Universität
Oldenburg
June 1, 2010

Extreme Fluctuations and Finite-Size Corrections in Spin Glasses and other Combinatorial Problems

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Find at: www.physics.emory.edu/faculty/boettcher

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Funding:

- NSF-DMR, Los Alamos-LDRD, Emory-URC

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Extremal Optimization (EO)

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    - *without* tuning any Control Parameters
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- How can we use it to optimize?
  - ➔ Extremal Driving:
    - Select and eliminate the "bad",
    - Replace it *at random*,
    - Eventually, only the "good" is left!

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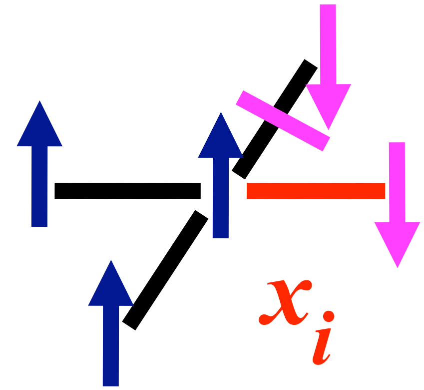
- Spin Glasses (eg. MAX-CUT):

$$\lambda_i = x_i \sum_j J_{i,j}\, x_j$$



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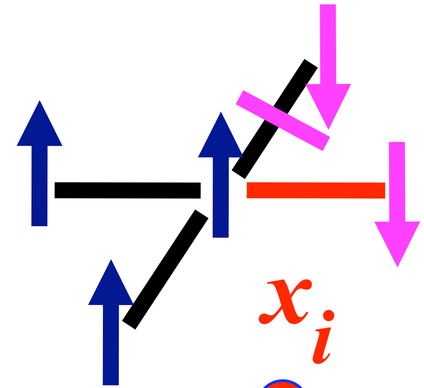
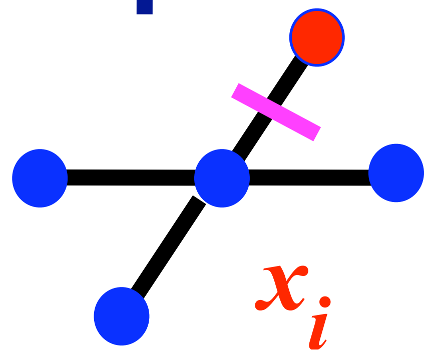


- Partitioning (eg. MIN-CUT):

$$\lambda_i = -\ (\text{\#-cut edges of } x_i)$$



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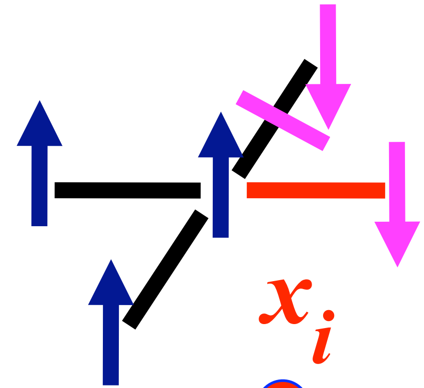

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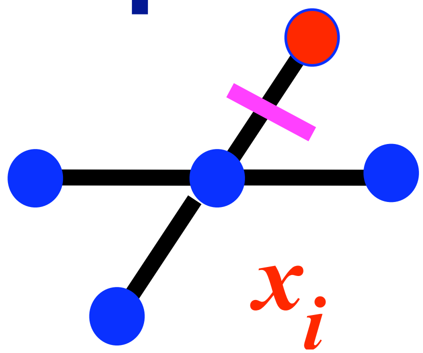

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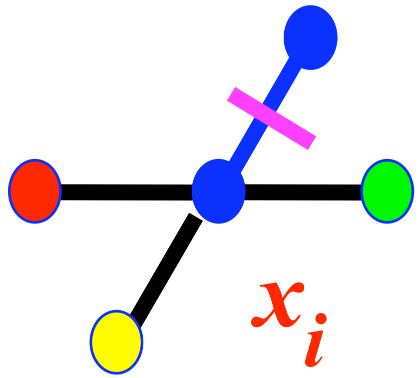

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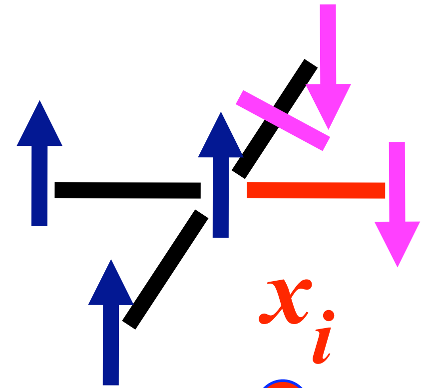

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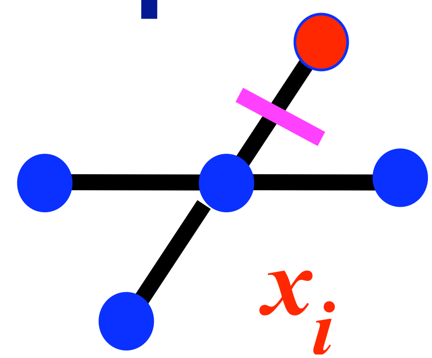

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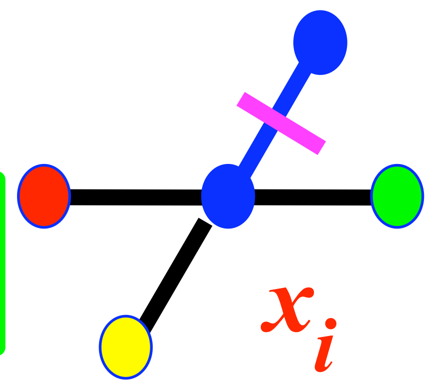

$$Cost \propto H = -\sum_i \lambda_i$$

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(7) Return: Best $C(S)$ found along the way!

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EO-run for Partitioning ($n=500$):

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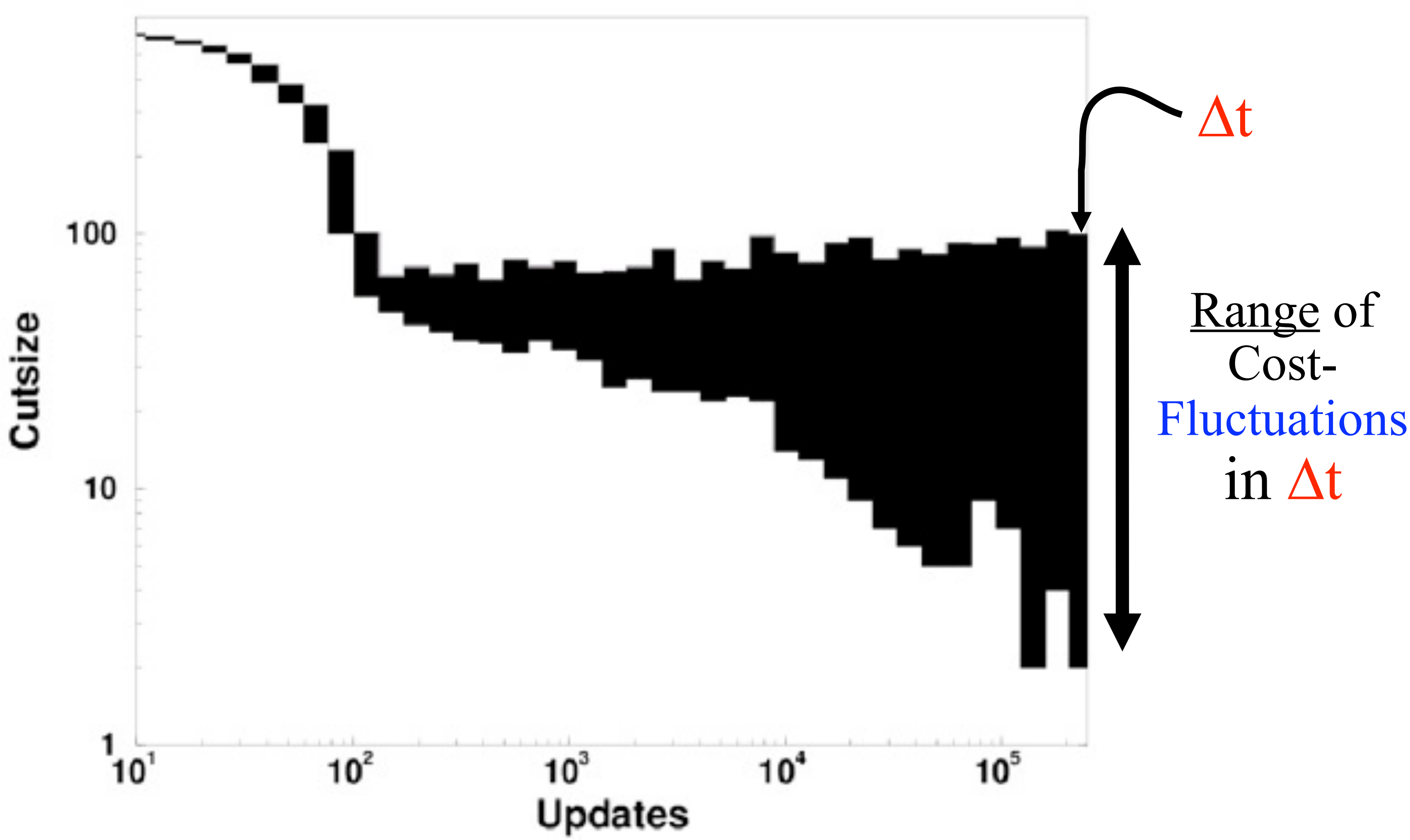

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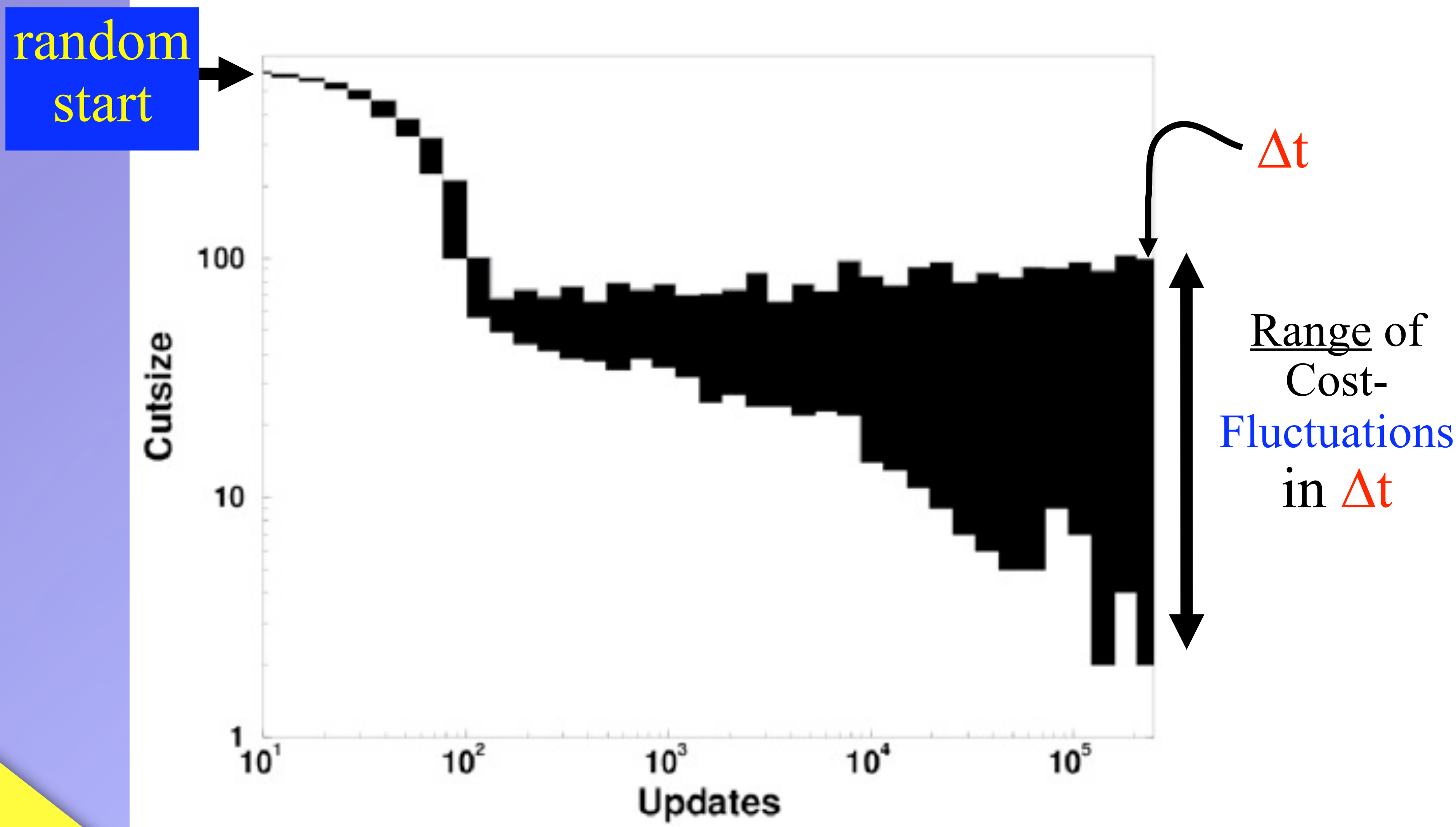

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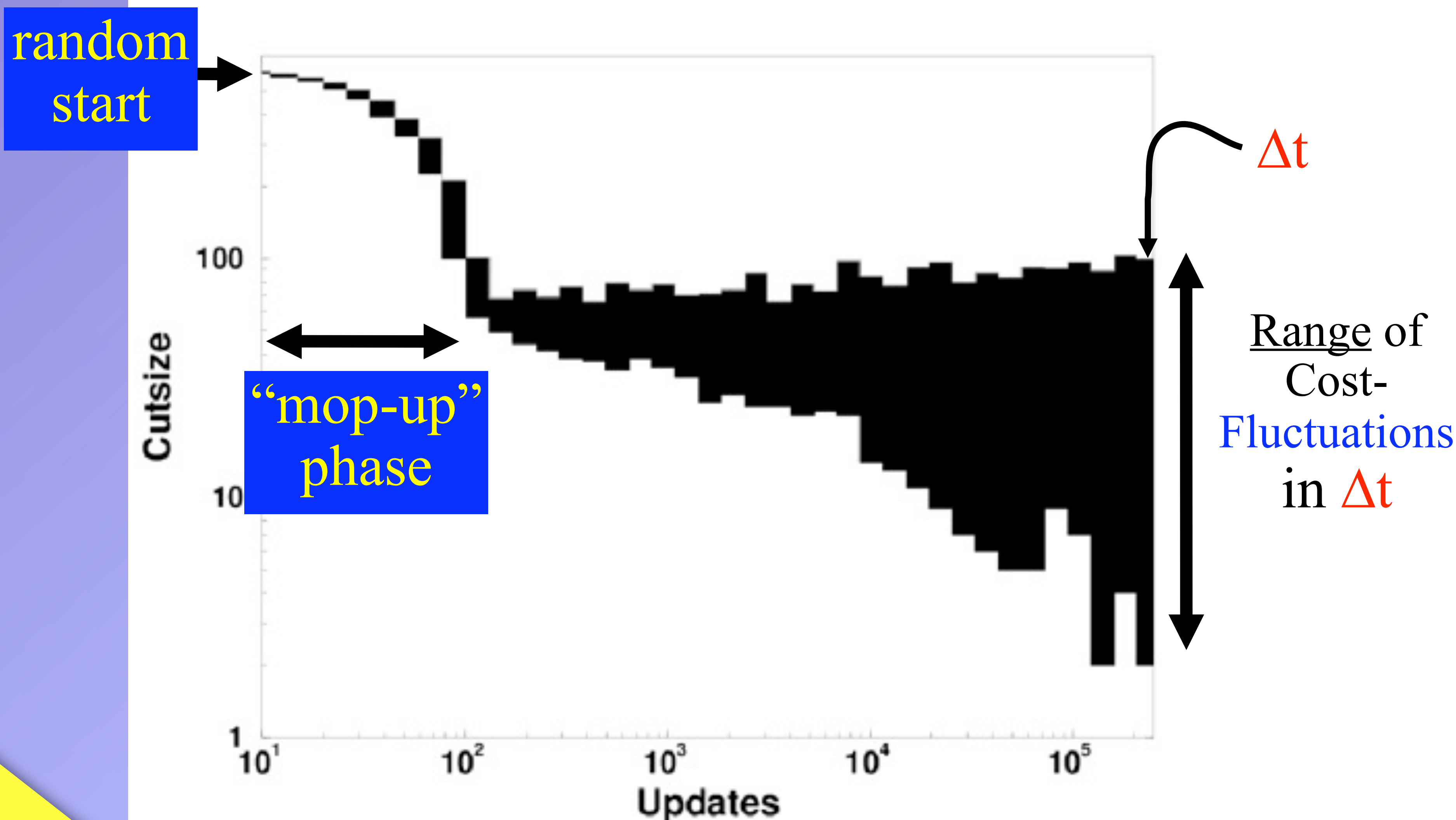

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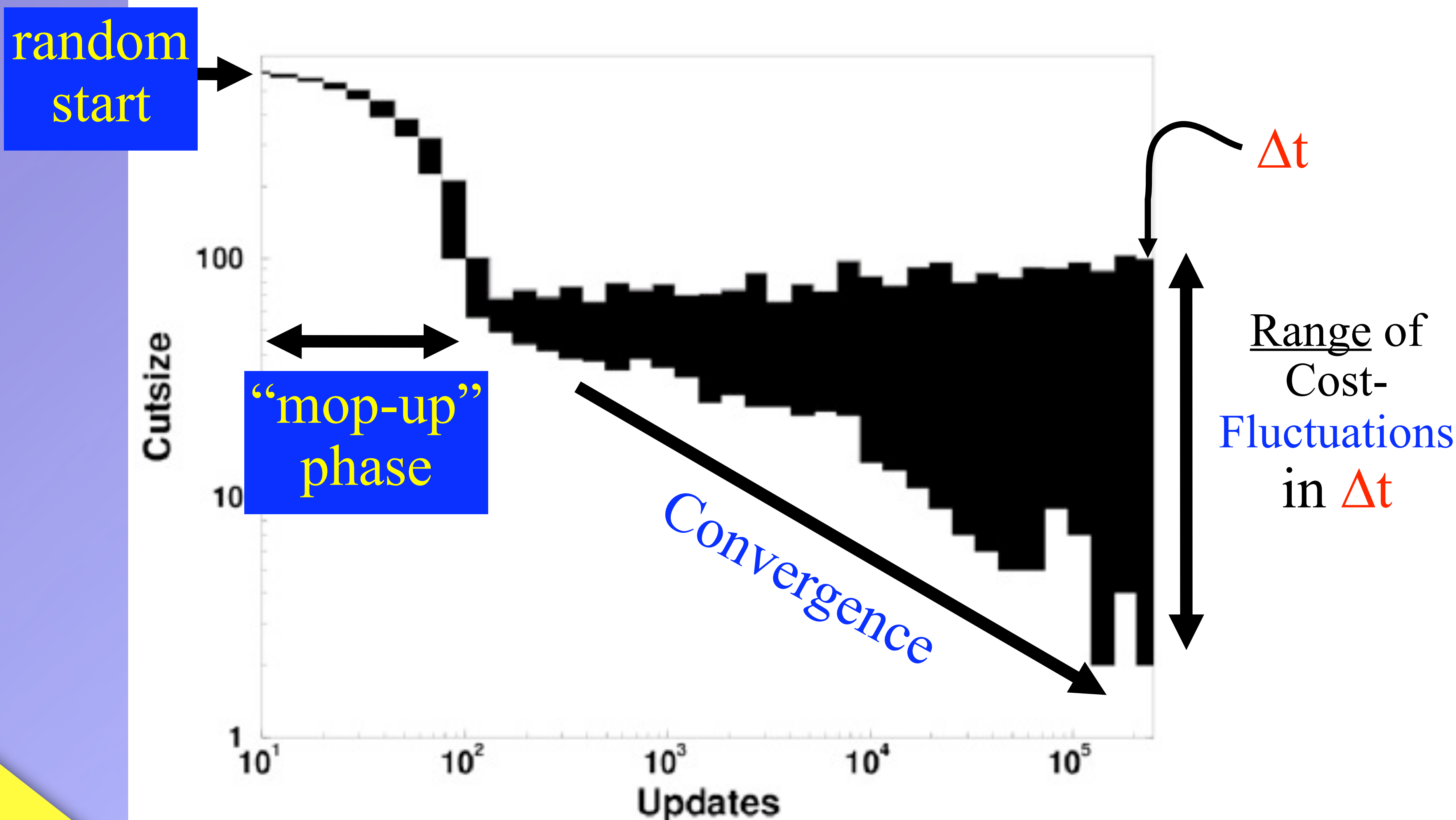

τ-EO - Searching at the "Ergodic Edge":

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For Ranks $\lambda_{\Pi(1)} \leq \ldots \leq \lambda_{\Pi(n)}$, update $i = \Pi(k)$ with

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For Ranks $\lambda_{\Pi(1)} \leq \ldots \leq \lambda_{\Pi(n)}$, update $i=\Pi(k)$ with scale-free, power-law distribution

$$P(k) \propto k^{-\tau}$$

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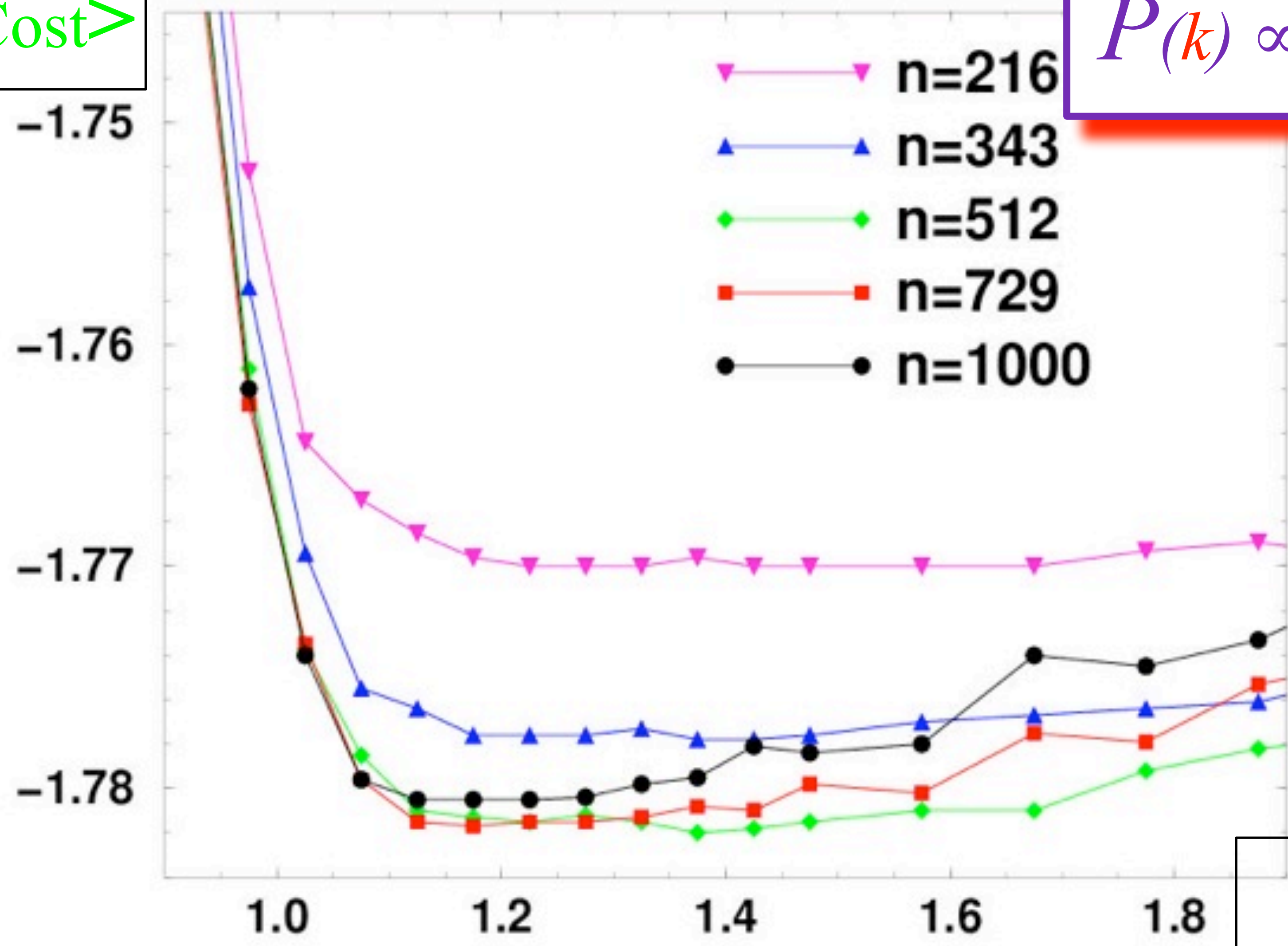
<Cost>



τ

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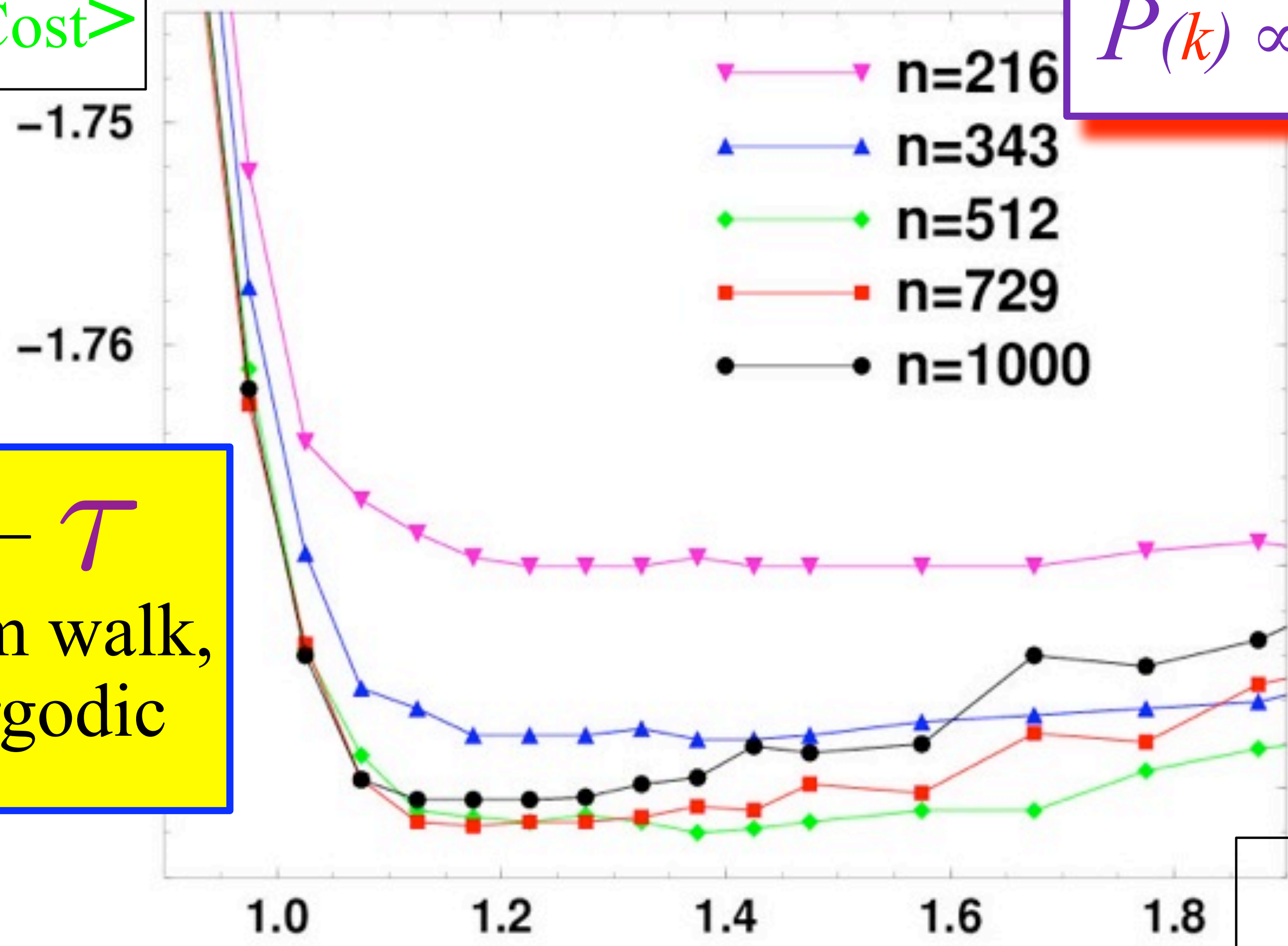

τ

$0 \leftarrow \tau$
random walk,
too ergodic

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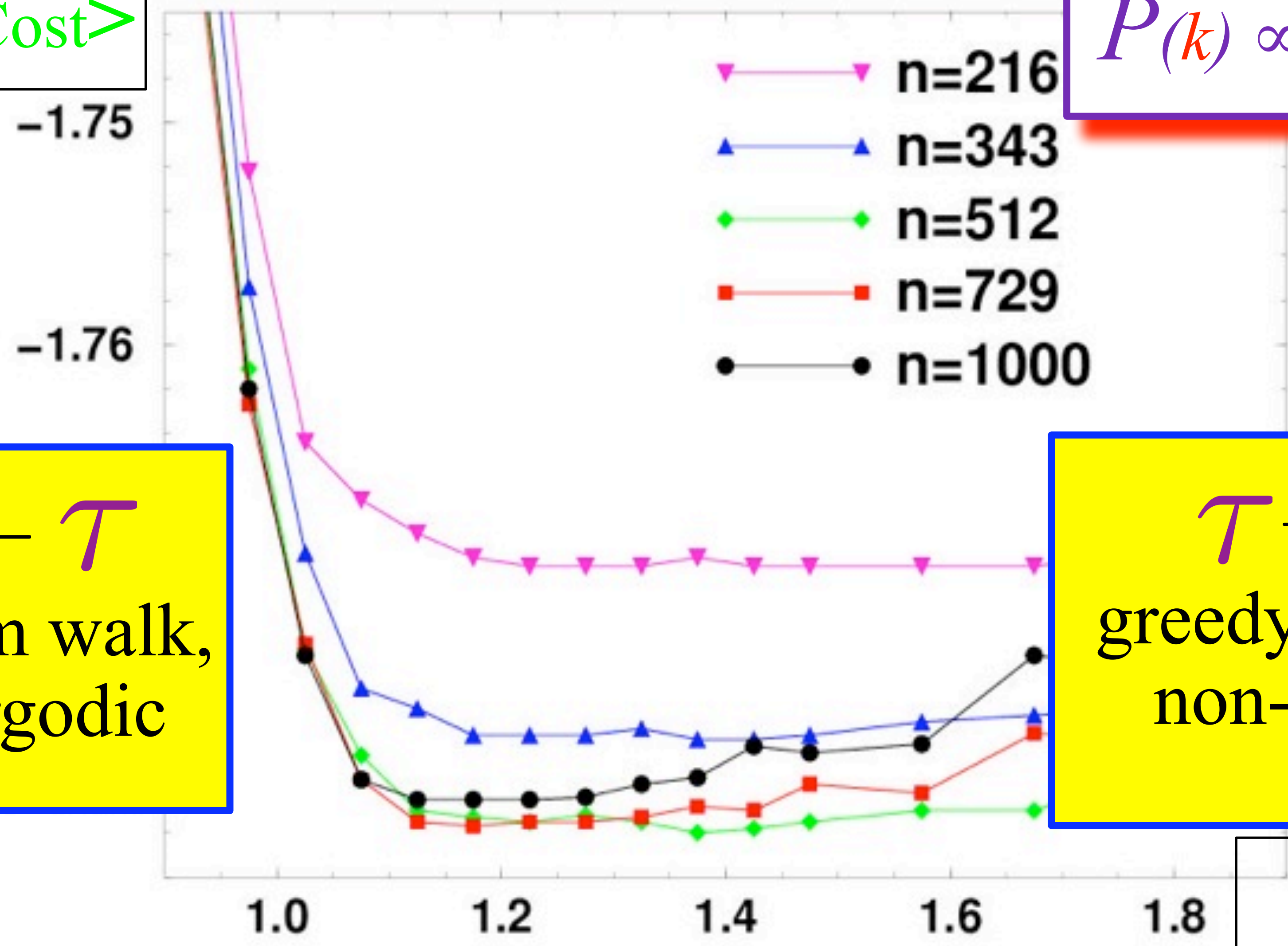

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$\tau \rightarrow \infty$
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τ

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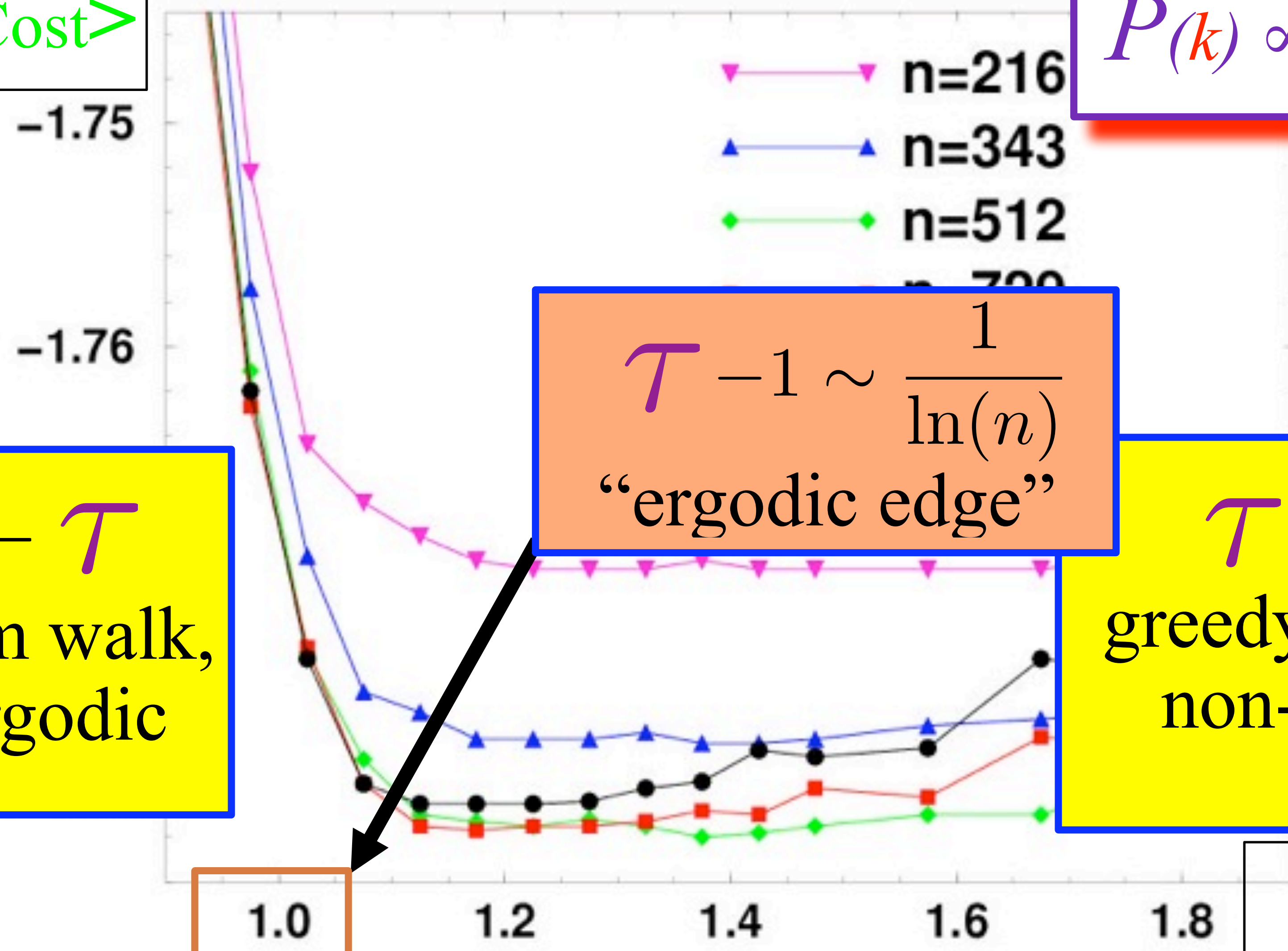

τ

$$\tau - 1 \sim \frac{1}{\ln(n)}$$

"ergodic edge"

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random walk,
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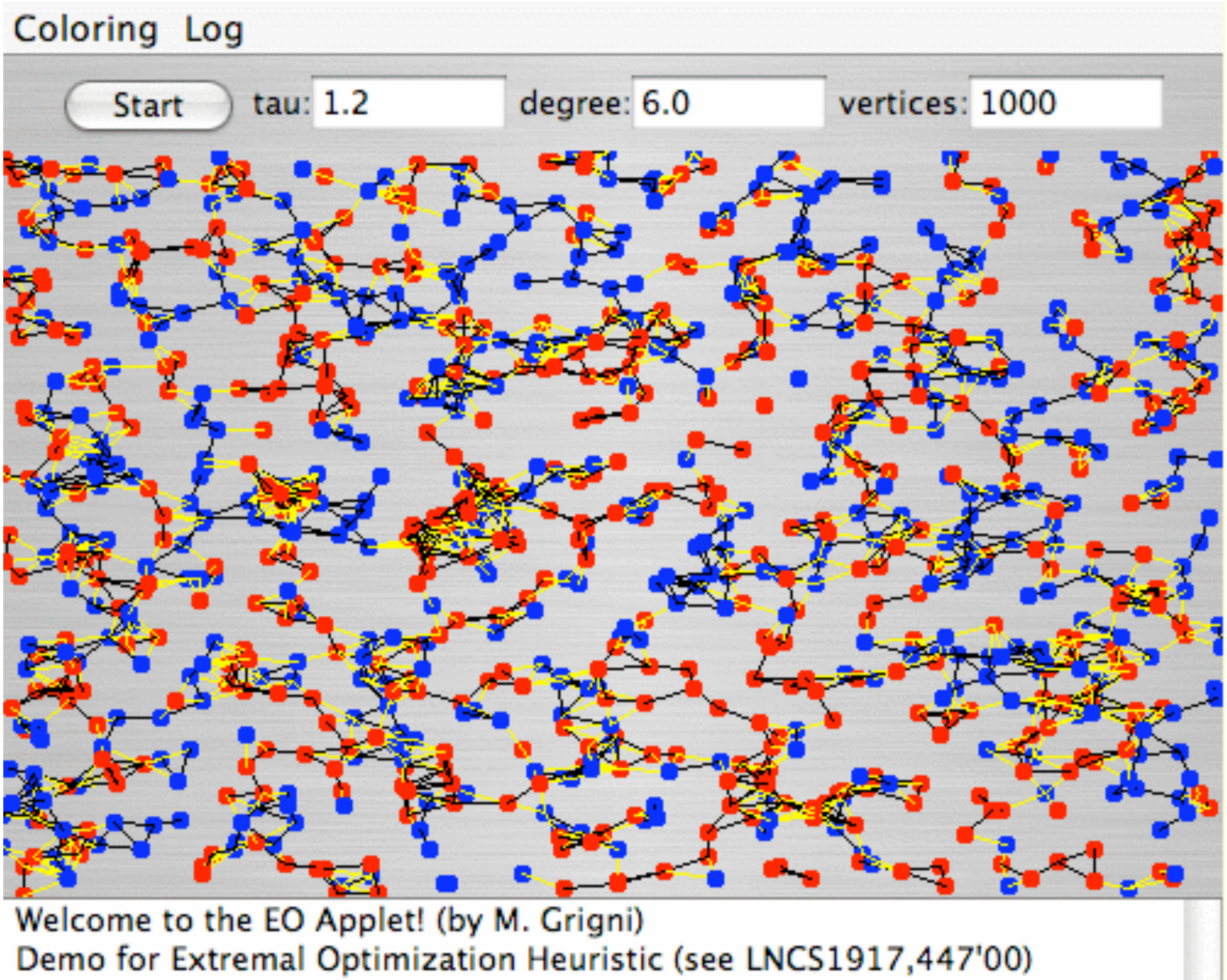
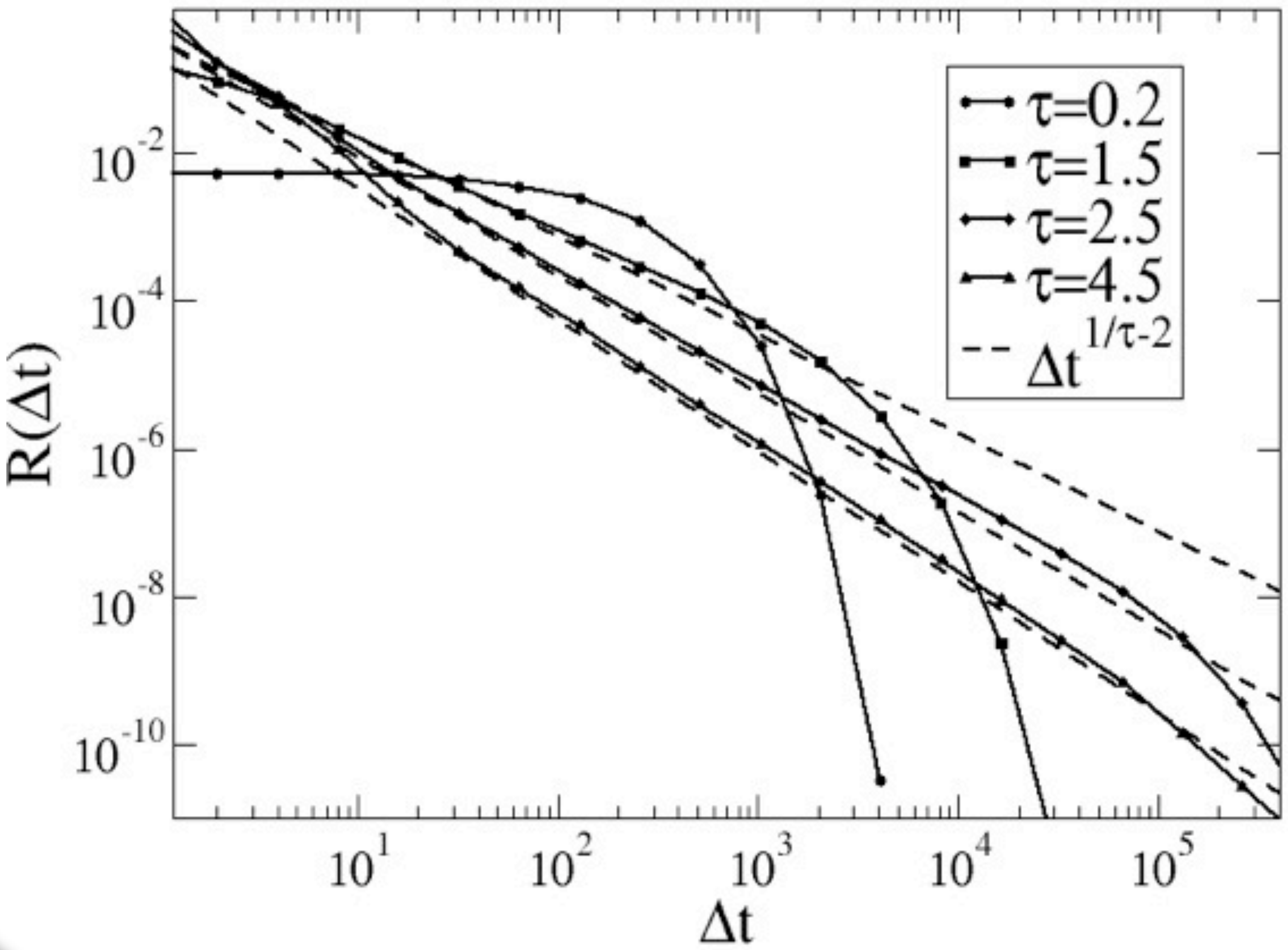
Animation of τ-EO for Graph-Partitioning



Dynamics of τ-EO: (for ±J-Spin Glass on 3-reg. Graph, N=256)

First-return time distribution R(Δt):



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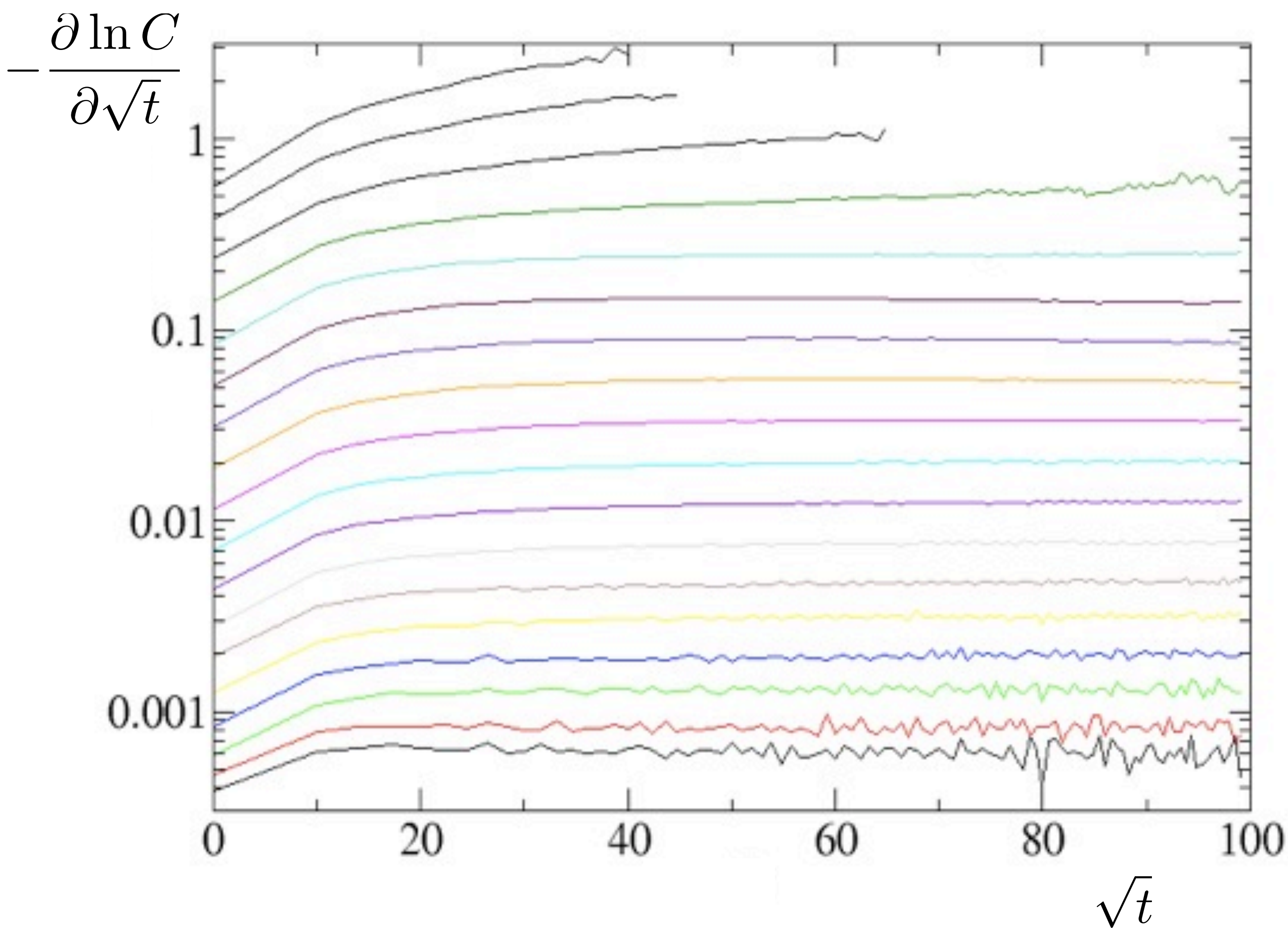

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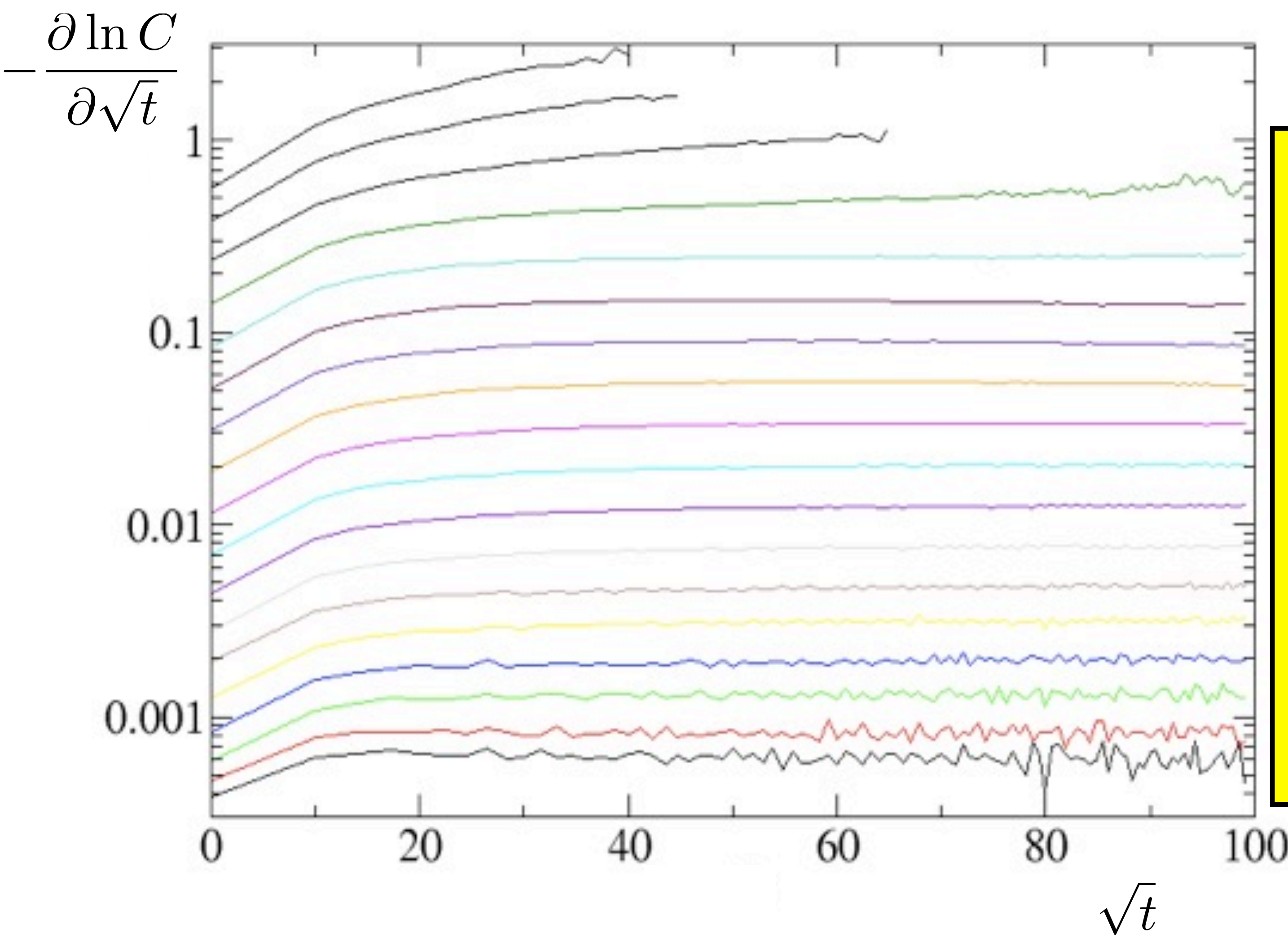

τ = 1.1

$$B_\tau \sim -\frac{\partial \ln C(t)}{\partial \sqrt{t}} \sim 1.6\, e^{-2.4\tau}$$

τ = 3.9

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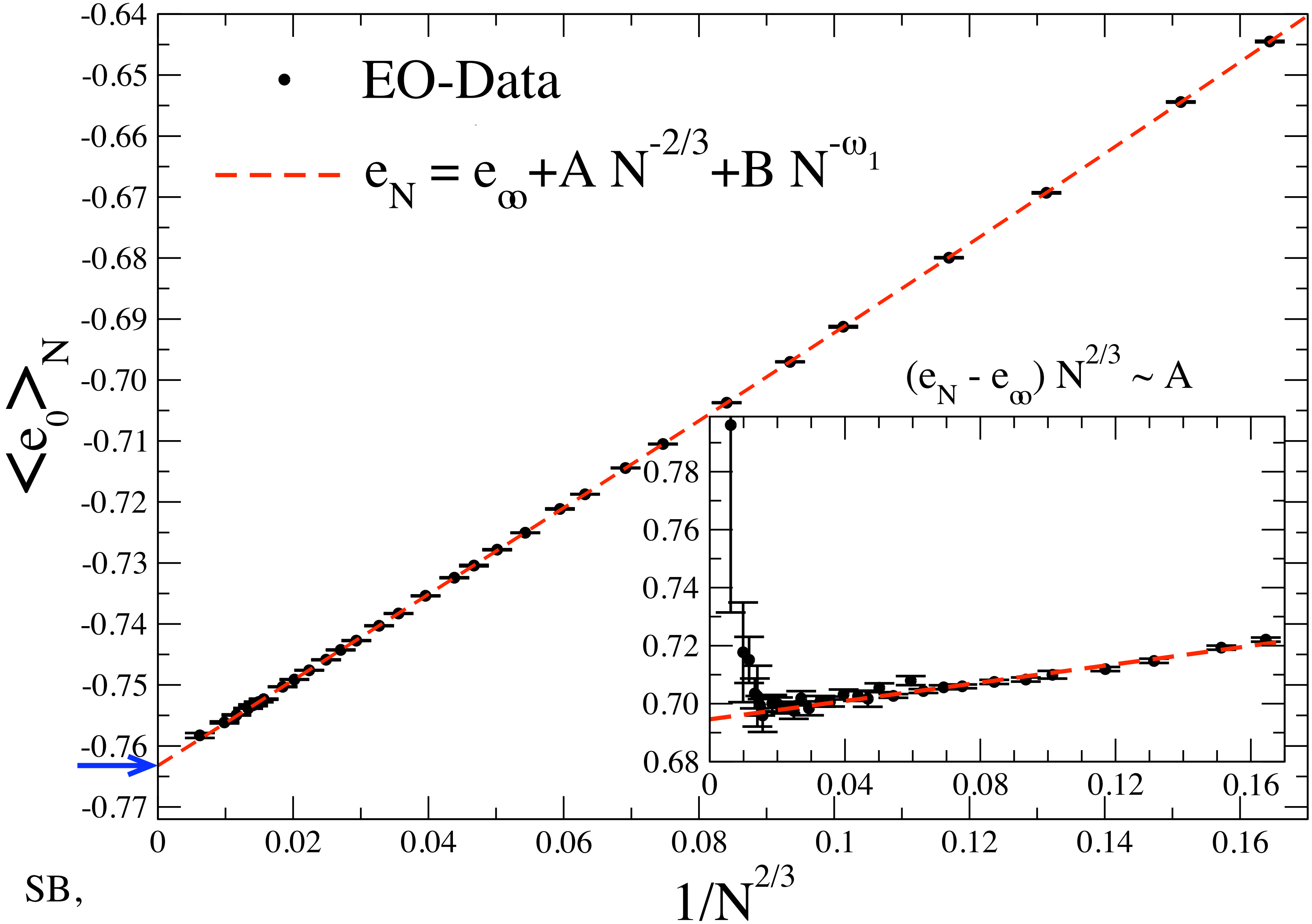
τ-EO for Sherrington-Kirkpatrick

- Mean-Field ($d\to\infty$) Spin Glasses:



SB, *Euro Phys J* B **46**, 501 (2005)

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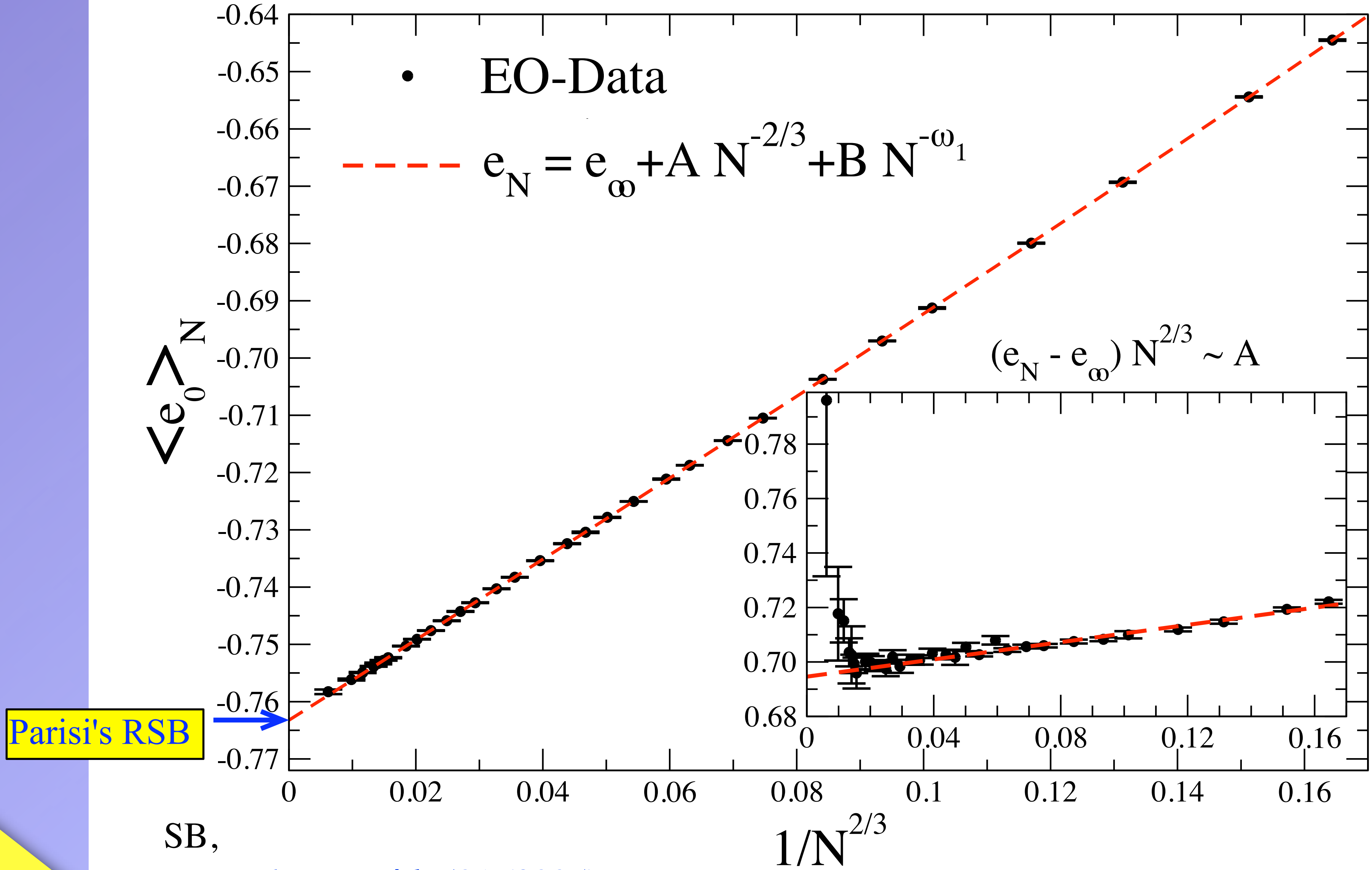

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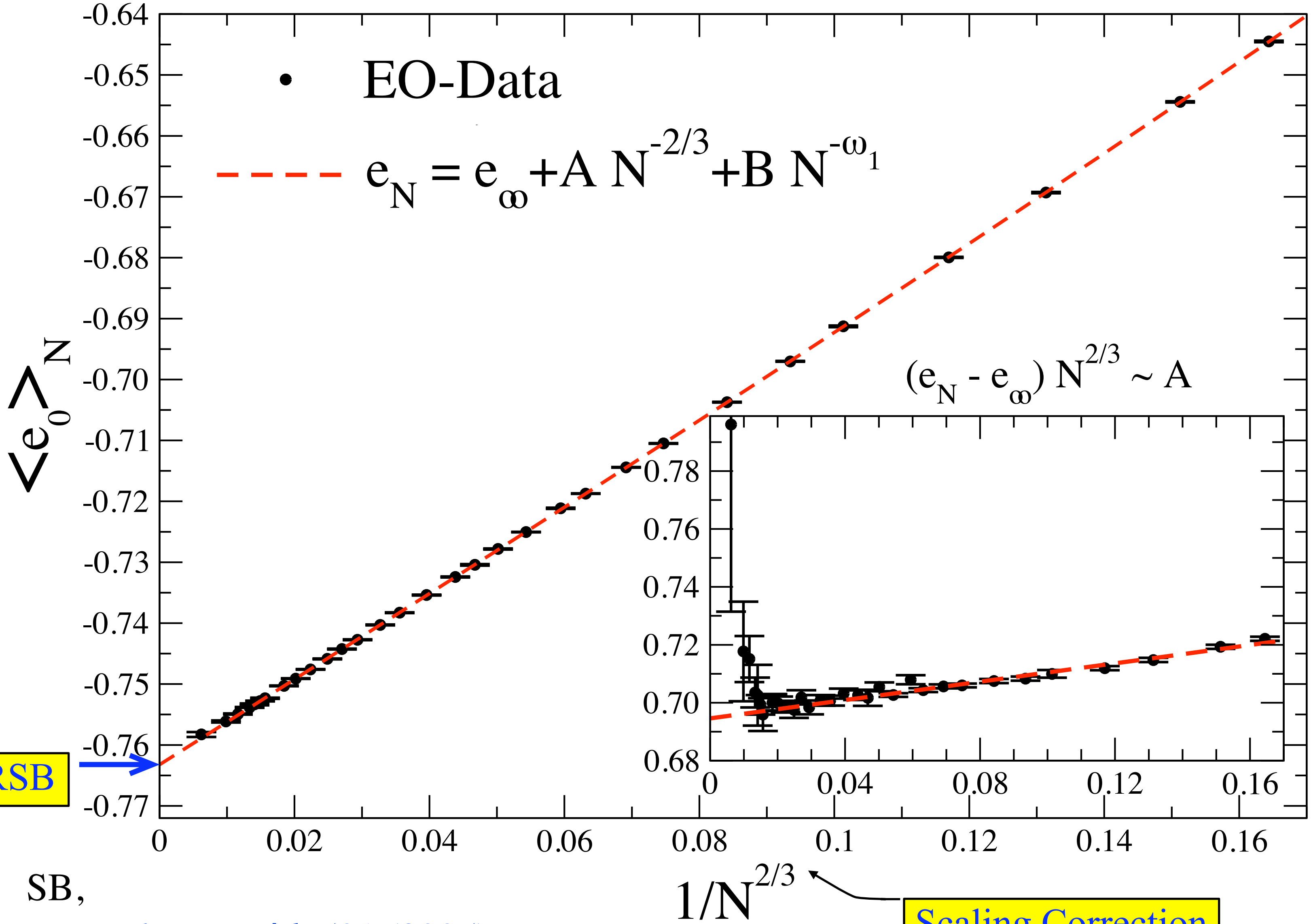

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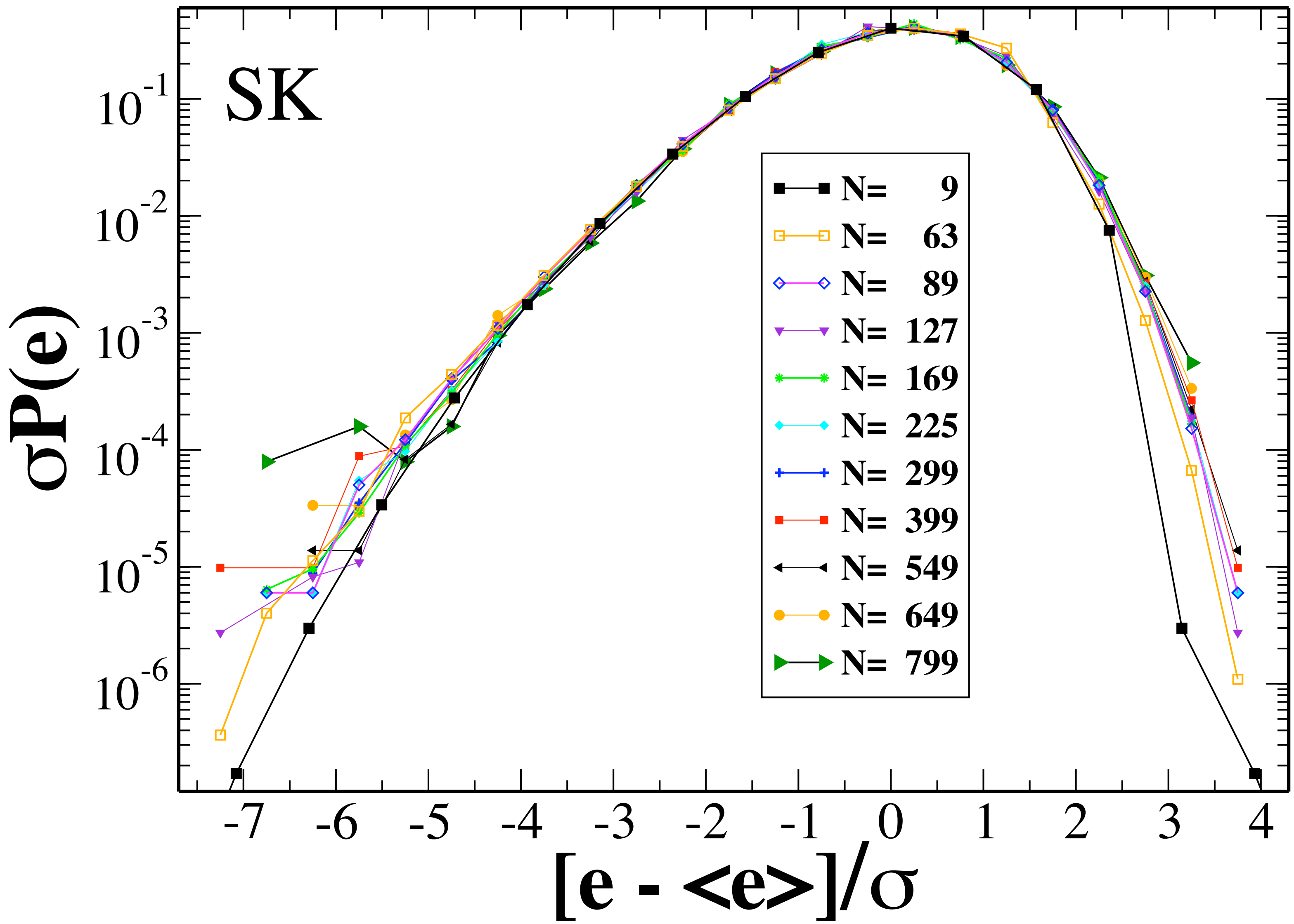

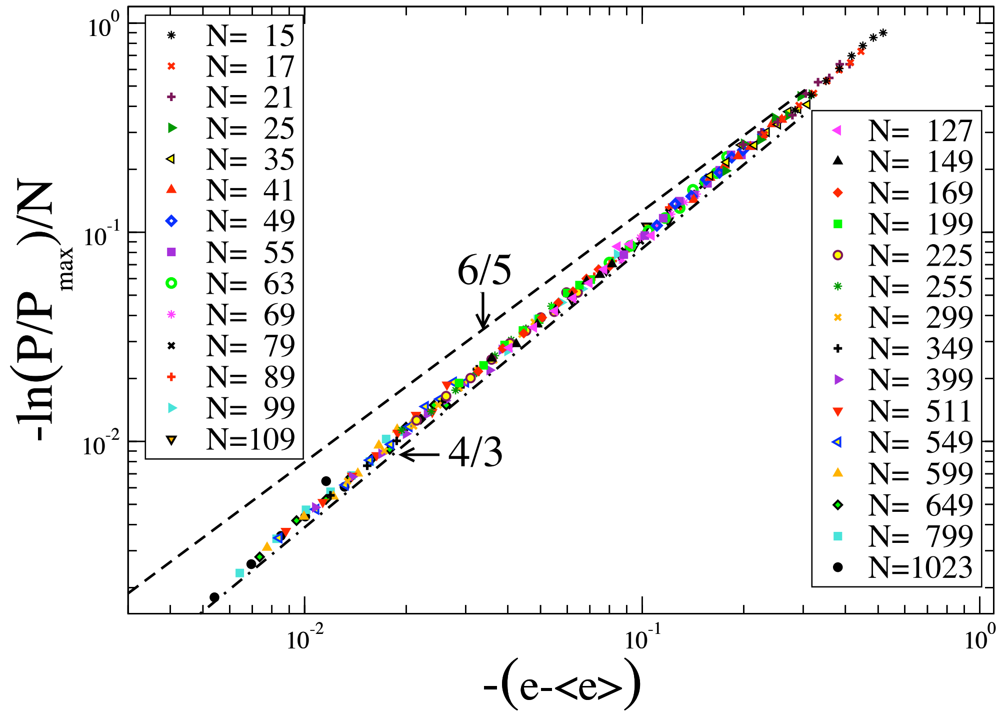

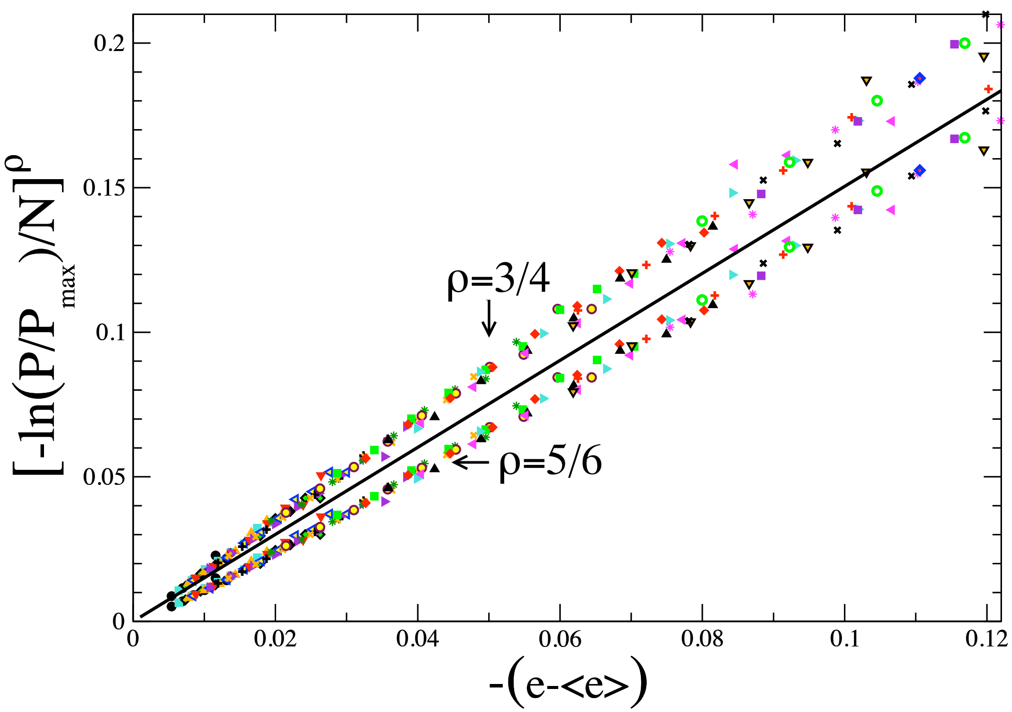

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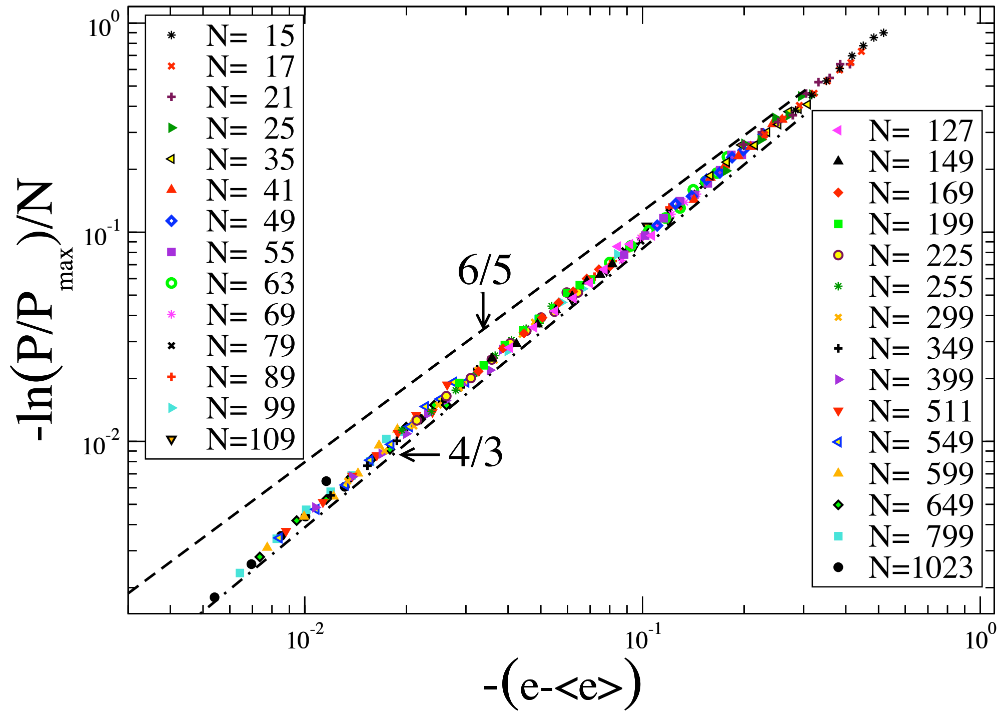

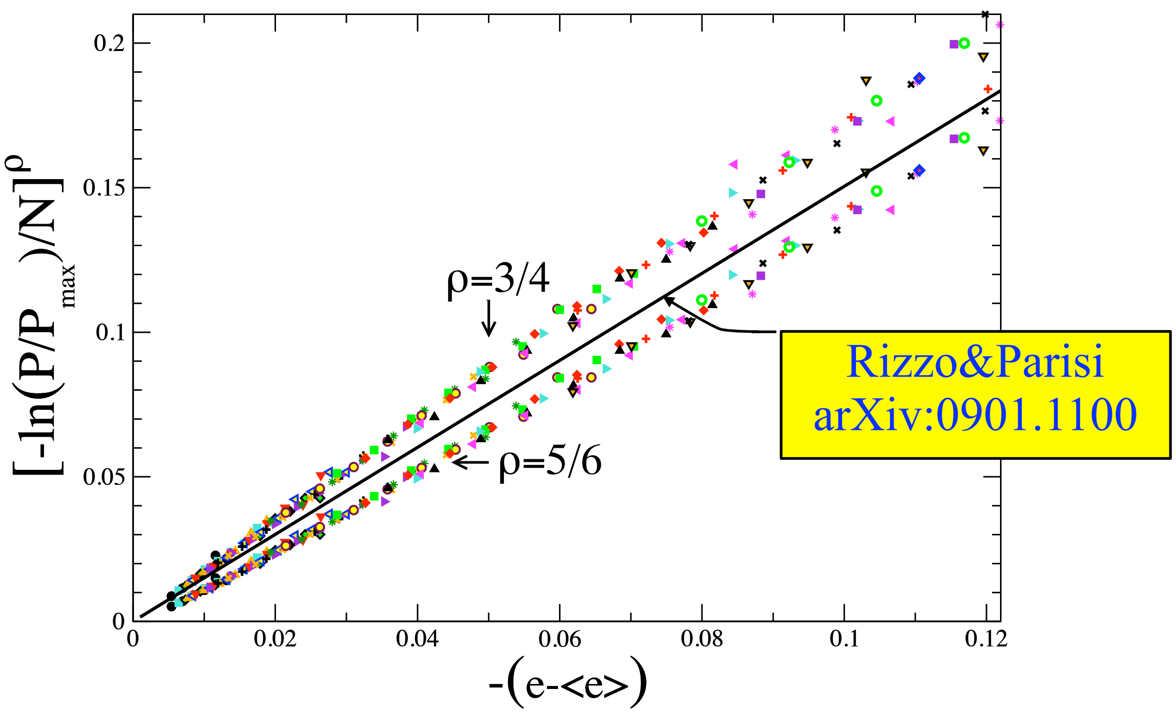

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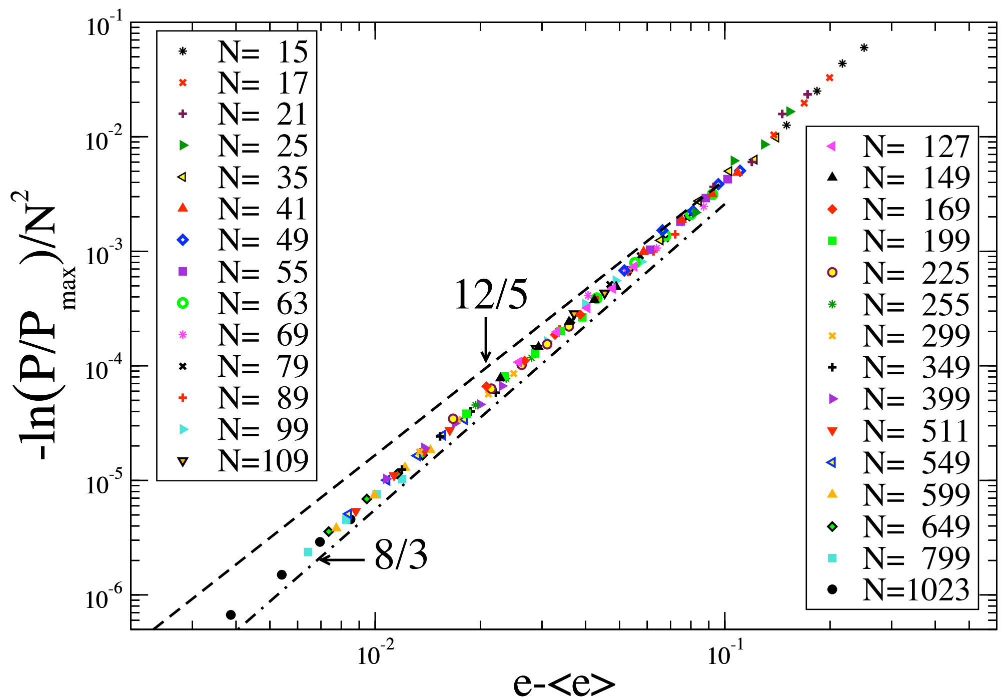
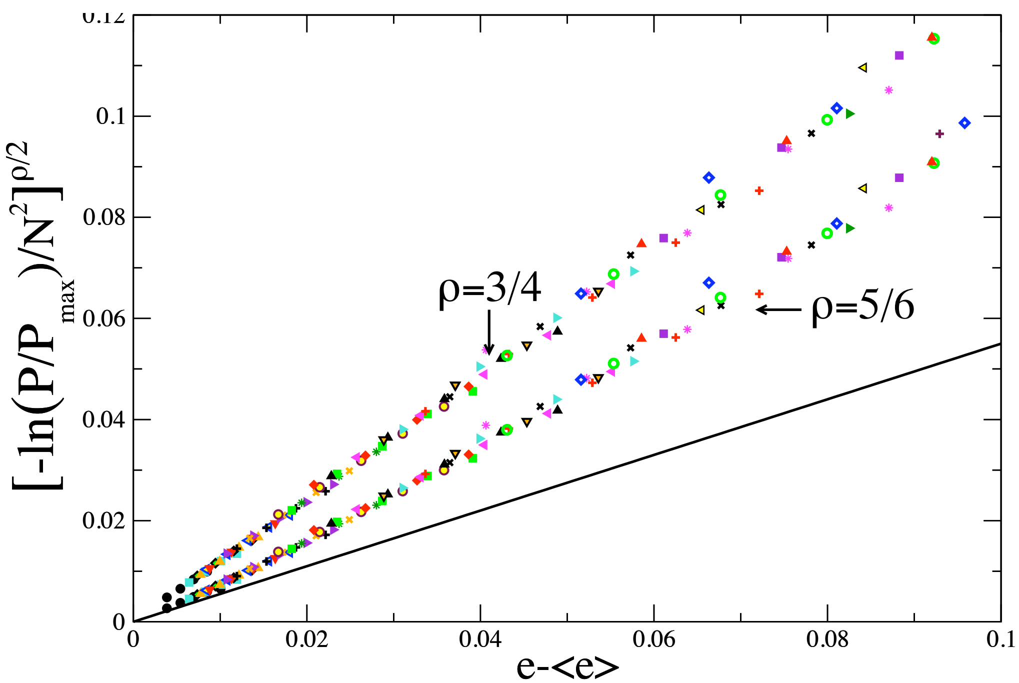

N= 15
N= 17
N= 21
N= 25
N= 35
N= 41
N= 49
N= 55
N= 63
N= 69
N= 79
N= 89
N= 99
N=109
N= 127
N= 149
N= 169
N= 199
N= 225
N= 255
N= 299
N= 349
N= 399
N= 511
N= 549
N= 599
N= 649
N= 799
N=1023
12/5
8/3
-ln(P/P_max)/N^2
e-<e>


ρ=3/4
ρ=5/6
[-ln(P/P_max)/N^2]^ρ/2
e-<e>

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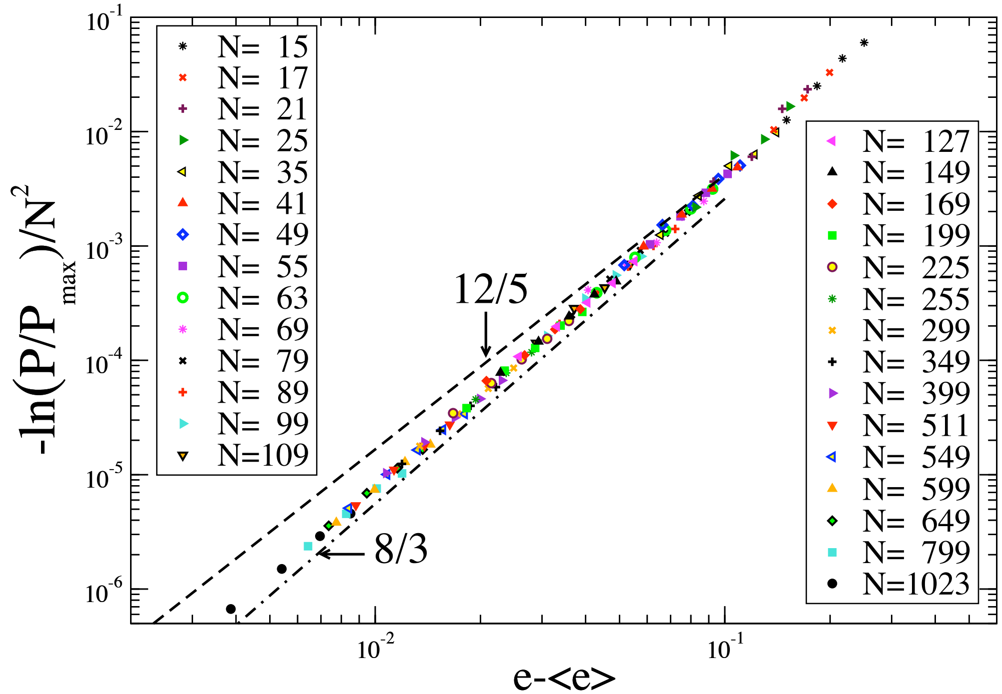

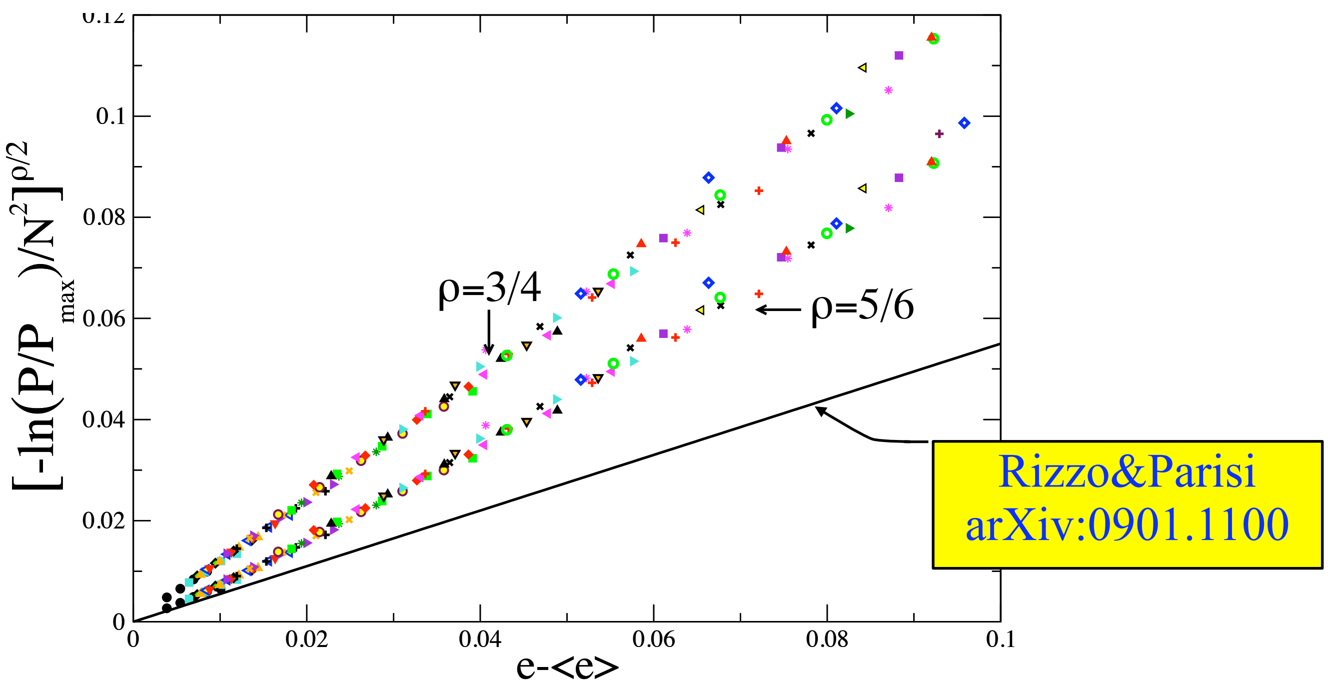

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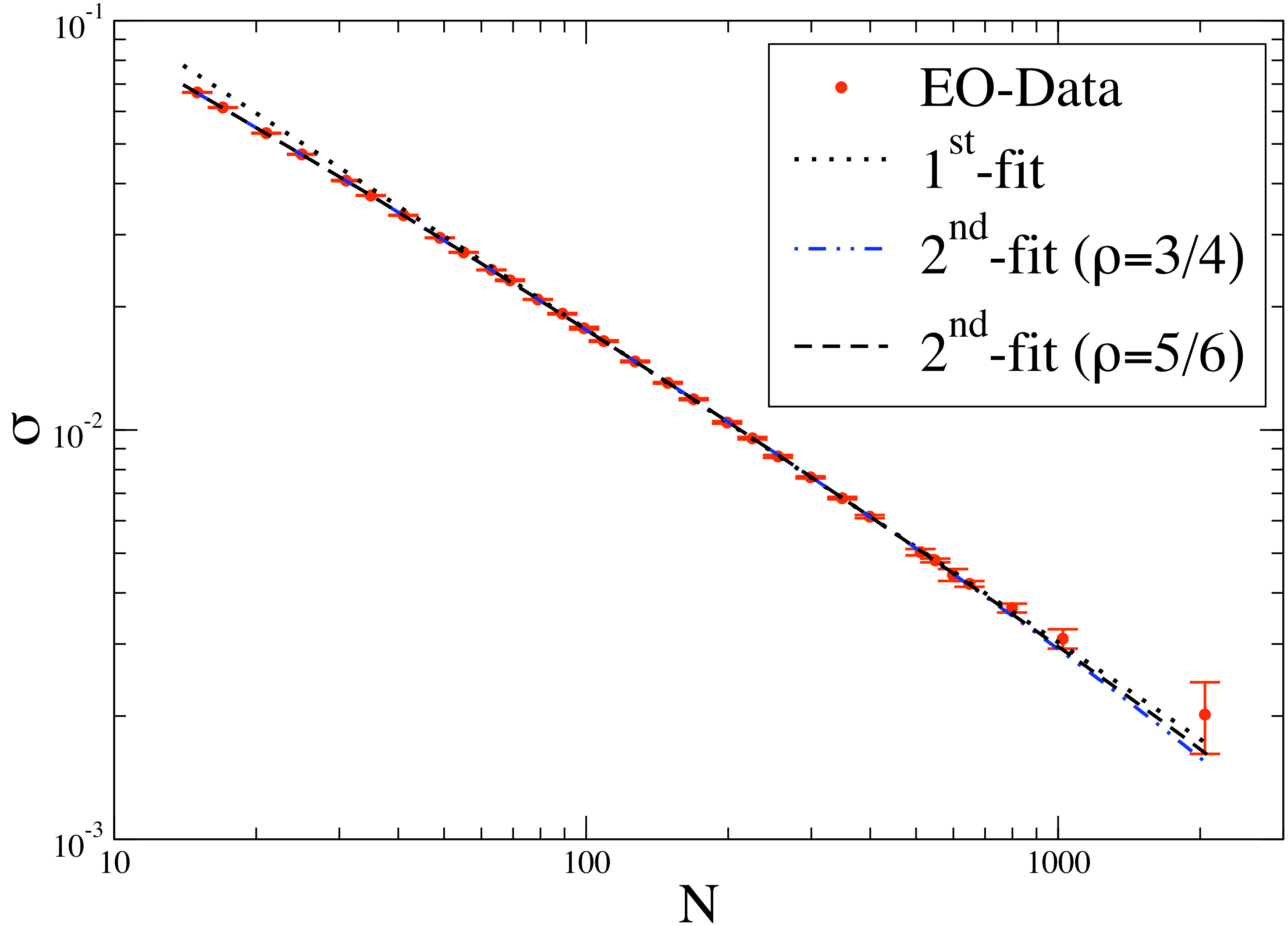
Fluctuation Exponent ρ



τ-EO for Sherrington-Kirkpatrick

- "Width" σ of the GS-Energy:

$$\sigma = \sqrt{\langle e_0^2\rangle - \langle e_0\rangle^2},$$

$$\sim A\,\frac{1}{N^{\rho}} + B\,\frac{1}{N^{\alpha}}, \qquad (\alpha > \rho),$$

$$\ln\sigma \sim -\rho\ln(N) + \ln(A) + \ln\left(1 + \frac{B}{A}N^{\rho-\alpha}\right),$$

$$-\frac{\ln\sigma}{\ln N} \sim \rho + a\,x + b\,x\,\exp\left[\frac{\rho-\alpha}{x}\right],$$

$$\left(x = \frac{1}{\ln N} \to 0\right).$$

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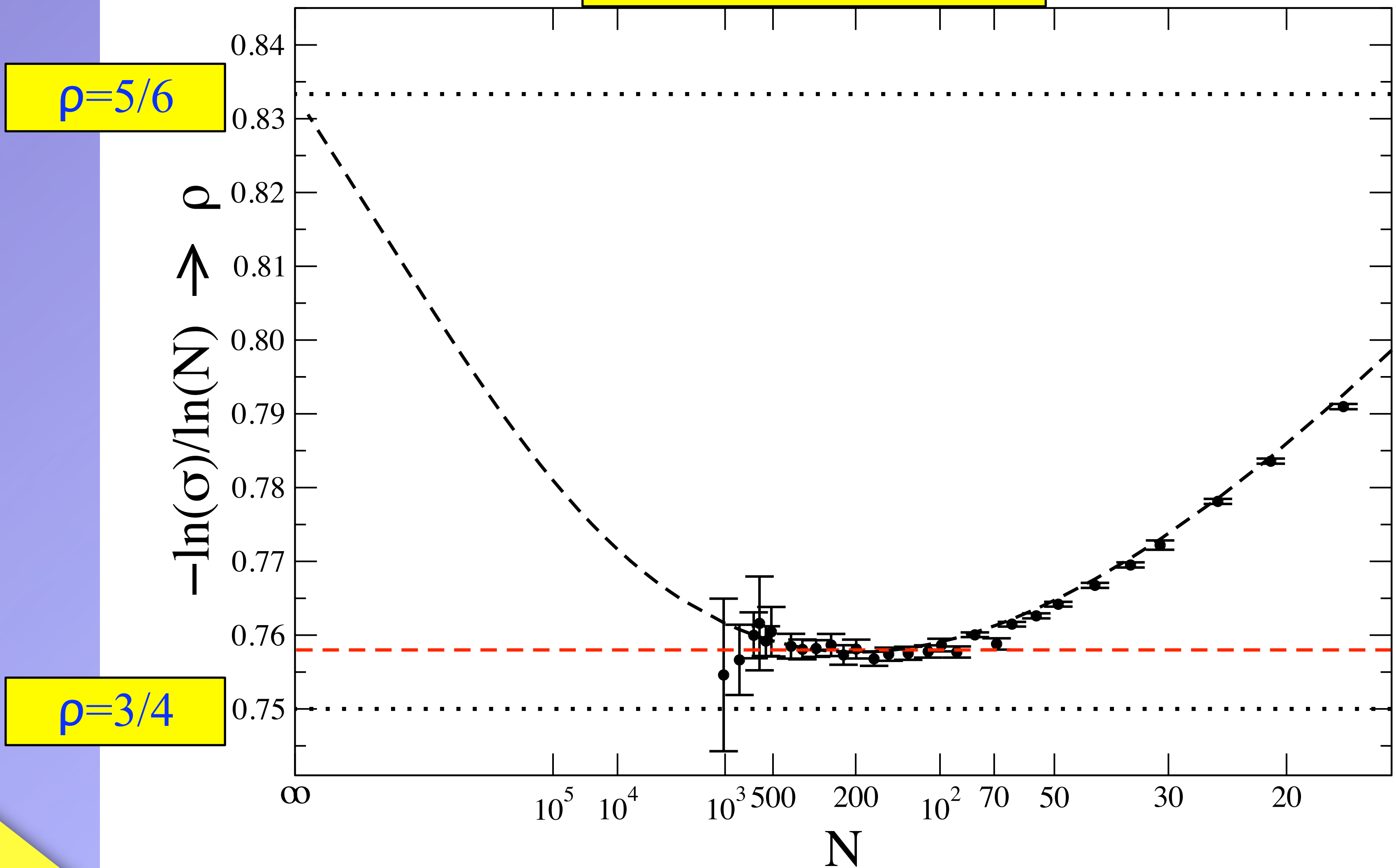
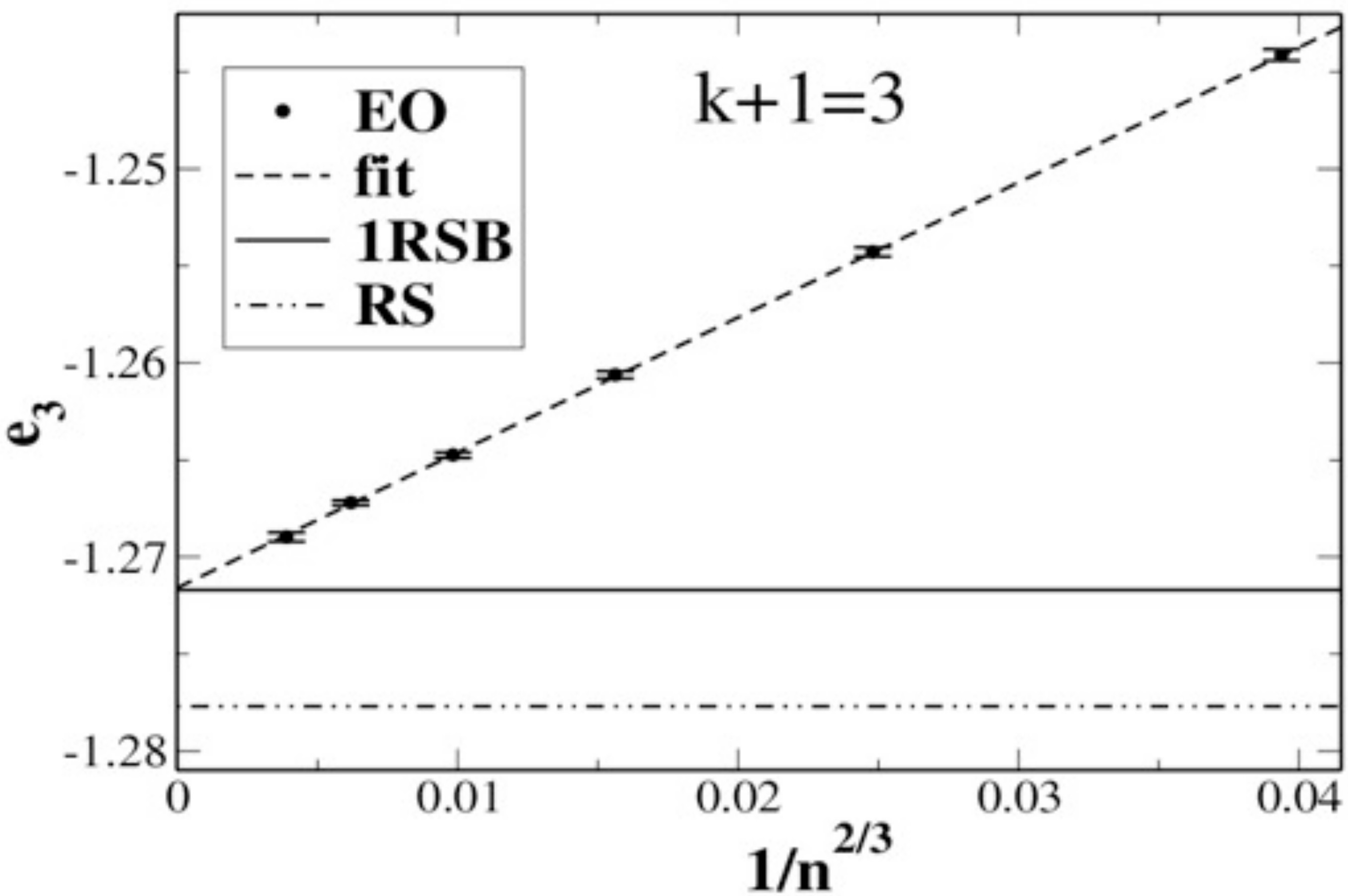


τ-EO for Bethe Lattices:

EO for 3-connected Bethe Lattice Glass w/ Replica Sym. Breaking:



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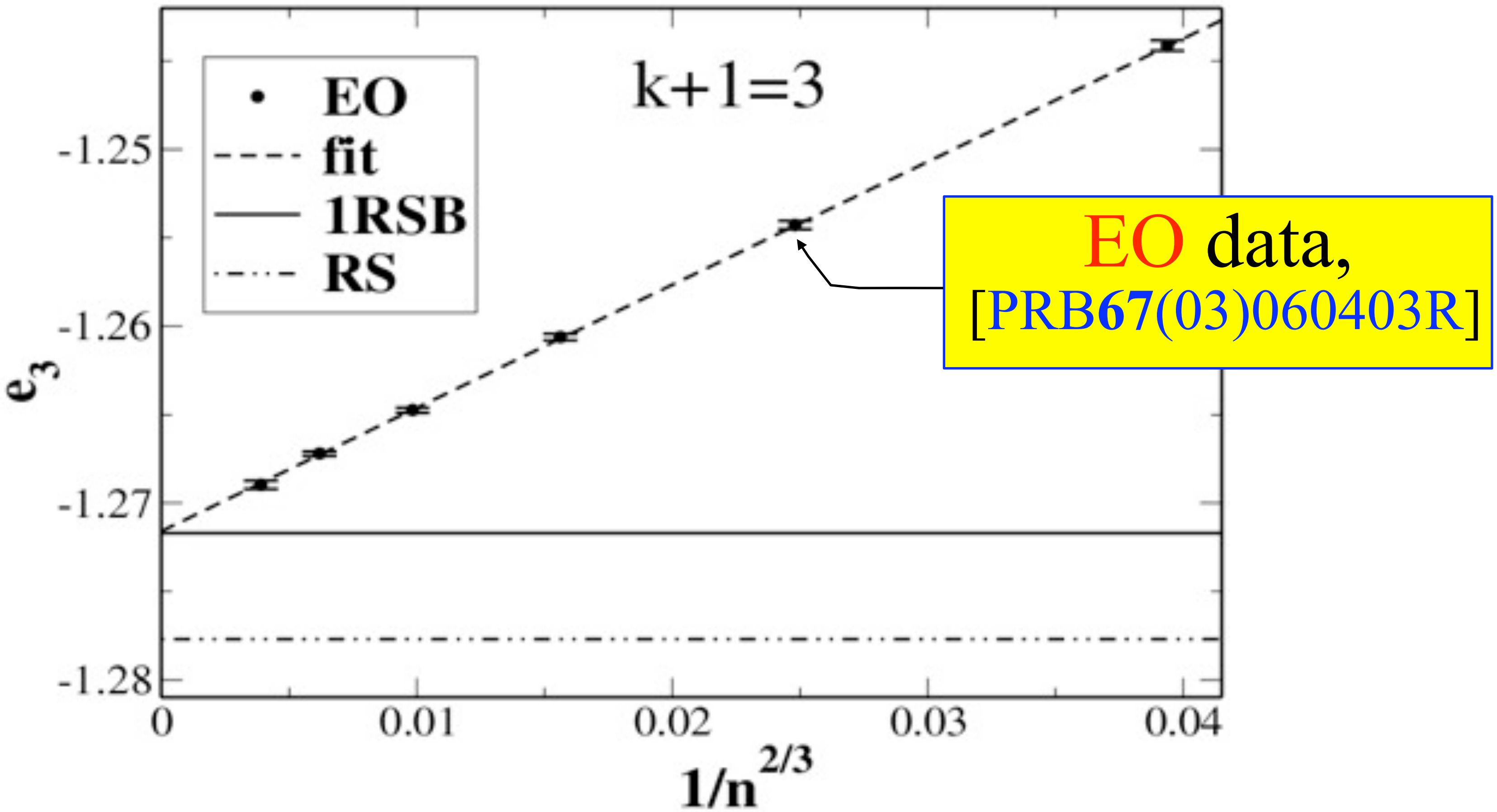

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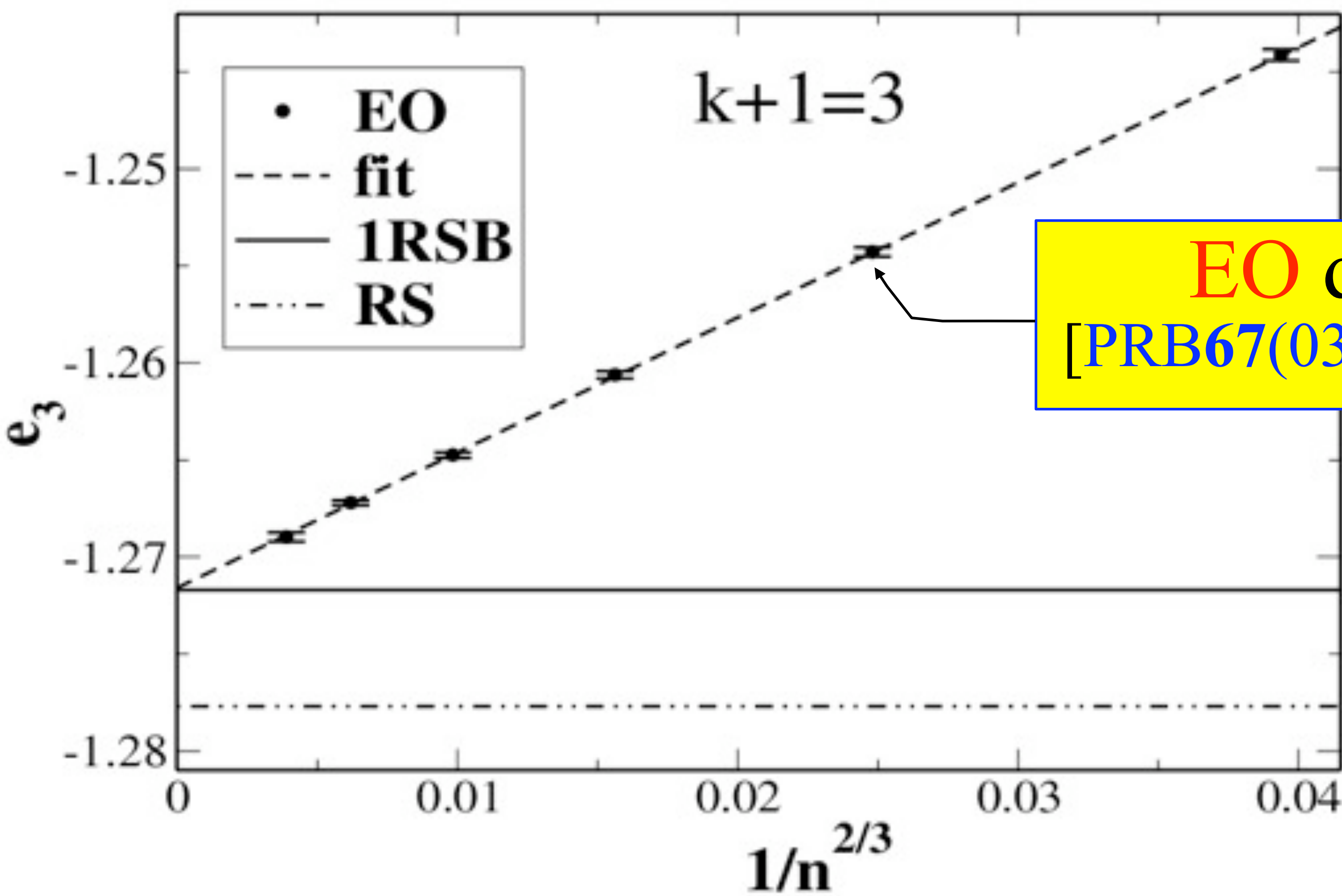

EO data,
[PRB**67**(03)060403R]

Replica Theory:
⇐ 1RSB,
⇐ no RSB,
[J.Stat.Phys.**111**(03)1]

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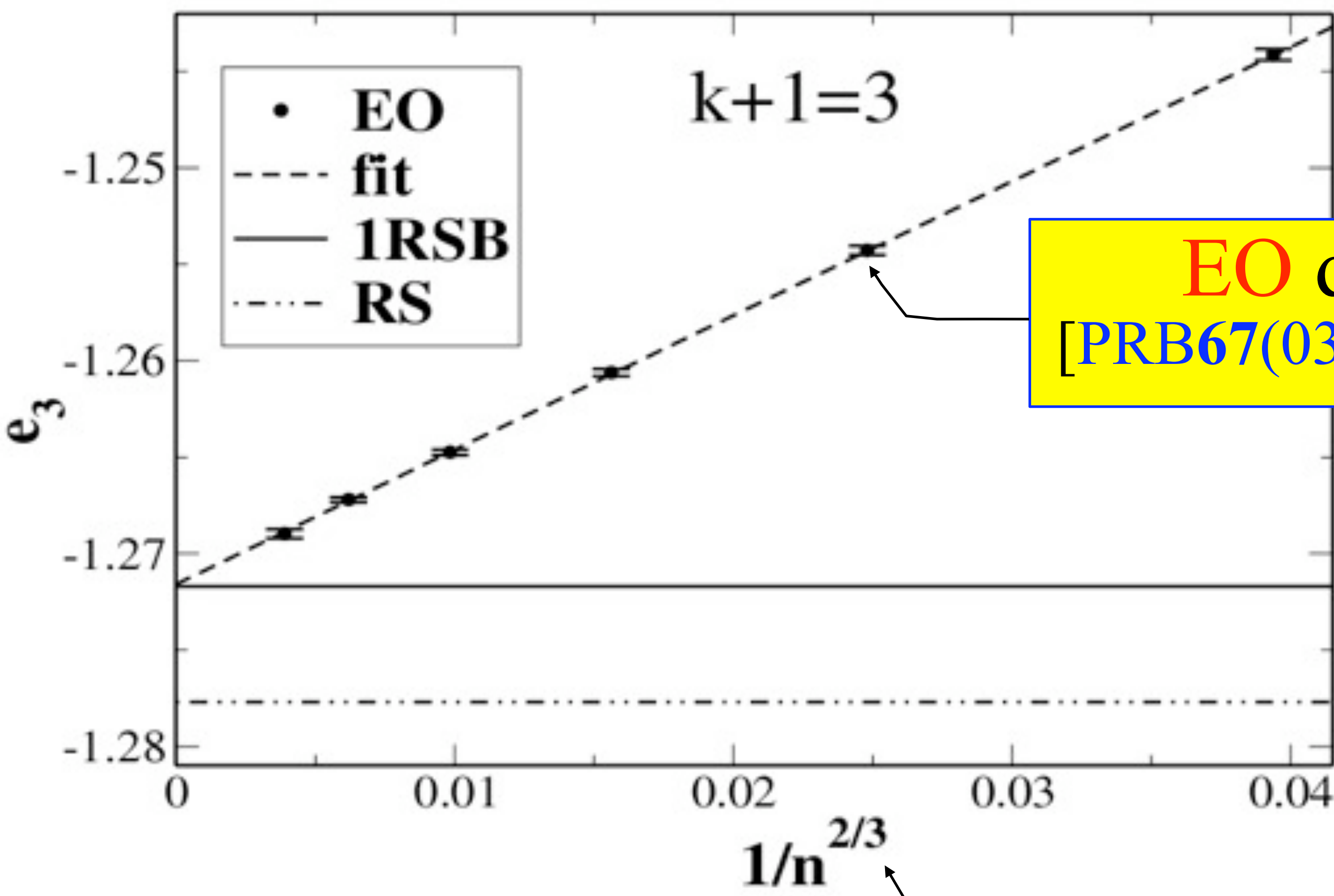

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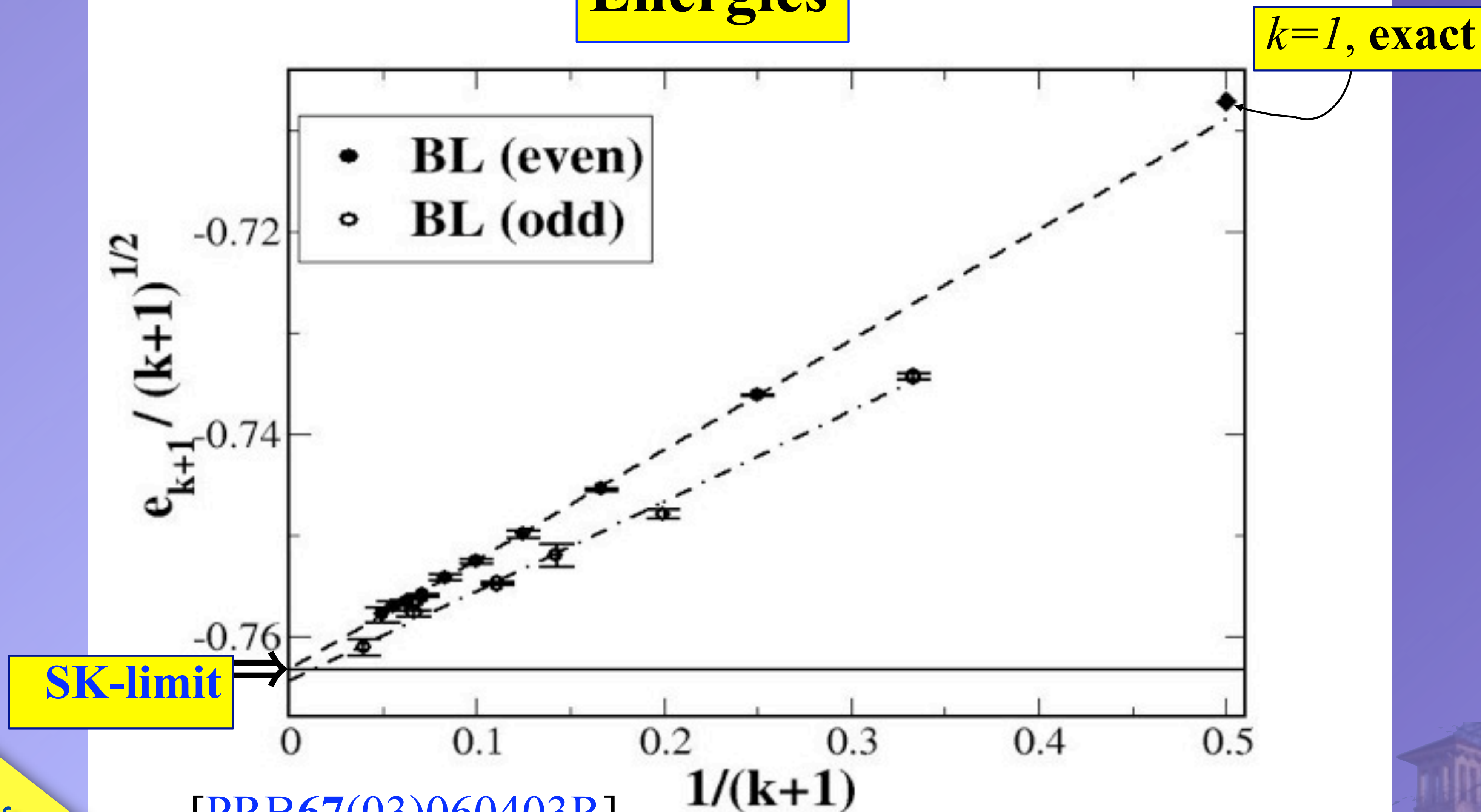
⇐ 1RSB,
⇐ no RSB,
[J.Stat.Phys.**111**(03)1]

Scaling corrections $\omega \approx 2/3$

τ-EO for Bethe Lattices:

EO for $(k+1)$-connected Bethe Lattice Glasses for $(k+1)\to\infty$:

Energies

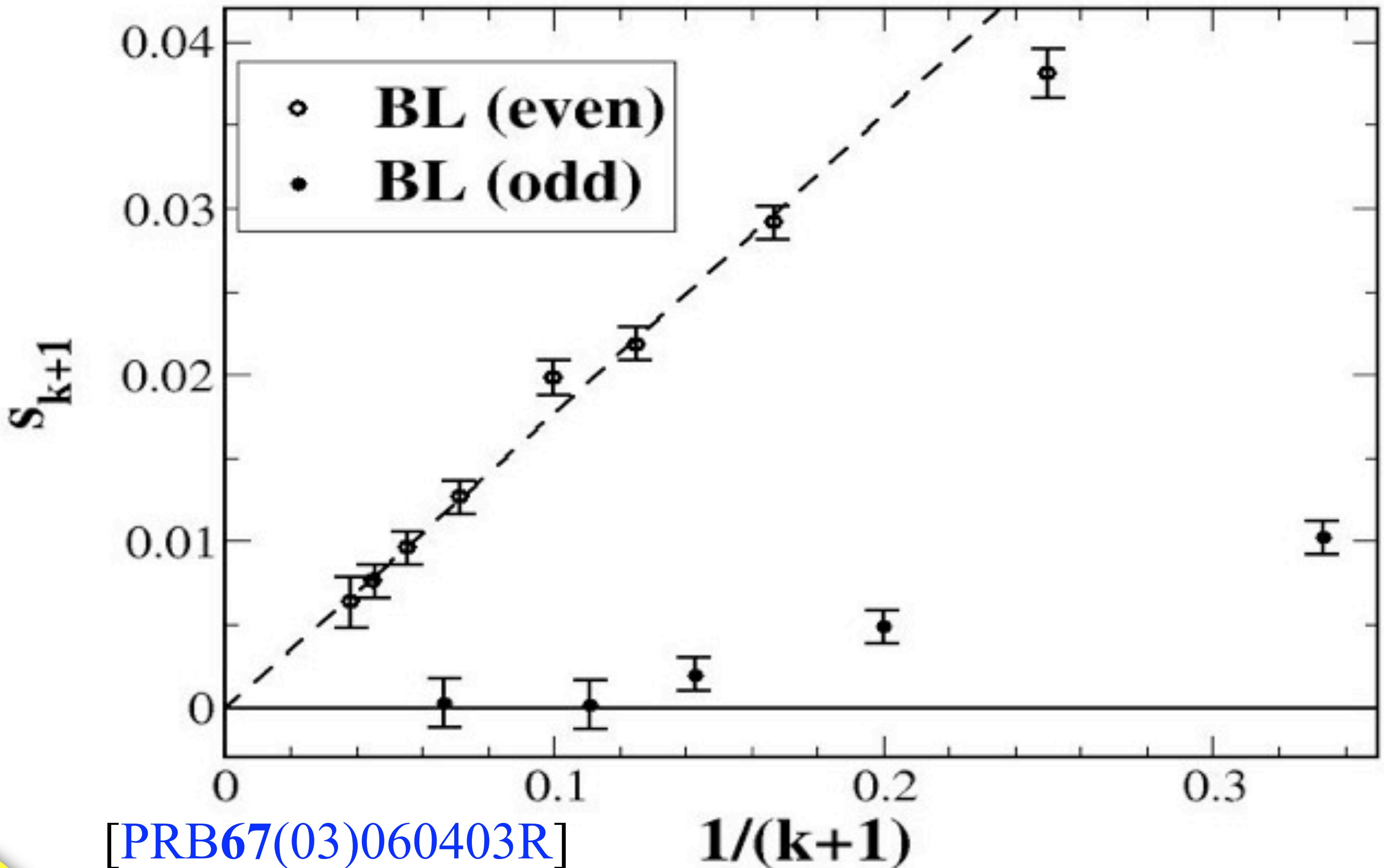


[PRB**67**(03)060403R]

τ-EO for Bethe Lattices:

EO for $(k+1)$-connected Bethe Lattice Glasses for $(k+1)\rightarrow\infty$:

Entropies

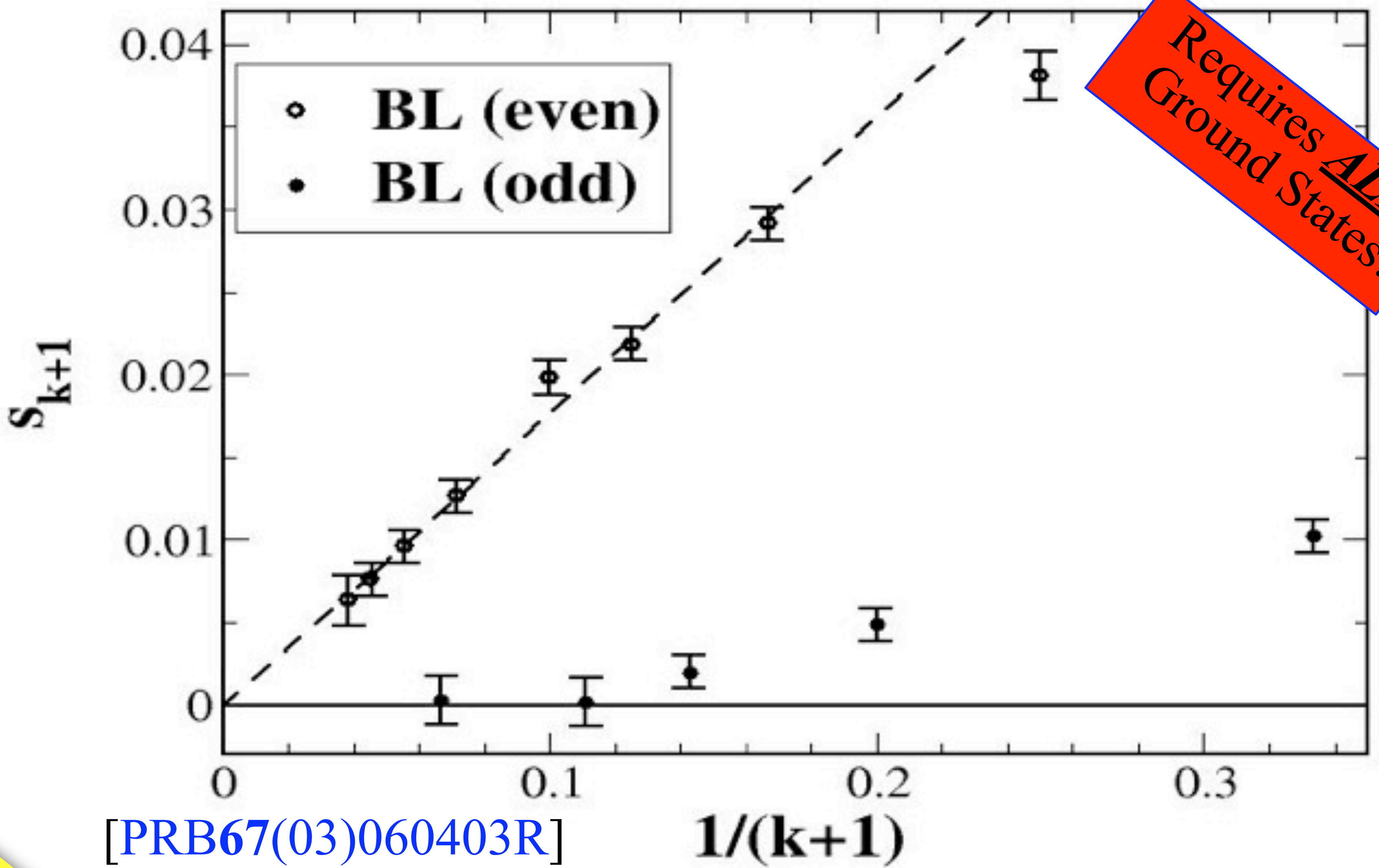


[PRB**67**(03)060403R]

τ-EO for Bethe Lattices:

EO for $(k+1)$-connected Bethe Lattice Glasses for $(k+1)\to\infty$:

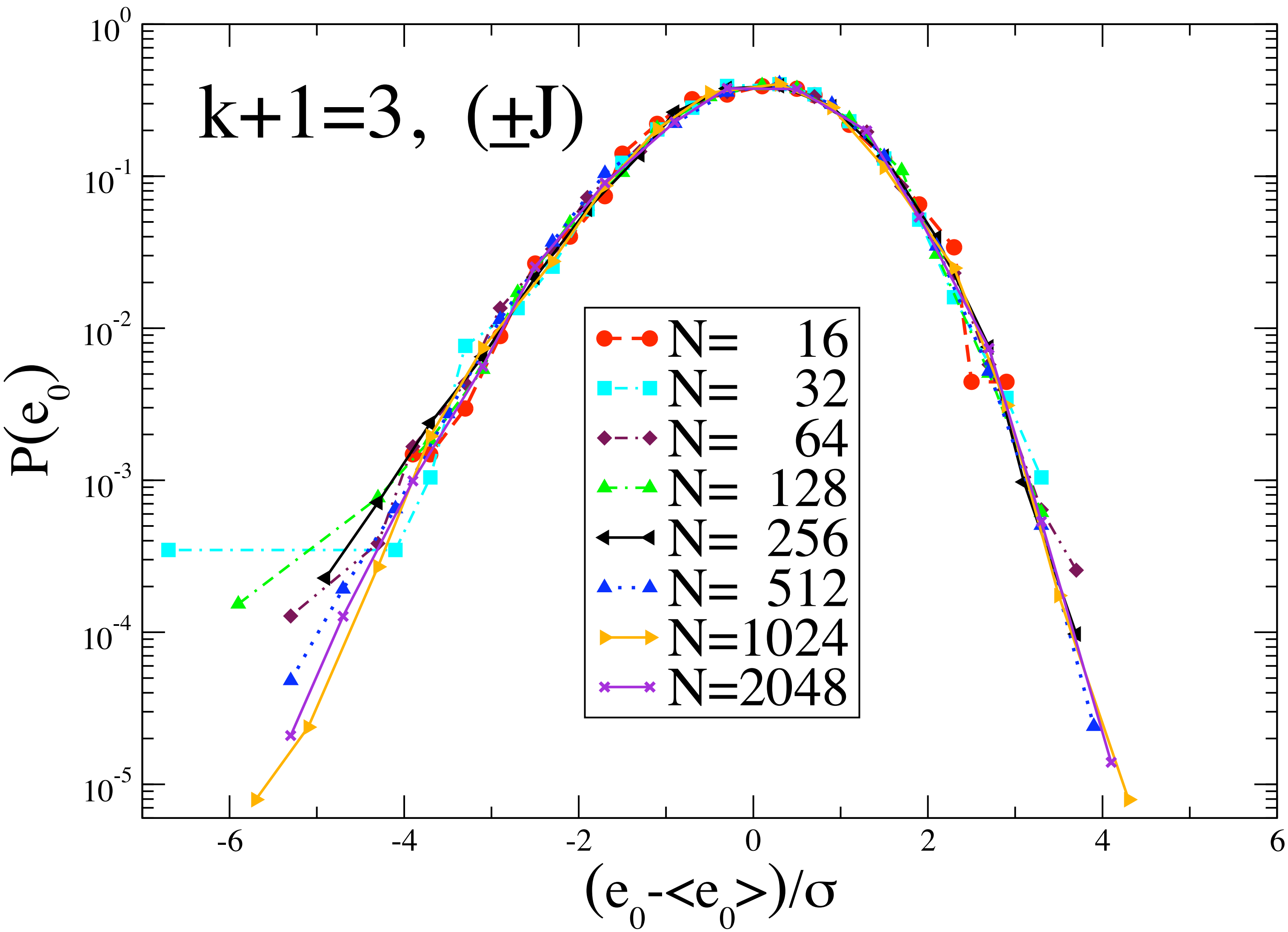
Entropies



[PRB**67**(03)060403R]

τ-EO for Bethe Lattices:

EO for 3-connected Bethe Lattice Glass:



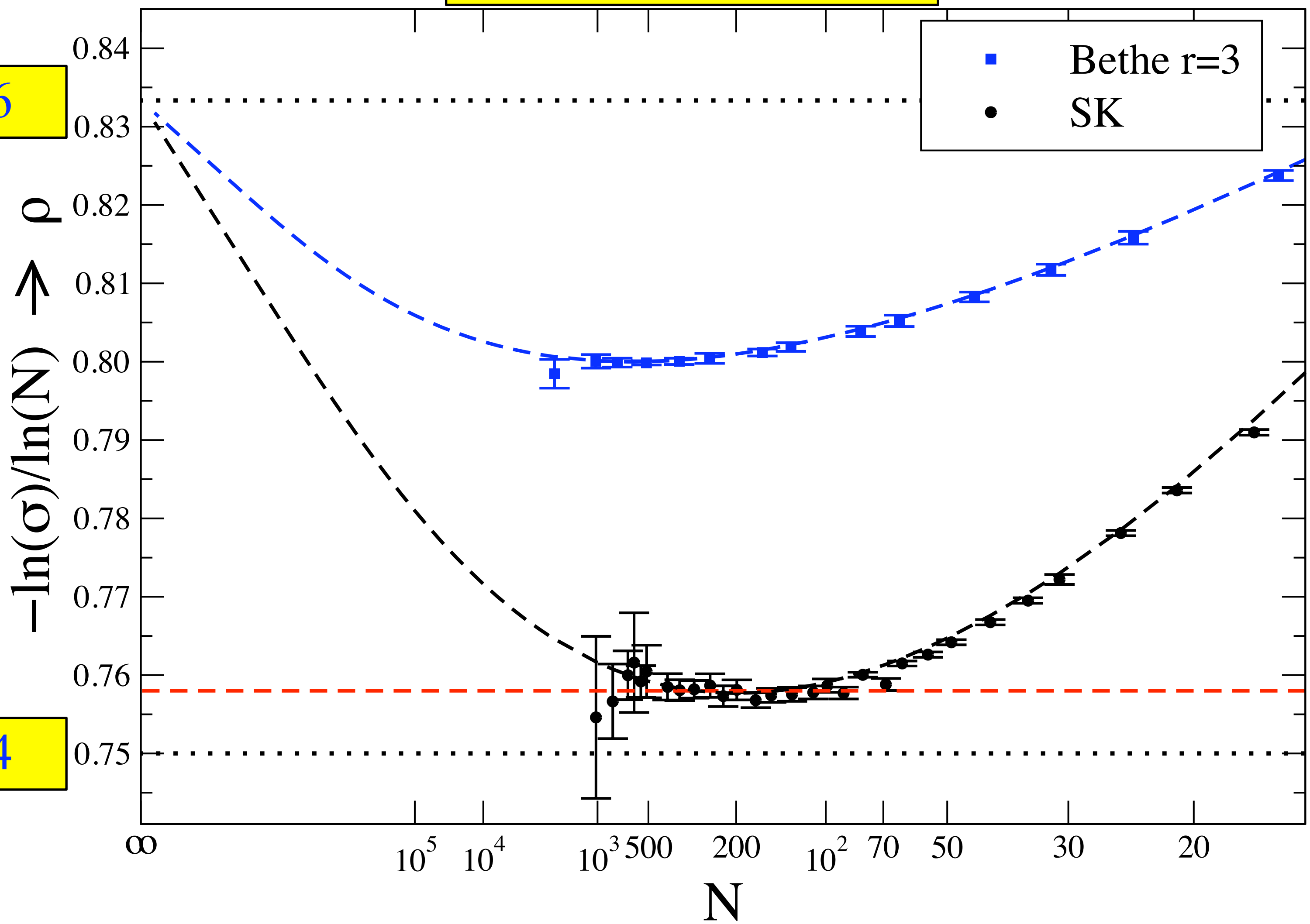
τ-EO for Sherrington-Kirkpatrick

- Mean-Field ($d \to \infty$) Spin Glasses:

Fluctuation Exponent ρ

ρ=5/6

ρ=3/4



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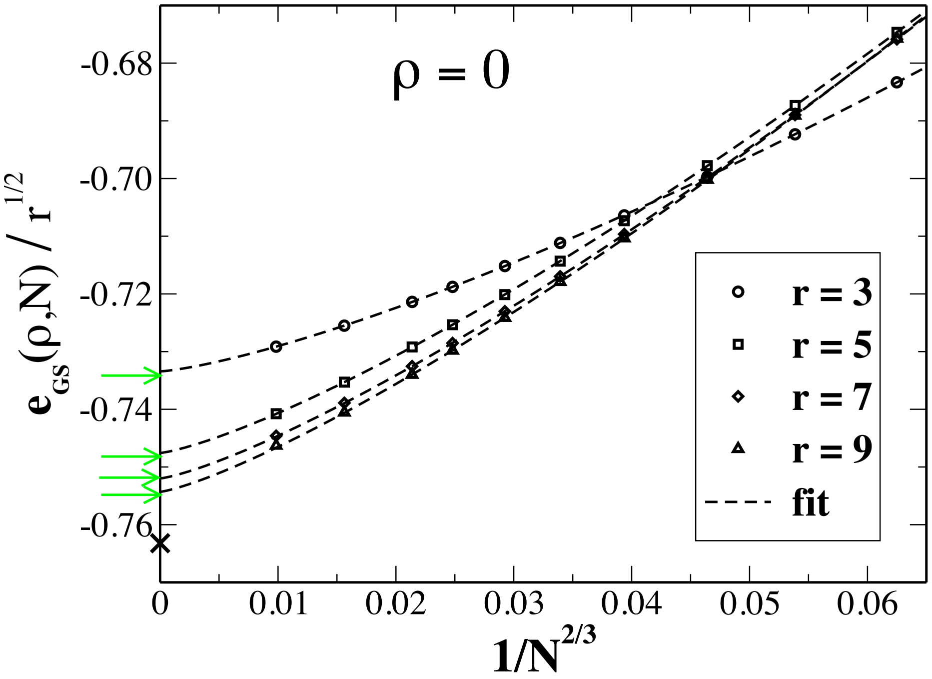
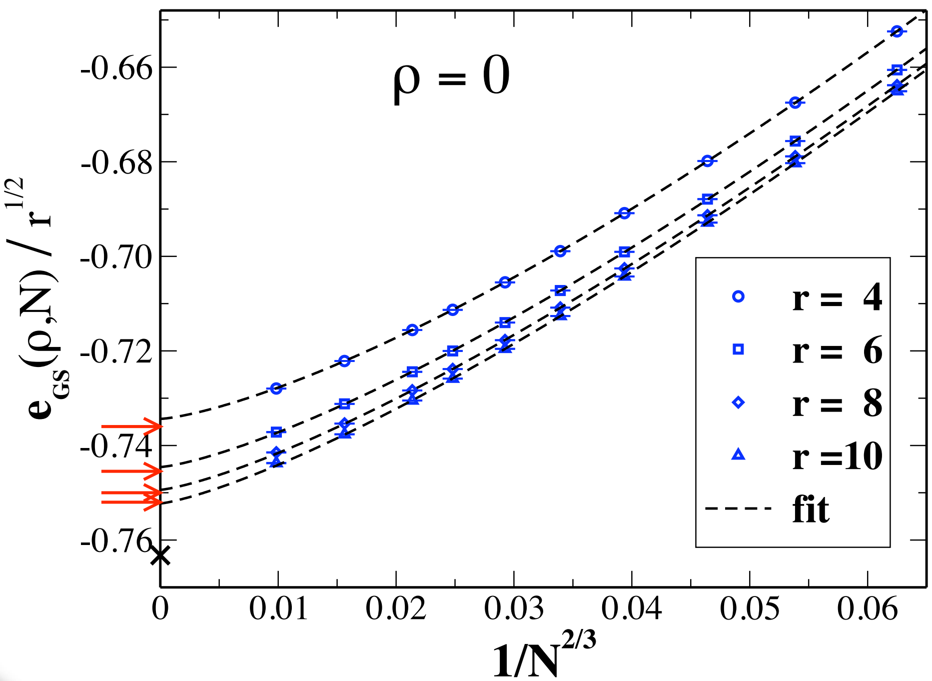
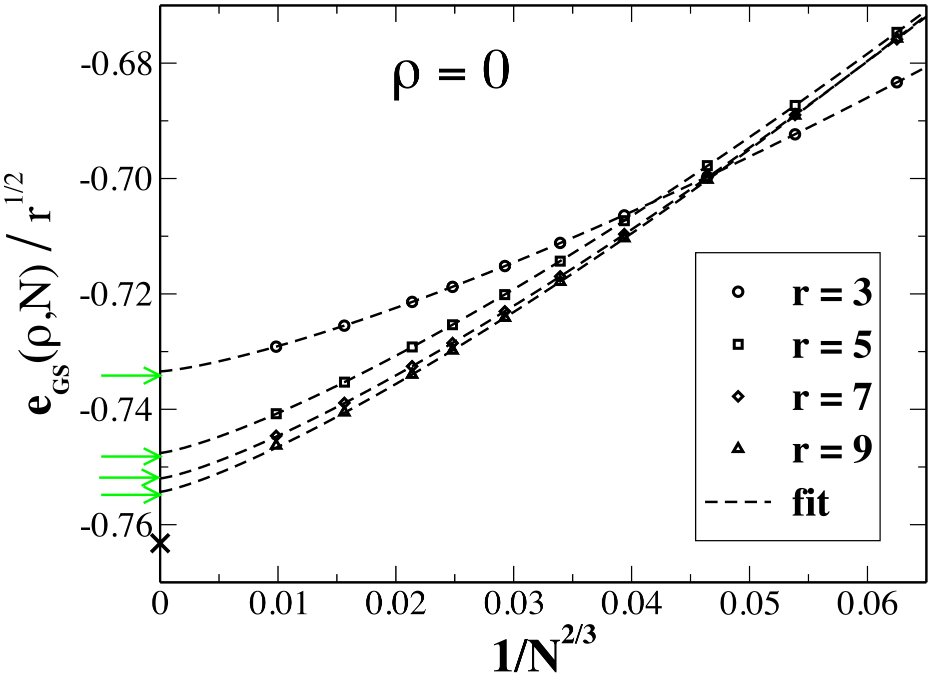


L. Zdeborova & SB,
J Stat Mech, P02020 (2010).





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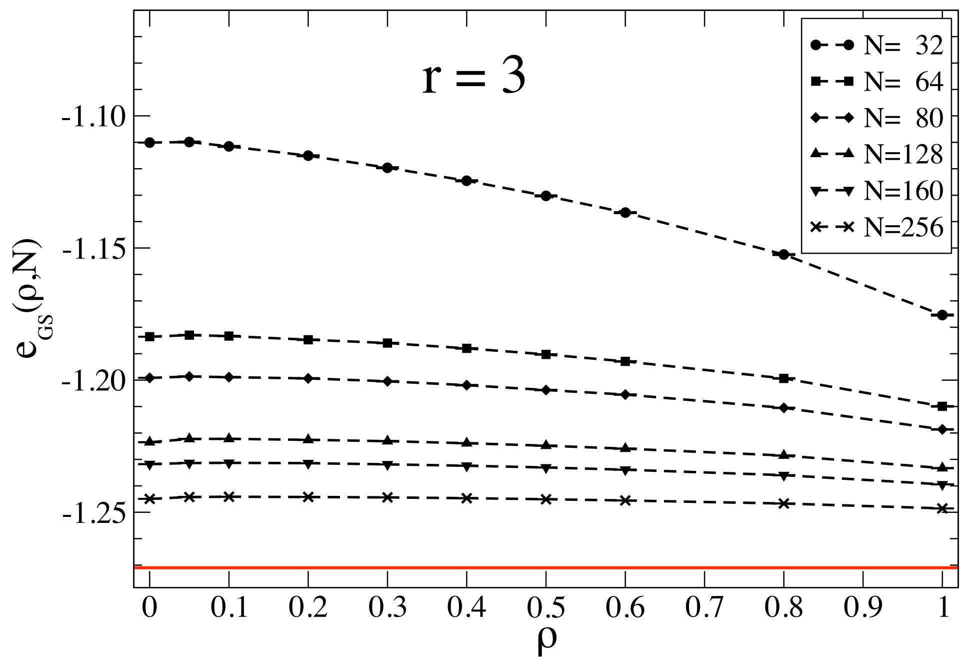

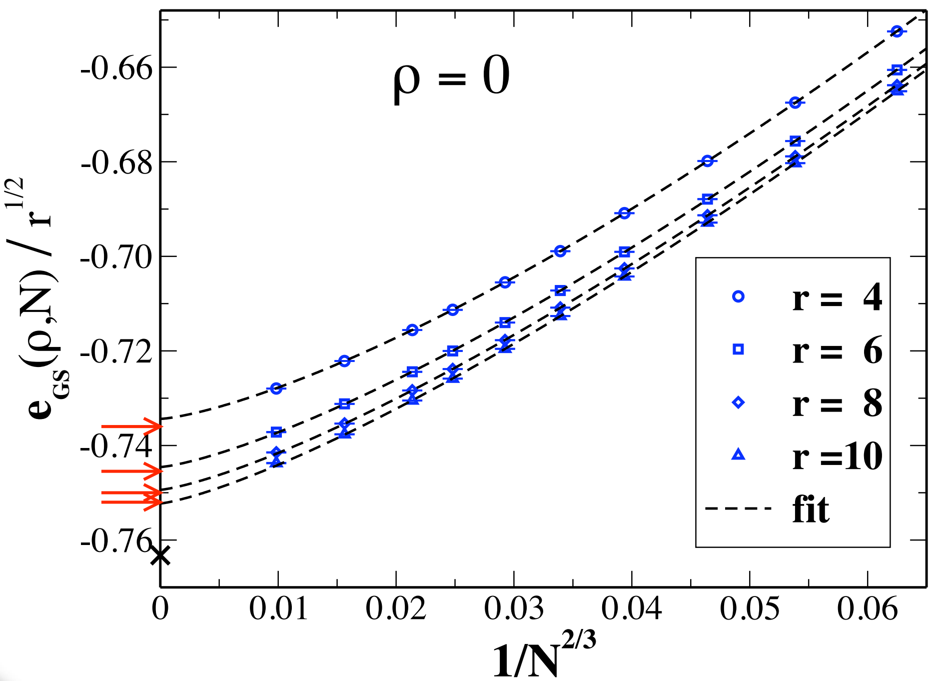

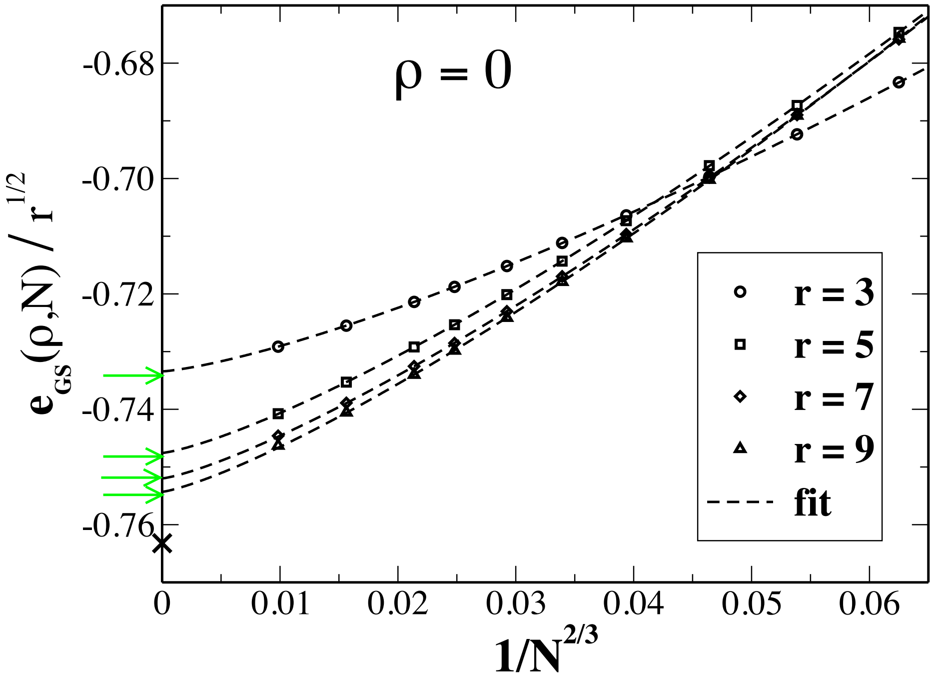

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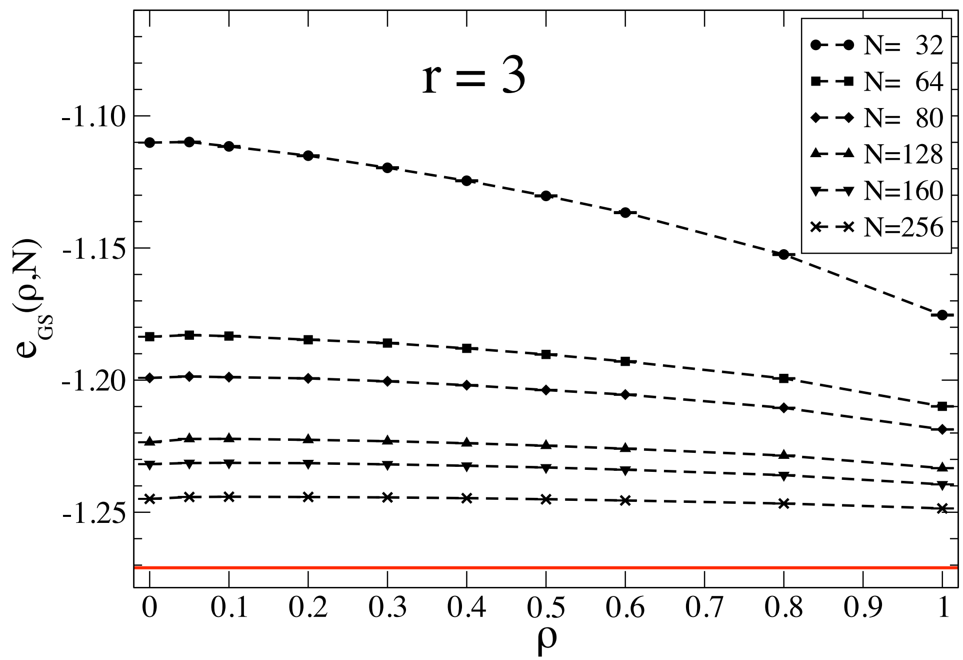
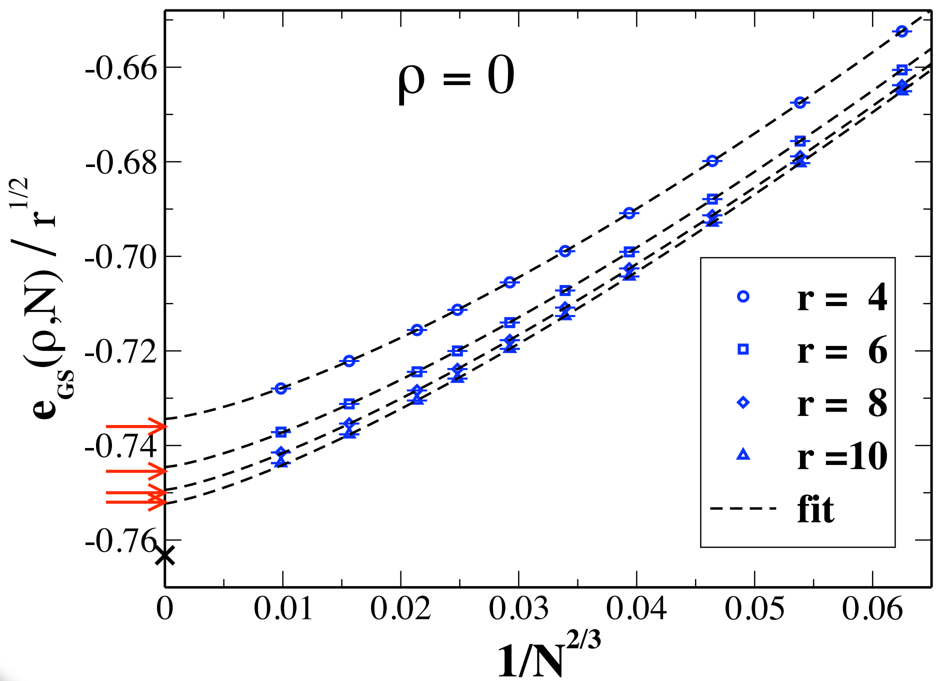
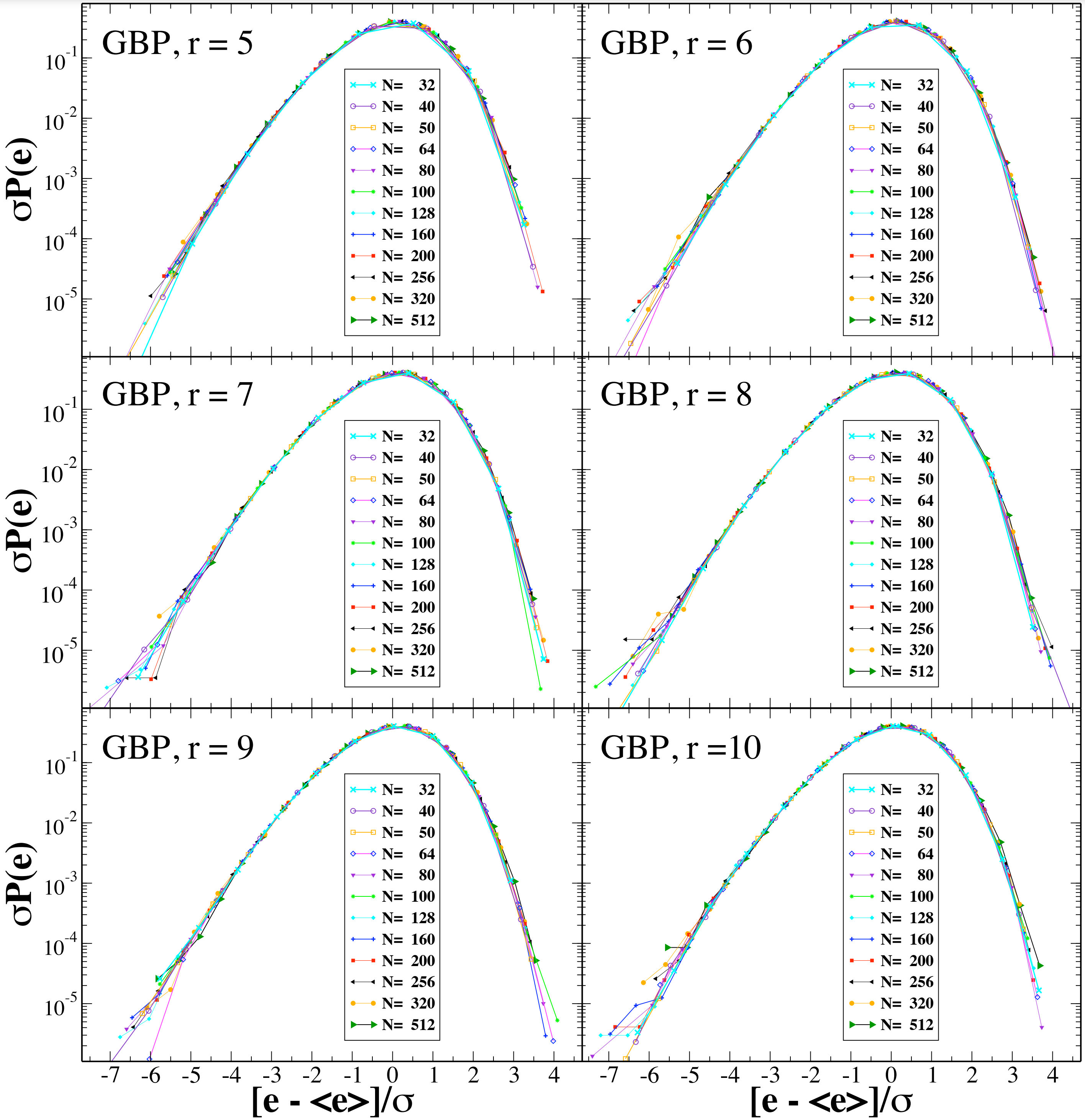




maximum cut <=> bisection!

Extreme Fluctuations and Finite-Size Corrections in Spin Glasses



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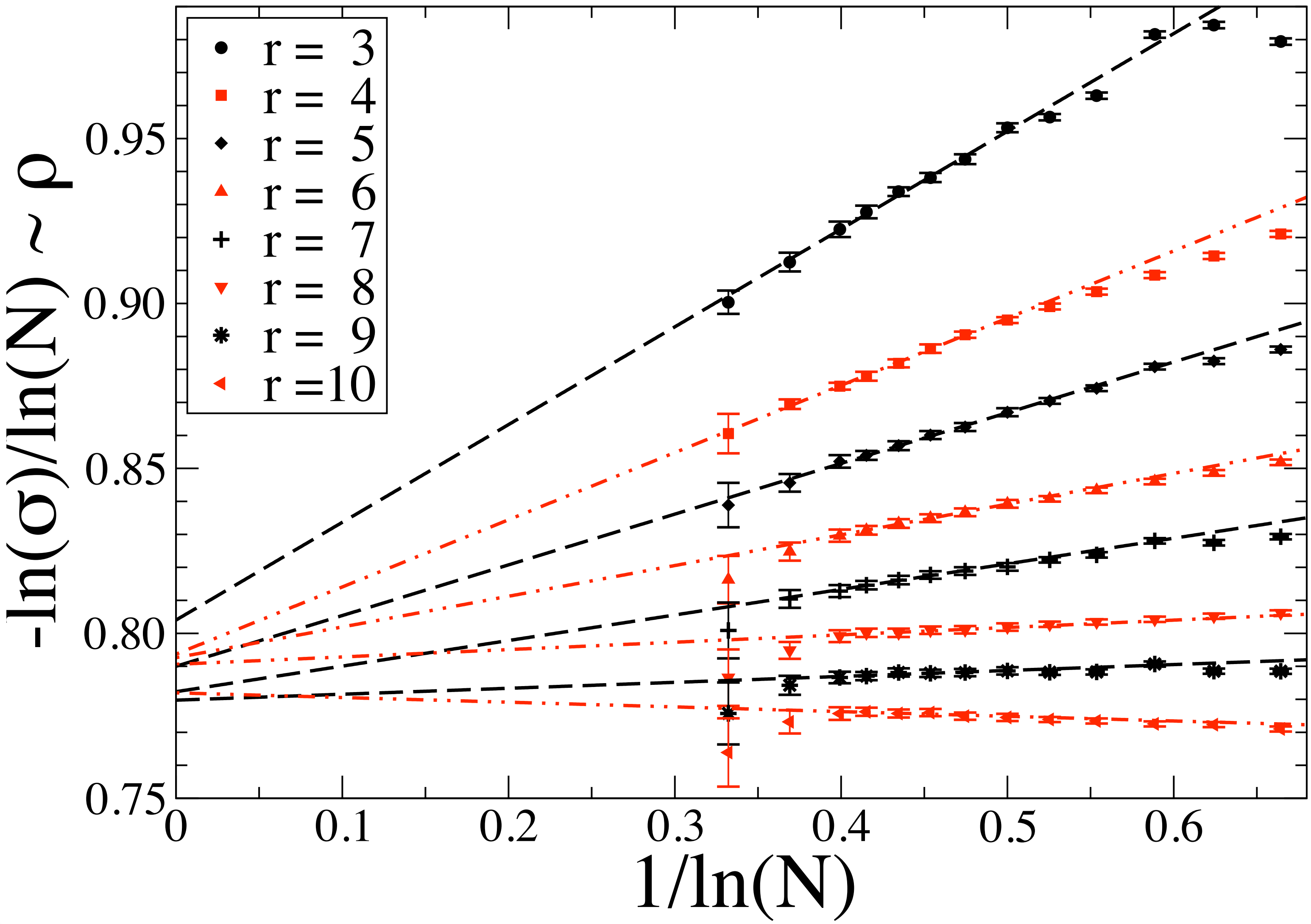
τ-EO for Bethe Lattice Spin Glasses:

- ±J Bonds:

Fluctuation Exponent ρ



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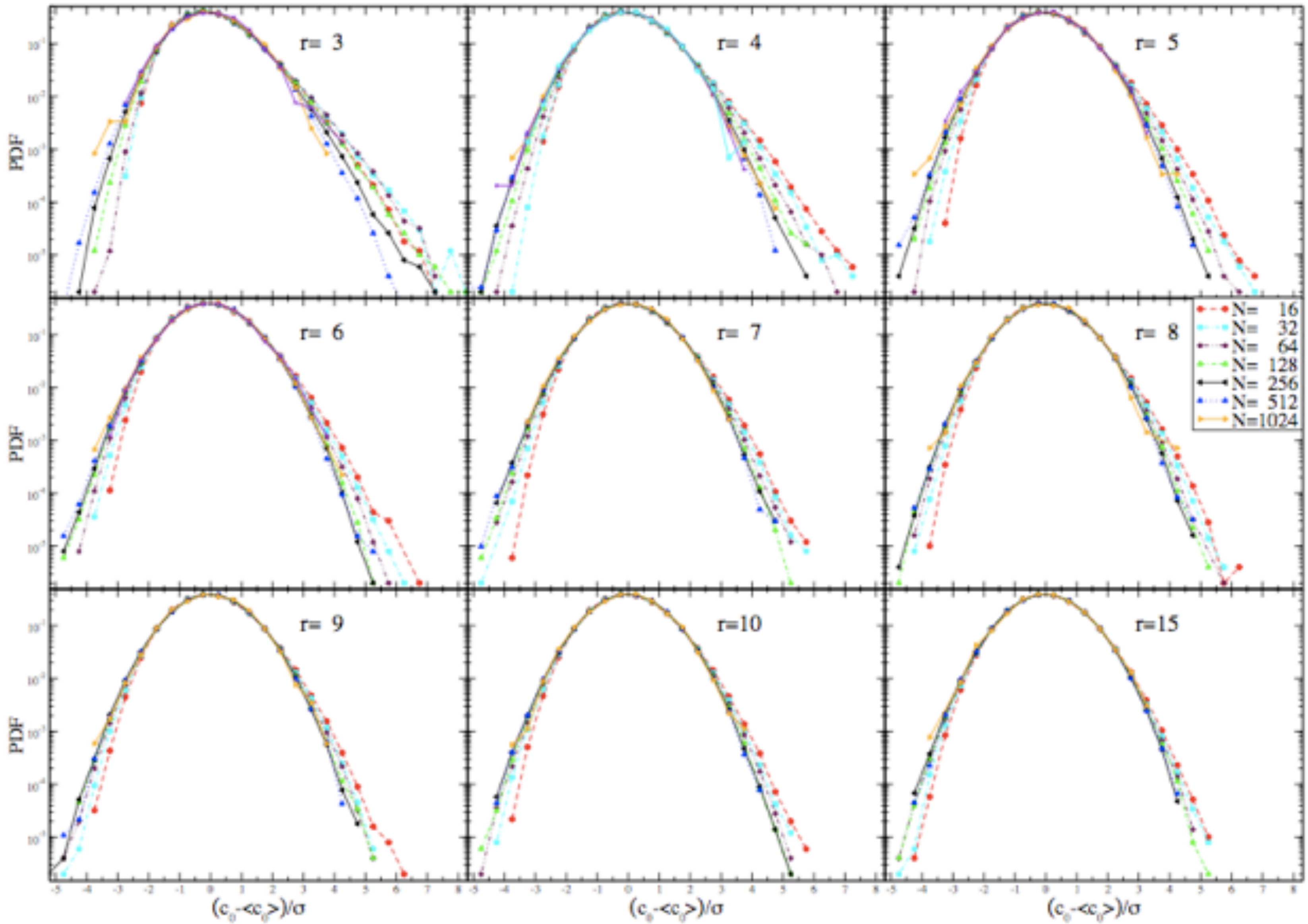
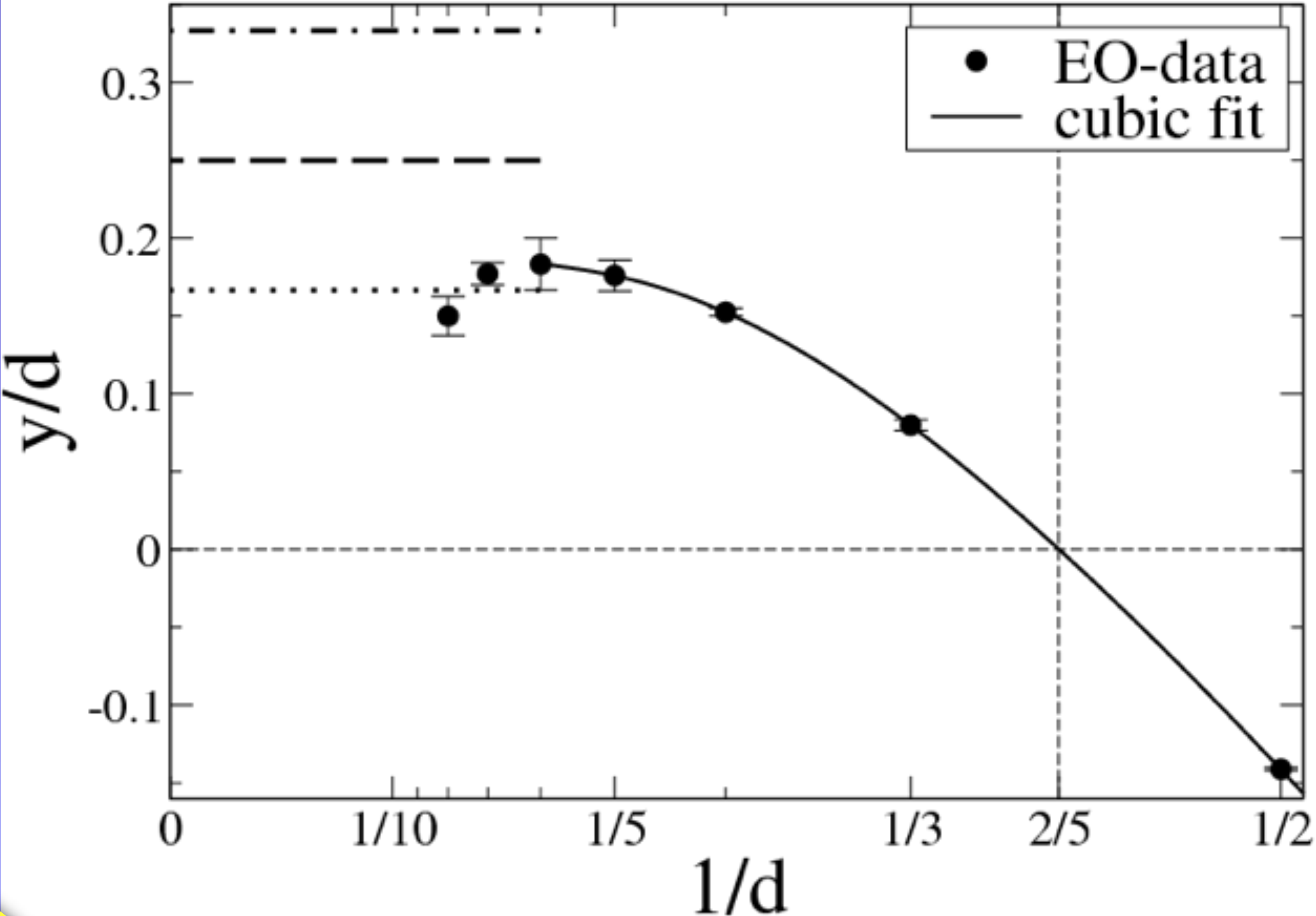
τ-EO for Bethe Lattice Spin Glasses:

- Gaussian Bonds:



Spin Glass on Dilute Hyper-Cubes (EA):

"Stiffness": $\sigma(\Delta E) \sim L^y$



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June 1, 2010

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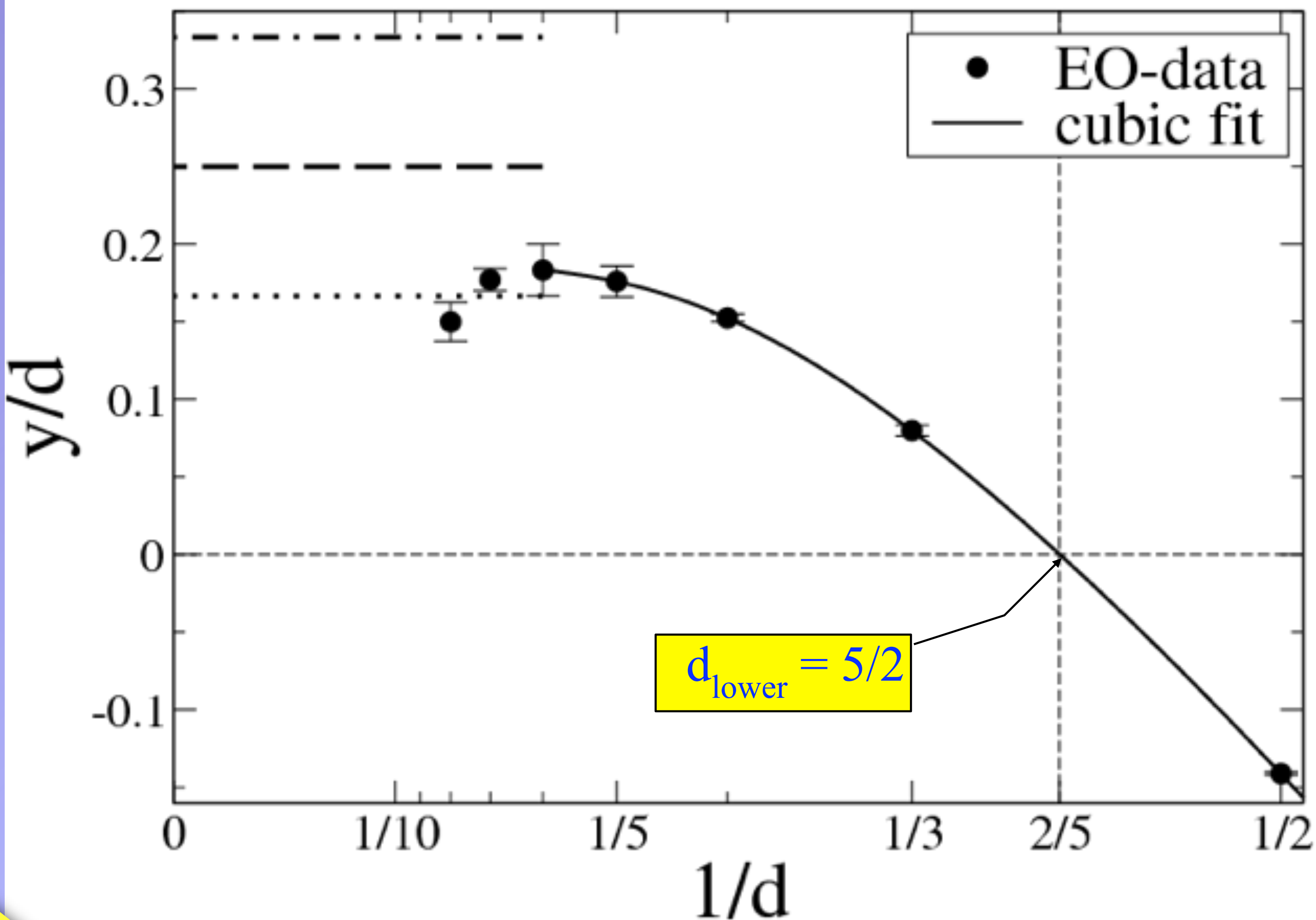

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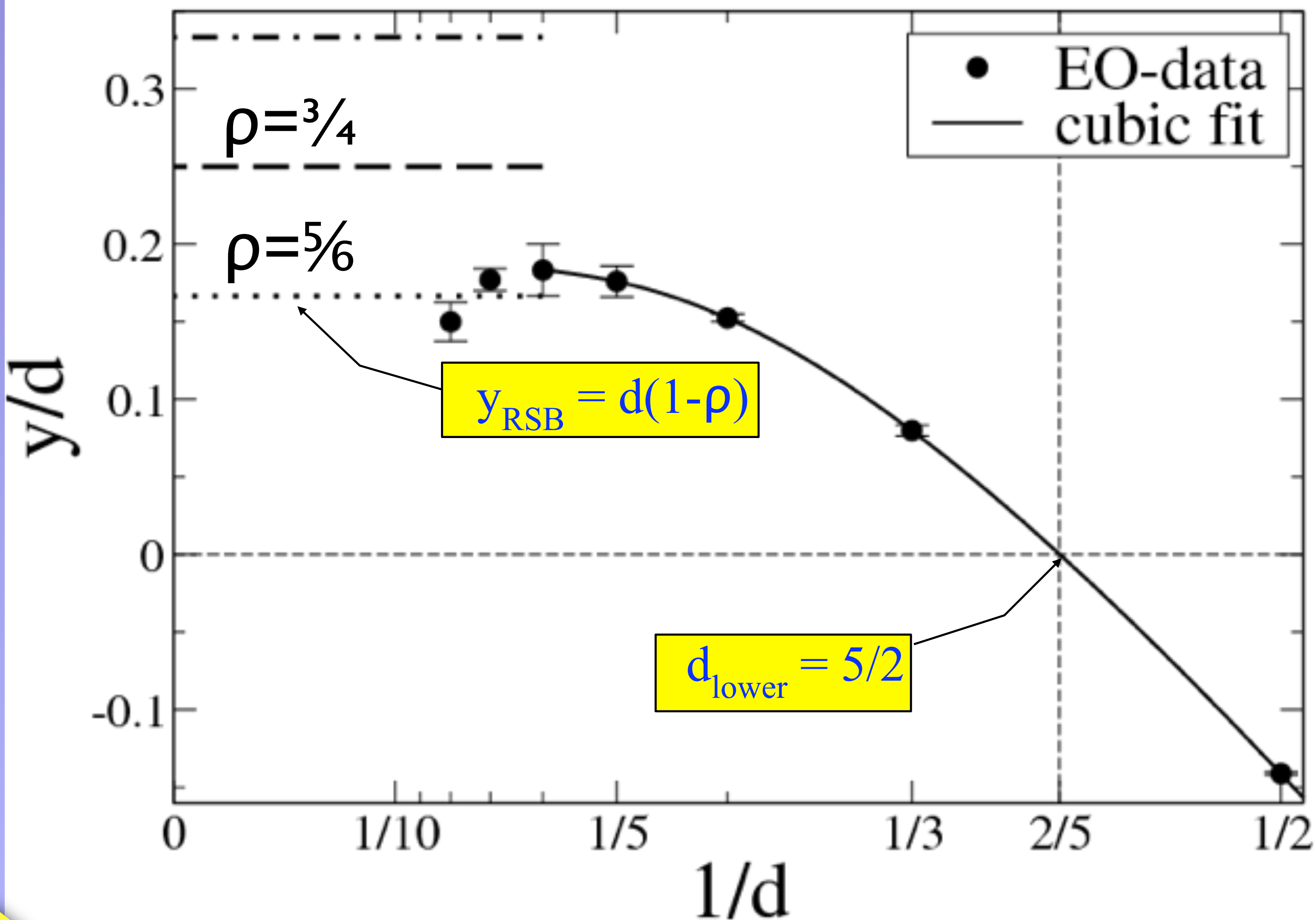

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Finite-Size Corrections in EA:

Ground State Energy: $E(L) \sim e_0 L^d + A L^y \qquad (L \to \infty)$

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$\omega = 1 - y/d$

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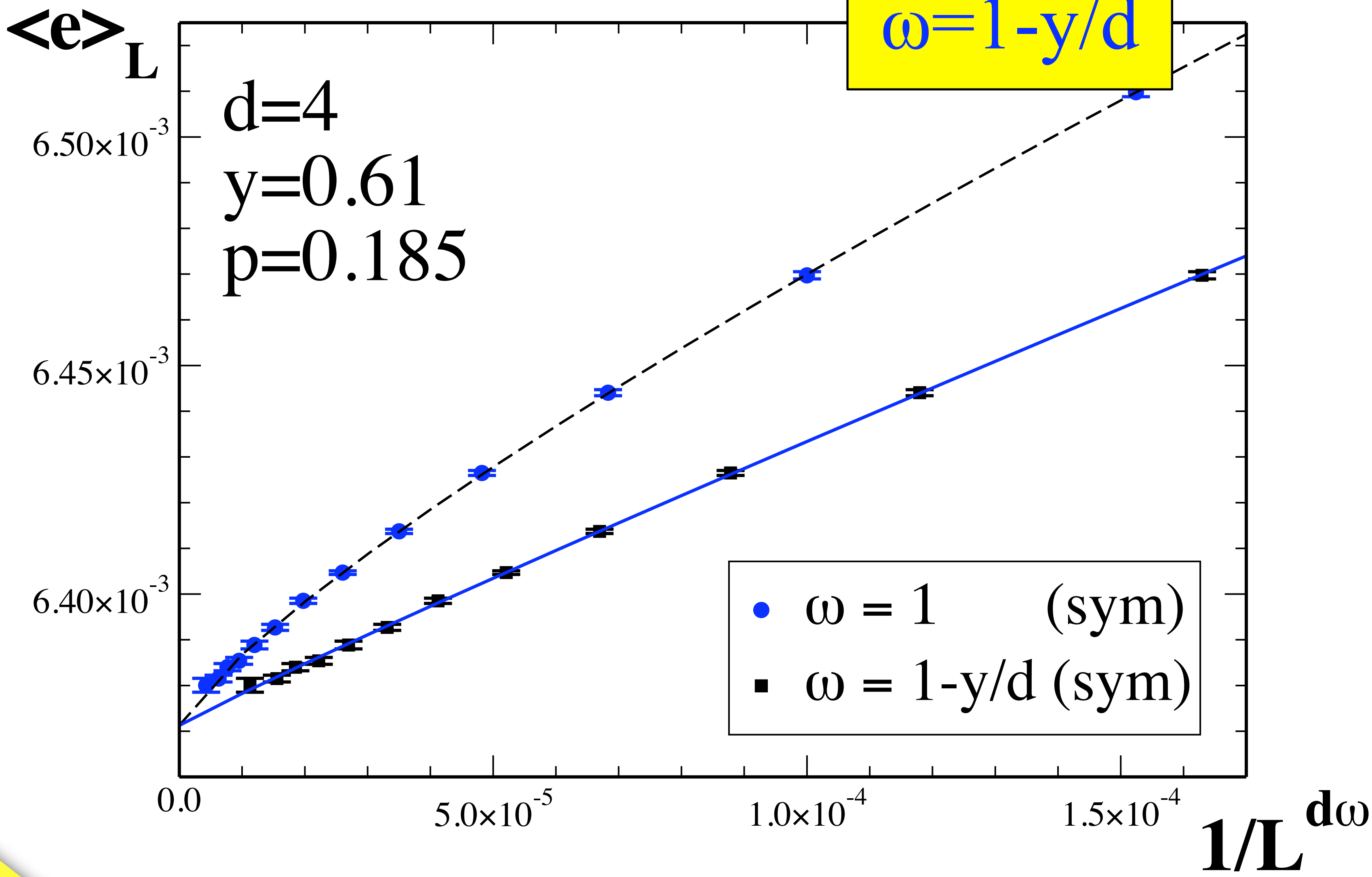

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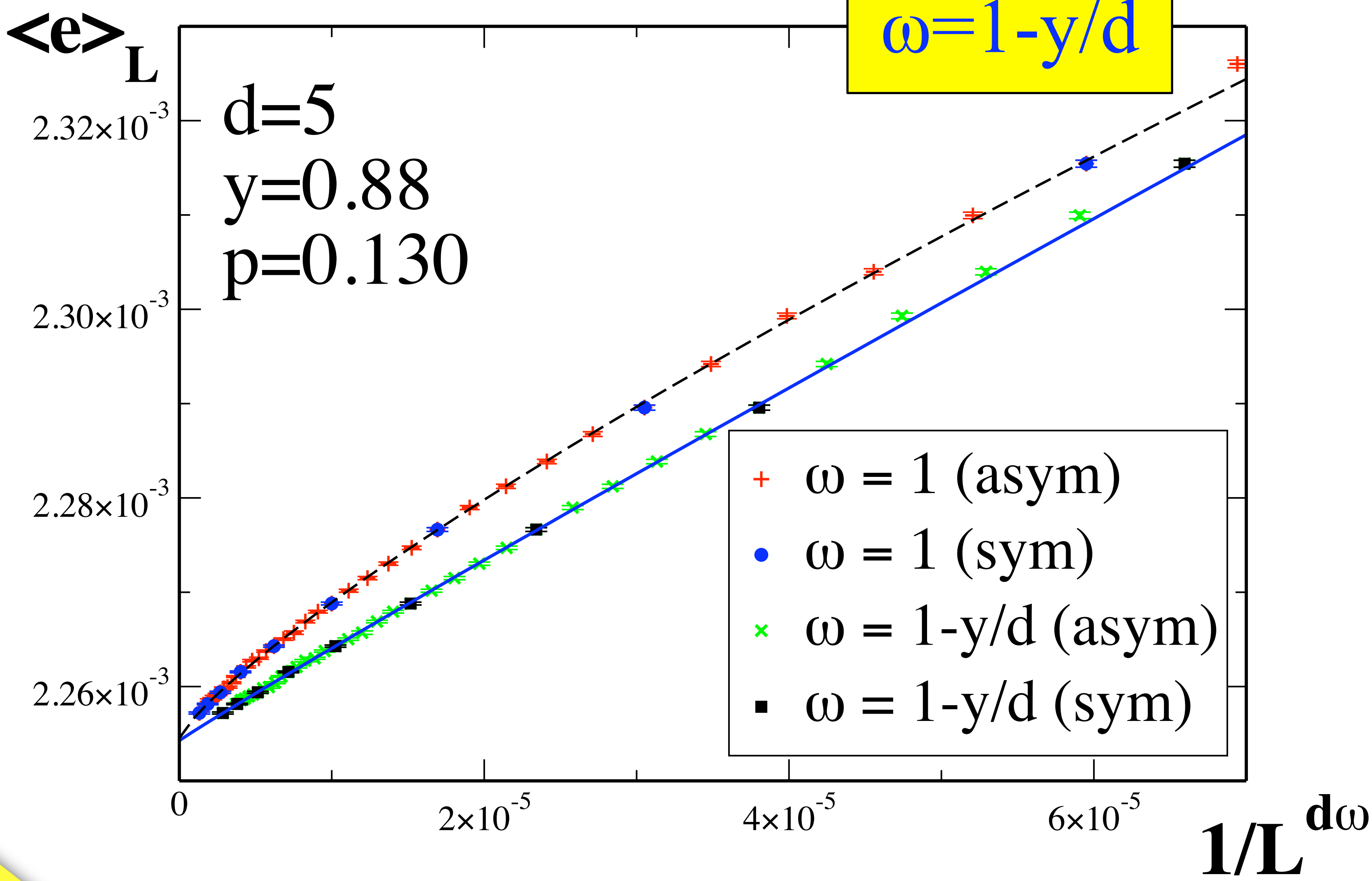

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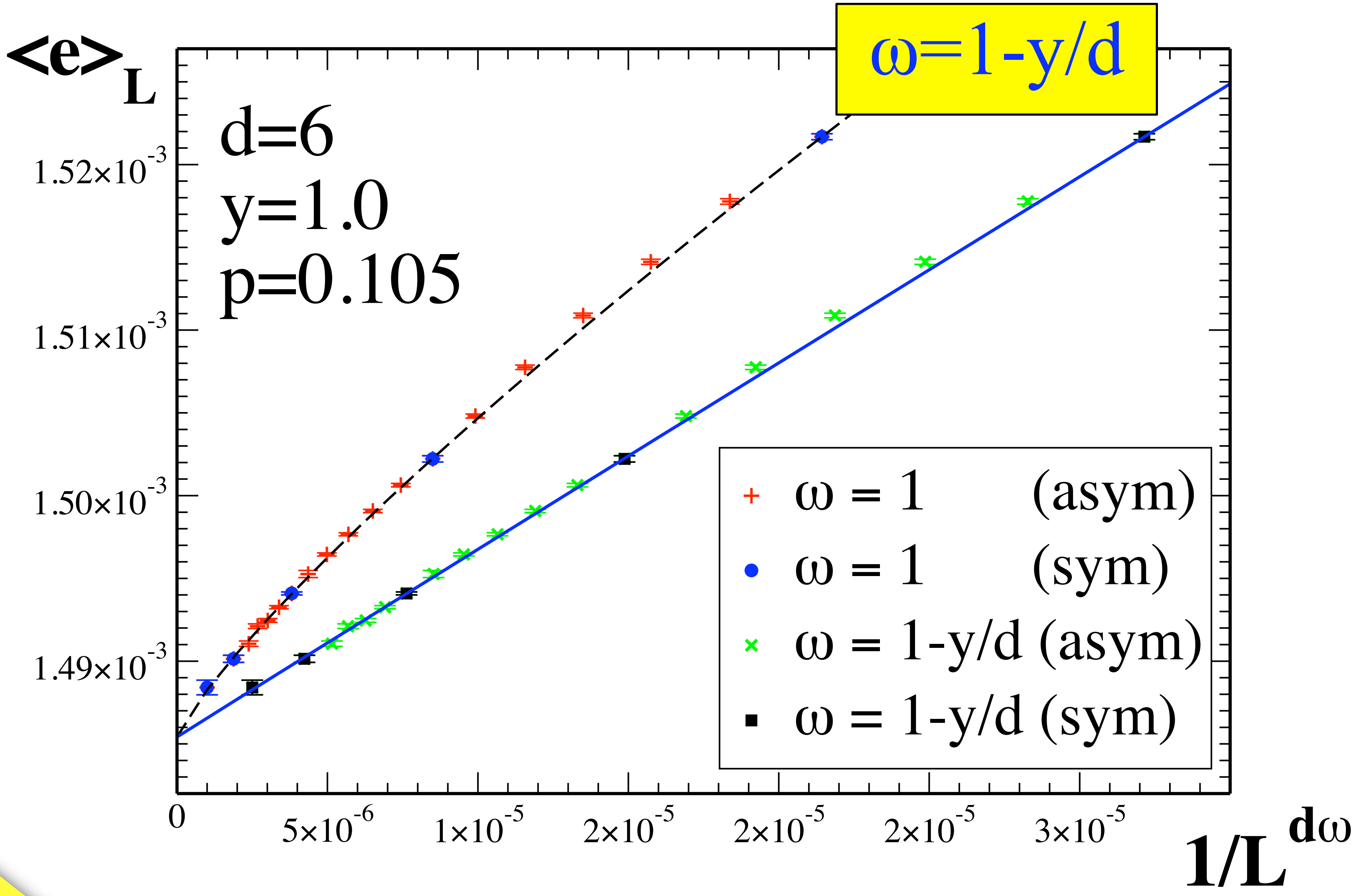

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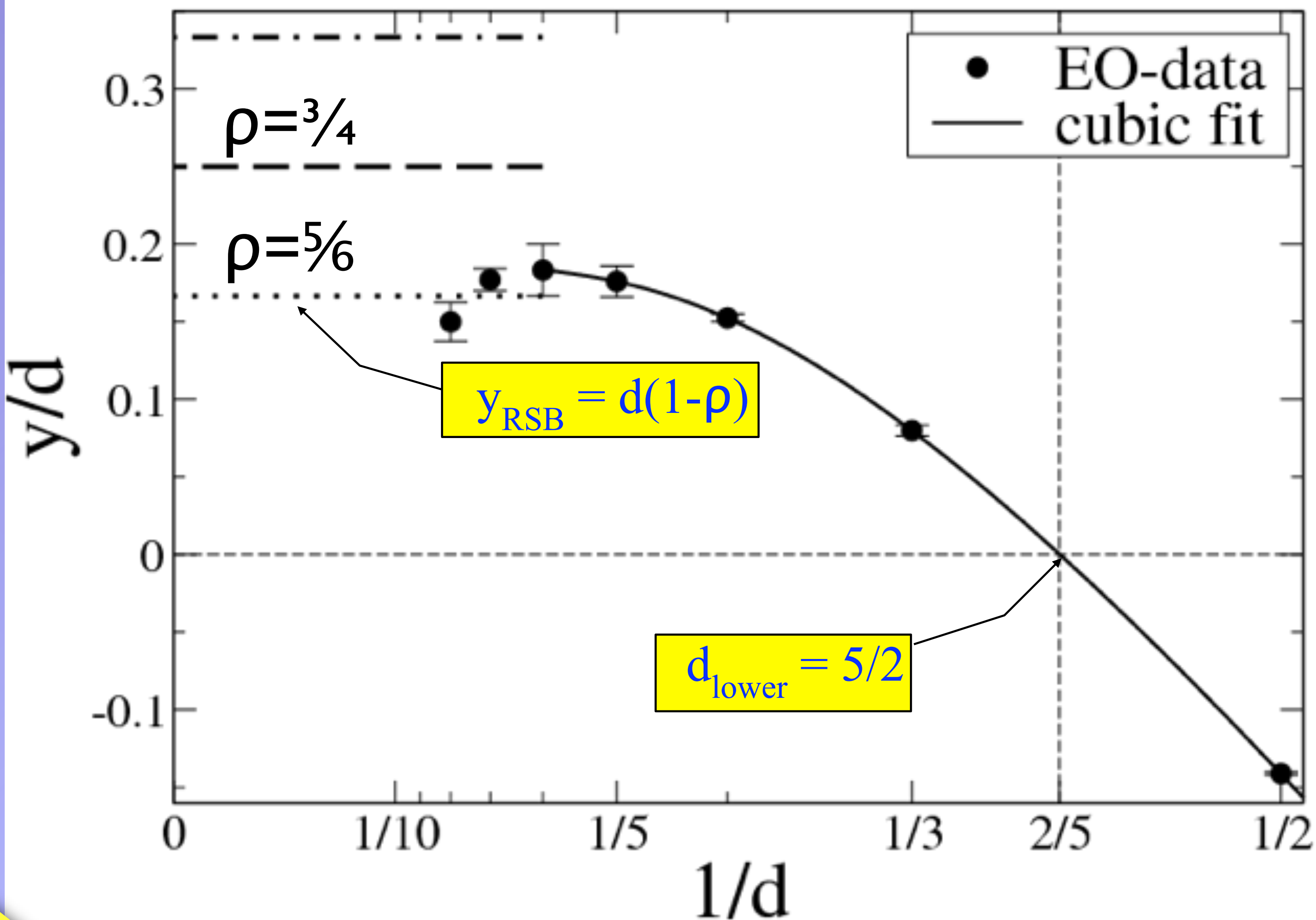

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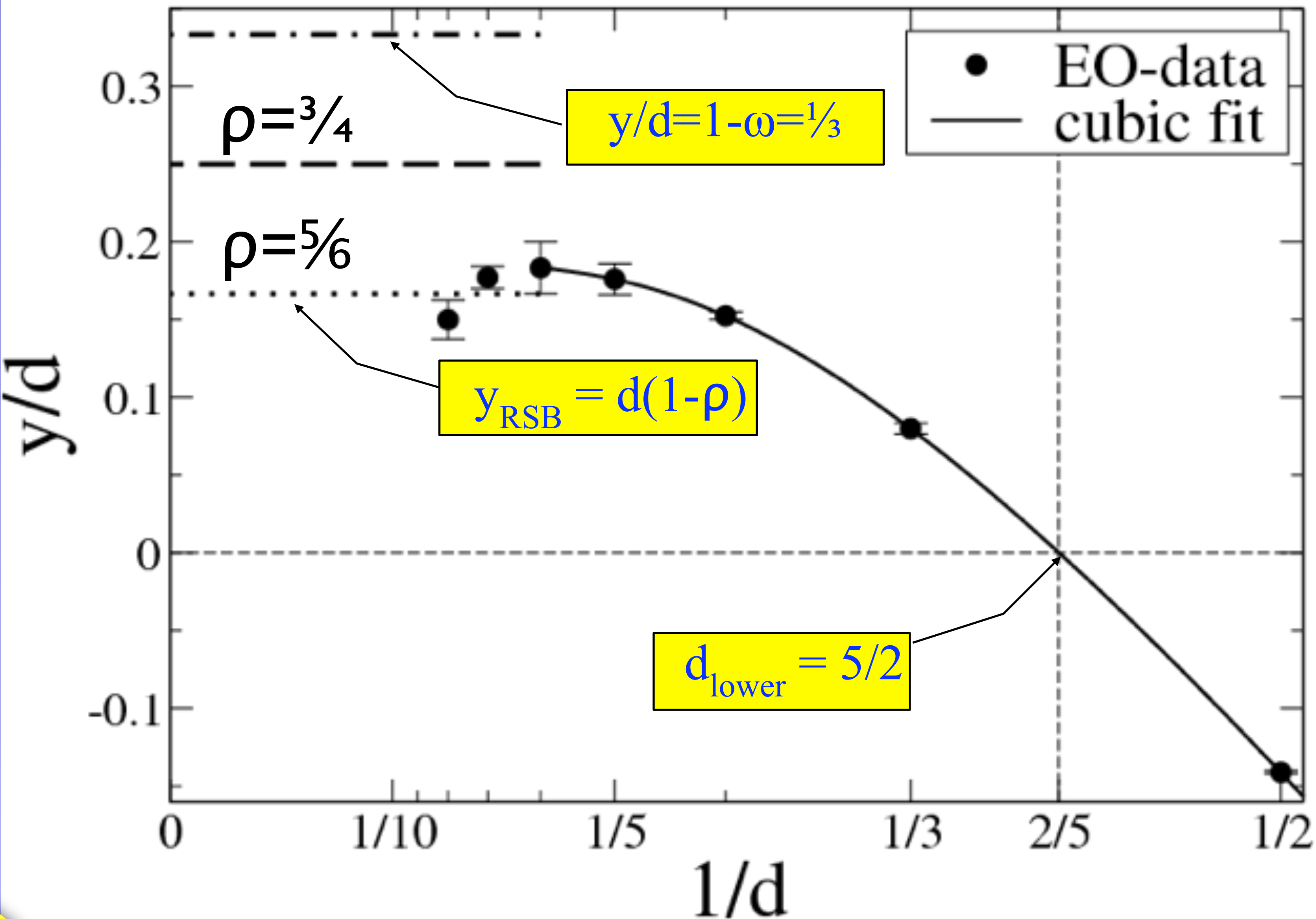

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