 6.6$ $W_c \approx 18.17$ W

An exponential number of paths (but less than k^n) contribute to transport and dissipation

Transport and dissipation only occur through few specific disorder-dependent paths

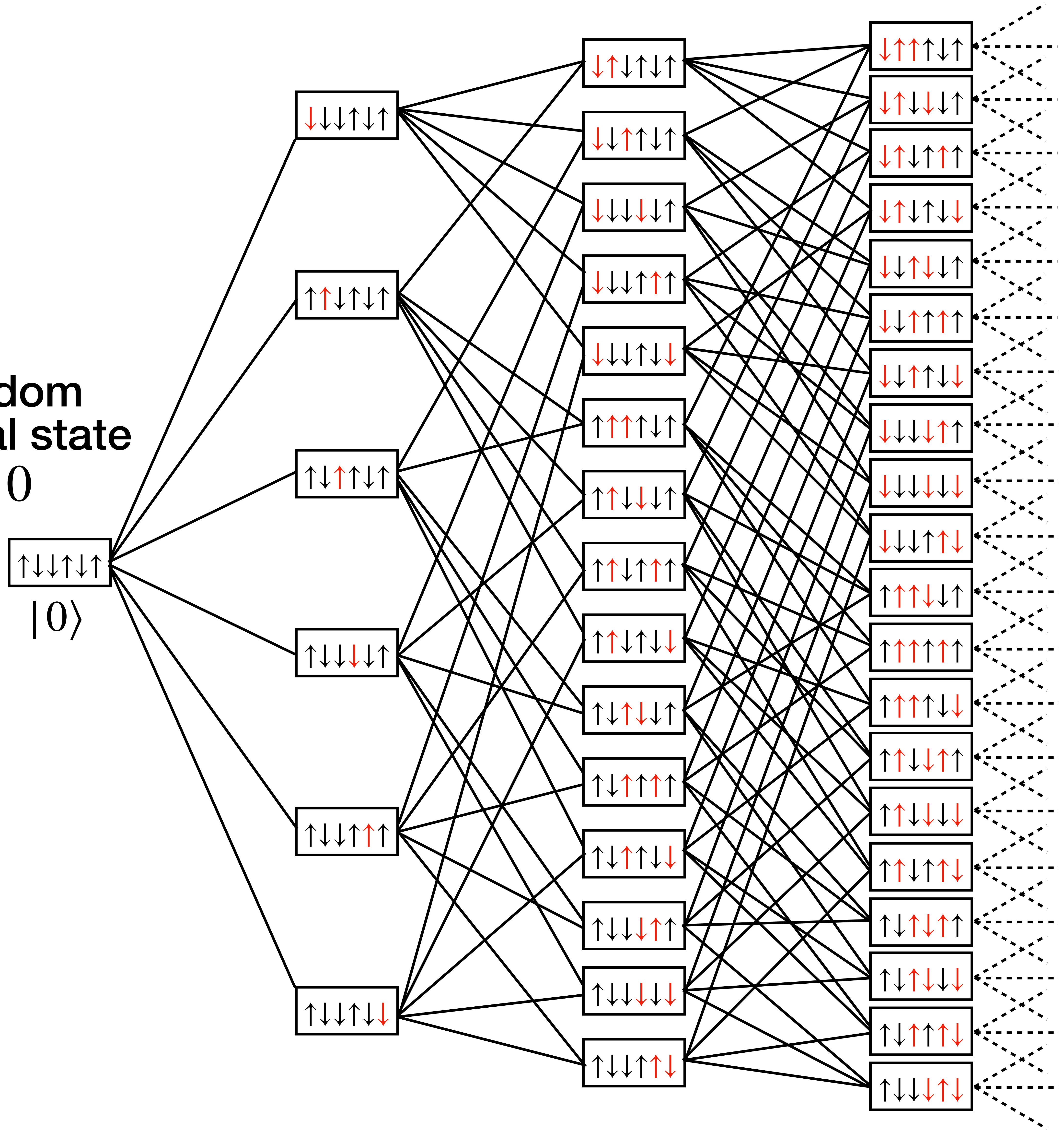
Anderson localization

Phase diagram

- Eigenstates of the Anderson model on the Cayley tree are multifractal even at small disorder **[Monthus & Garel ’09 & ’11; Biroli & Tarzia ’20; Tikhonov & Mirlin ’16; Sonner & al ’17]** and the level statistics is not given by RMT
- Ergodicity is restored in the thermodynamic limit when the loops are reintroduced (e.g. on sparse random graphs)
- The FSA provides un upper bound for AL: $W_c^{\rm fsa} \simeq 2etk \log k > W_c \simeq 4tk \log k$ **[Abou-Chacra & al ’73]**

Interacting case: The Imbrie model

Random initial state $E \approx 0$



$\binom{n}{n/2} \simeq \frac{2^n}{\sqrt{n}}$ configurations $|f\rangle \to n/2$ spin flips

$$\sum_i \sigma_i^z(0)\,\sigma_i^z(f) = 0$$

$n/2!$ paths from $|0\rangle$ to $|f\rangle$

$$Z(\beta) = \sum_{f=1}^{\binom{n}{n/2}} |G_{0f}|^\beta$$

$Z(2) \propto$ probability that the system has decorrelated from the initial condition (Fisher-Lee conductivity from $|0\rangle$ to the "equator")

$$\phi_q(\beta) = \frac{2}{\beta n}\overline{\log Z(\beta)} \qquad \phi_a(\beta) = \frac{2}{\beta n}\log \overline{Z(\beta)}$$

Quenched and annealed free-energy

Previous numerical studies of the spectral statistics for $10 \le n \le 16$ indicate that the MBL transition should occur in the interval $W_c \in [3.5,4]$ **[Abanin & al '19; Roy & Logan '21]**

- Exact computation of $G = [E - H]^{-1}$ $(n \le 16)$
- Forward-scattering approximation $(n \le 24)$ at strong disorder

Summary of the numerical results

Summary of the numerical results

$S_{conf} \simeq \log 2$
$\beta_\star > 2$

- Full ergodicity
- ETH - RMT
- Normal transport

Summary of the numerical results

Summary of the numerical results

$S_{conf} \simeq \log 2$
$\beta_\star > 2$

$W_{ergo} \approx 2.5$

- Full ergodicity
- ETH - RMT
- Normal transport

$W_g \approx 3.5$

$\beta_\star < 2$
$S_{conf} = 0$
$\phi(2) > 0$

- Many-body configurations only hybridize with few O(1) resonances far away in the Hilbert space
- Poisson statistics?

$W_c \approx 8$

$\phi(2) < 0$

W

- Genuine MBL phase
- Absence of transport and dissipation
- Poisson statistics

Summary of the numerical results

Typical samples are localized but rare resonances still exist in rare realizations of the disorder (Griffiths region)

$W_{ergo} \approx 2.5$ $W_g \approx 3.5$ $W_c \approx 8$ W

- Full ergodicity
- ETH - RMT
- Normal transport

- Many-body configurations only hybridize with few O(1) resonances far away in the Hilbert space
- Poisson statistics?

- Genuine MBL phase
- Absence of transport and dissipation
- Poisson statistics

Summary of the numerical results

Summary of the numerical results

- Fully ergodic regime at small disorder where $S_{conf} \simeq \log 2$ and all configurations $|f\rangle$ contribute to $Z(2)$
- W_c drifts to higher disorder when the system size is increased. The genuine MBL transition occurs at much stronger disorder that has been suggested in previous studies **[Morningstar & al '21; Sierant & al '20]**
- The MBL transition estimated numerically in previous studies seems to coincide with the "glass transition" where $\beta_\star$ crosses 2
- A very broad disorder range where typical samples are localized but rare resonances still exist **[Morningstar & al '21]**

A tentative phase diagram (for finite-size systems)

[Morningstar & al ’21; Althuler & al ’97]

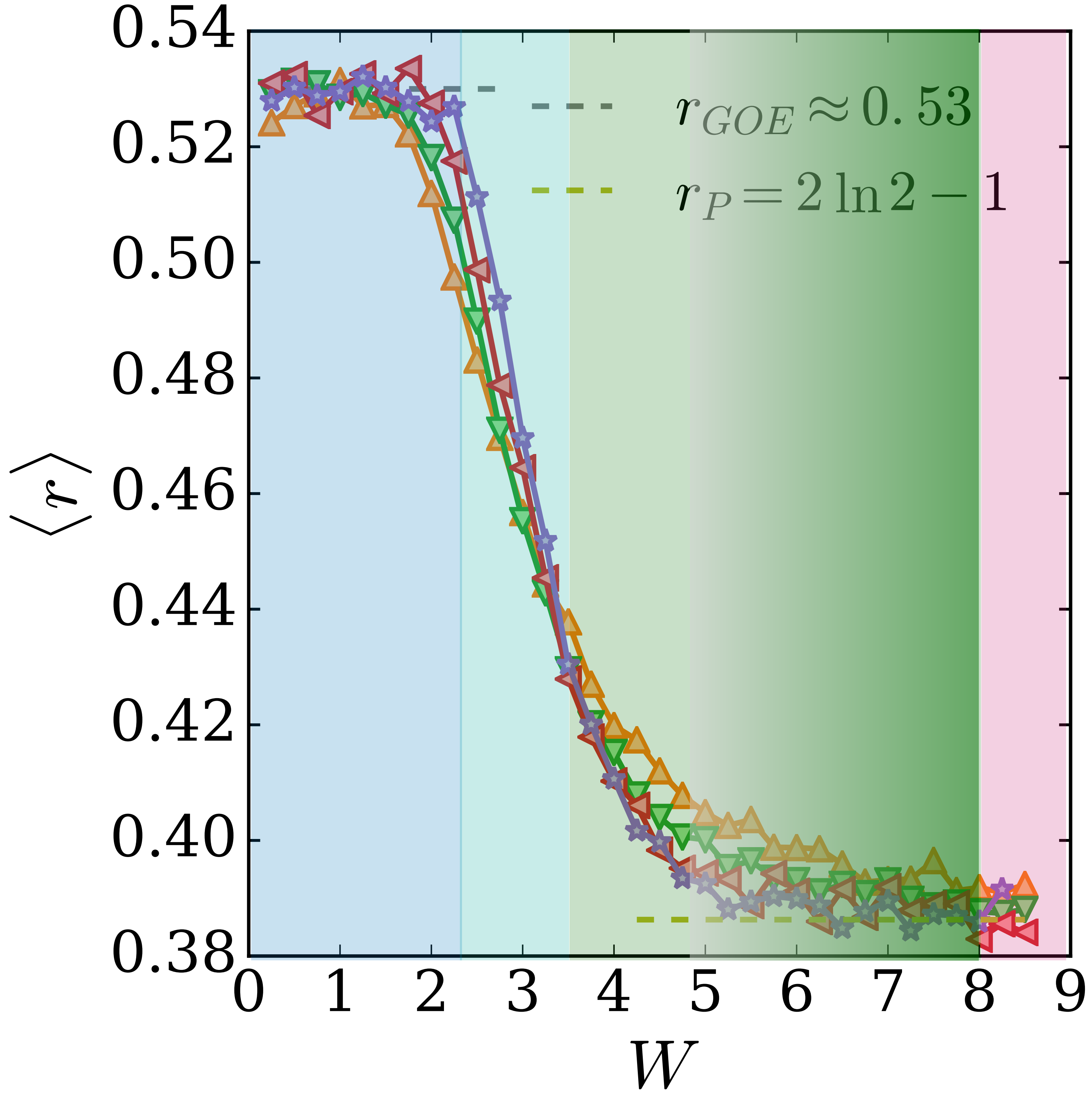
$W_{ergo} \approx 2.5$ $W_g \approx 3.5$ $W_c \approx 8$ W

- Full ergodicity
- ETH - RMT
- Normal transport

- Highly heterogeneous transport and dissipation
- Intermediate statistics?

- Many-body configurations only hybridize with few O(1) resonances far away in the Hilbert space
- Poisson statistics?
- Griffiths region: Typical samples appears as localized but resonances are still found in rare samples

- Genuine MBL phase
- Absence of transport and dissipation
- Poisson statistics



[Abanin & al ’19]

$n = 10, \ldots, 16$

Summary & Perspectives

- A new interpretation of the MBL transition in terms of the freezing glass transition of the paths leading to decorrelation in the Hilbert space [Monthus & Garel ’09 & ’11; Biroli & Tarzia ’20; Kravtsov & al ’18; Lemarié ‘19]
- Several MBL regimes in finite size samples [Morningstar & al ’21]. The true MBL transition occurs at much stronger disorder than previously reported [Morningstar & Huse ’21; Šuntajs & al ’20; Sierant & al ’20; Sels '21]
- A complementary mechanism for slow transport and anomalous diffusion in the bad metal regime. Rare insulating segments with anomalously large escape times in real space *vs* delocalization along rare paths in the Hilbert space [Biroli & Tarzia ’17]
- A new set of tools inherited from DPRM to inspect the statistics of resonances, which unveil new features of MBL

➡ Thermodynamic limit? The glass transition is just a crossover? (e.g. Anderson model on the RRG)

➡ Quasiperiodic systems? (The only source of randomness is the choice of the initial condition)

➡ $d > 1$?

➡ Repeat the analysis varying the length of the “polymers” (i.e. the number of spin flips)?
Relationship with the characteristic length of the LIOMs?

➡ Depinning transition? Avalanches? Chaos?

➡ Implications on the dynamics and on the multifractality of the eigenstates? [De Tomasi & al ’21]