Transport in heterogeneous media – diffusion, fractals and anomalous dynamics

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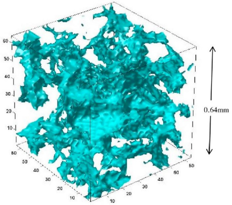
Ludwig
Maximilians
Universität

Outline

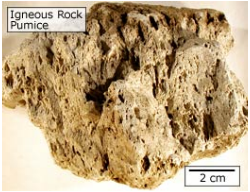
1. Motivation
2. Ant in the labyrinth
   - fractals
   - percolation
   - transport
3. Transport in heterogeneous media
   - Molecular Dynamics simulations
   - Mean-square displacement
   - VACF – two dimensions
   - Continuum Percolation Theory
   - Dynamic Scaling Hypothesis
   - Corrections to scaling

Motivation- Transport in Disordered Media



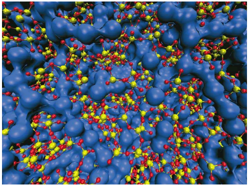
sandstone

Okabe & Blunt, PRE (2004)



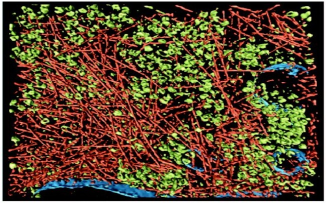
pumice

M. Nyman, TERC



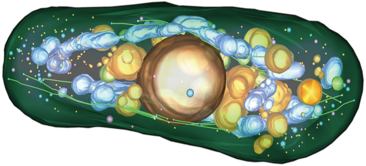
Na-silicate

A. Meyer *et al*, PRL (2004)



cellular crowding

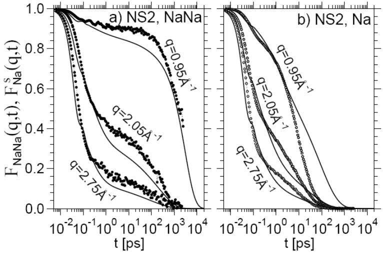
O. Medalia *et al* (2002) Science



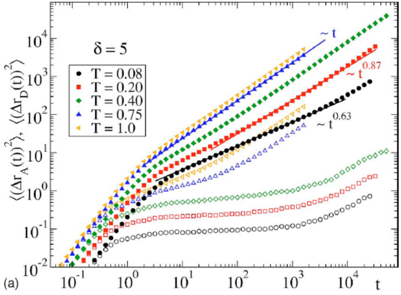
yeast

J. Höög, EMBL

Dense mixtures

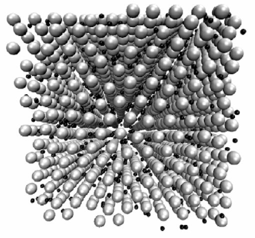


Na-Silicate T. Voigtmann, J. Horbach



disparate soft spheres

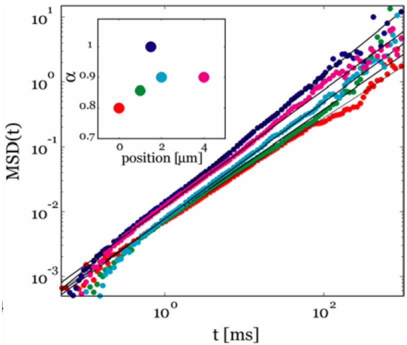
A. Moreno, J. Colmenero

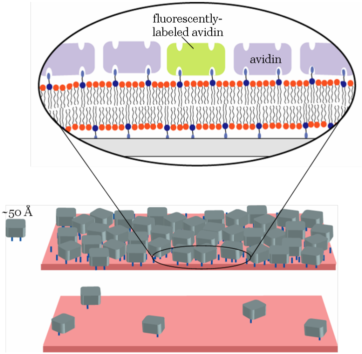


subdiffusive behavior in
strongly disparate mixtures

disparate Yukawa particles N. Kikuchi, J. Horbach

Membranes





- avidin binds irreversibly to biotinylated supported lipid bilayer (SLB)
- Fluorescent correlation spectroscopy (FCS)

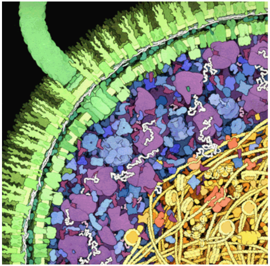
M. Horton, J. Rädler, LMU

subdiffusive behavior in crowded membranes

Molecular crowding

- Molecular crowding

"Molecular crowding is more accurately termed the excluded volume effect, because the mutual impenetrability of all solute molecules is its most basic characteristic. This nonspecific steric repulsion is always present, regardless of any other attractive or repulsive interactions that might occur between the solute molecules." R. John Ellis 2001



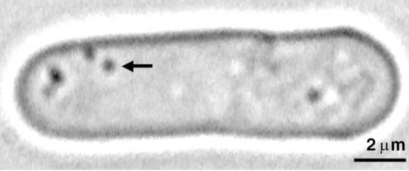
E-coli

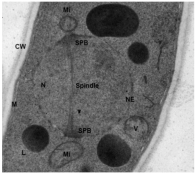
D. Goodsell

- 30% volume fraction by sugars, lipids, membranes
- anomalous transport in the cell
- chemical reactions are slow
- apparent density-dependent exponents?
  - alternatively: huge crossover regimes
  - origins: static heterogeneities, random traps, polymer networks

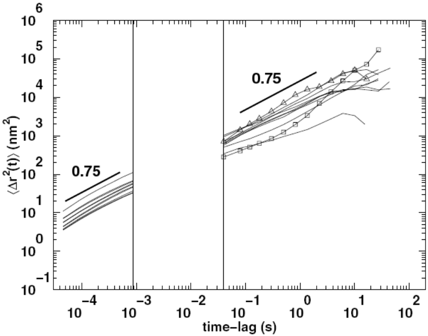
related: M. Weiss, M.J. Saxton

Living Cells





living fission yeast

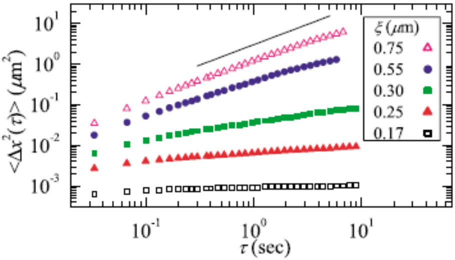


- small lipid granules that occur naturally in the cytoplasm
- particle tracking in video microscopy

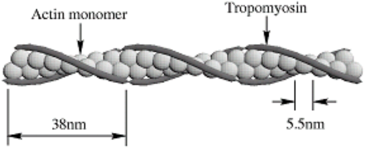
Iva Tolić-Nørrelyke, MPI-CBG

subdiffusive behavior in crowded cells

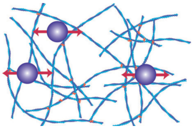
Biopolymer Networks



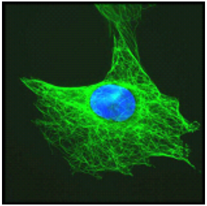
Wong *et al*, PRL 2004



- multiple particle tracking
- entangled F-actin filament networks
- particle diameter *a* comparable to mesh size ξ



Bausch, Kroy



F-actin

subdiffusive behavior in crosslinked networks

Origin of anomalous transport?

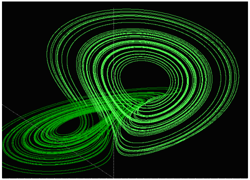
- dense system constitutes course of obstacles
- long-living heterogeneities
- many length scales induce hierarchy of time scales
- Complex systems – also other mechanisms
  - distribution of sticking times
  - spectrum of relaxation times in polymers
  - glassy dynamics
  - phase separation
  - non-equilibrium, aging
  - living, active systems

1. Motivation

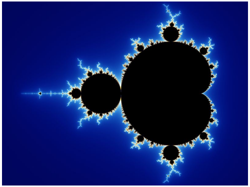
2. Ant in the labyrinth
   - fractals
   - percolation
   - transport

3. Transport in heterogeneous media
   - Molecular Dynamics simulations
   - Mean-square displacement
   - VACF – two dimensions
   - Continuum Percolation Theory
   - Dynamic Scaling Hypothesis
   - Corrections to scaling

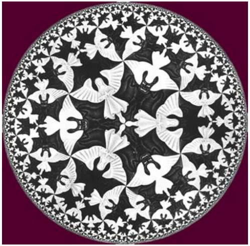
fractals – picture gallery



Lorenz attractor

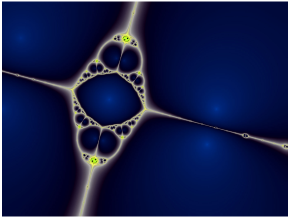


Mandelbrot set



Devils and Angels

M.C. Escher

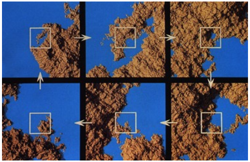


bubbles fractal

www.fractalarts.com

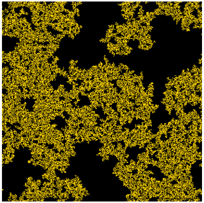
fractals – picture gallery

Romanesco

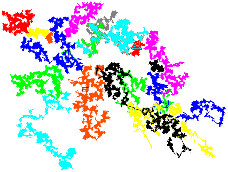


coastline

Sierpińksi

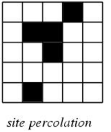


site percolation

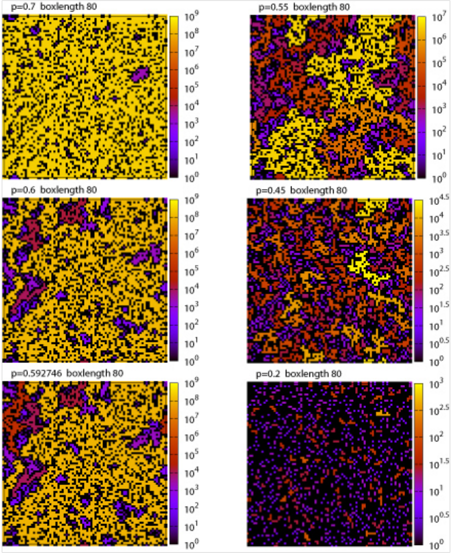


continuum percolation

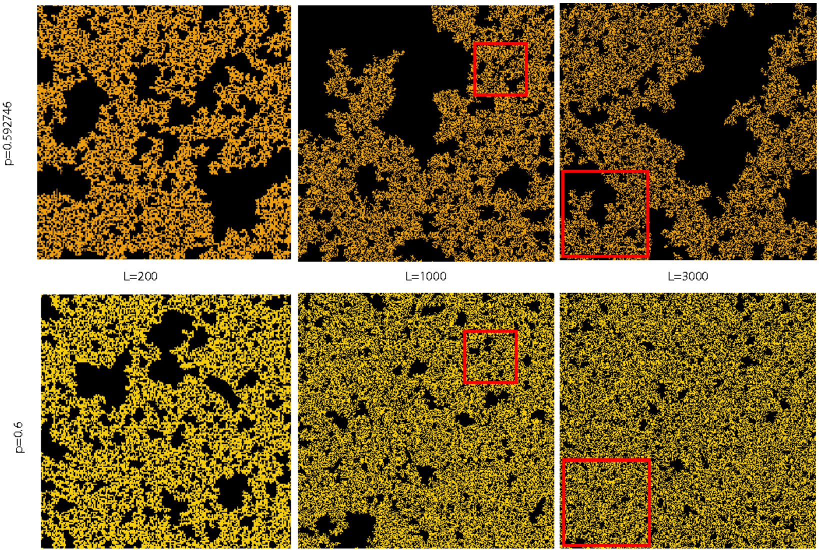
Site Percolation



- sites are occupied with probability p
- Occupied sites form clusters
- An infinite cluster is present above some threshold
- correlation length $\xi(p)$: size of the largest finite cluster



Self-similarity



Fractal dimension

- self-similarity at criticality
- mass of the infinite cluster

$$M(r) \sim r^{d_f}$$

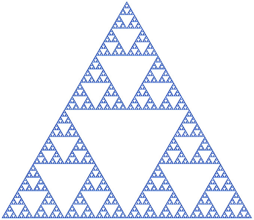
d_f is the fractal dimension

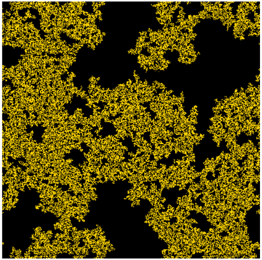
- Sierpiński gasket

$$d_f = \frac{\log 3}{\log 2}$$

- percolation

$$d_f = 91/48 \approx 1.9 \quad (d = 2)$$
$$d_f \approx 2.53 \quad (d = 3)$$

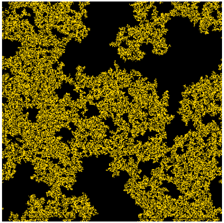




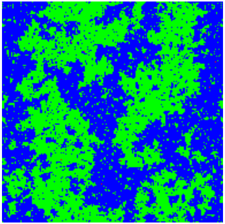
critical exponents

- self-similarity implies
  - correlation length $\xi \sim |p - p_c|^{-\nu}$
  - infinite cluster $P_\infty \sim (p - p_c)^\beta$
  - mean finite cluster size
    $\ell \sim |p - p_c|^{-\nu+\beta/2}$
  - cluster size distribution at $p = p_c$
    $\boxed{n_s \sim s^{-\tau}}$
- similar to continuous phase transitions
  - p plays rôle of temperature
  - P_∞ order parameter
- scaling relations
  - Fisher exponent and fractal dimension

$$\tau = 1 + d/d_\mathrm{f} \qquad d_\mathrm{f} = d - \beta/\nu$$



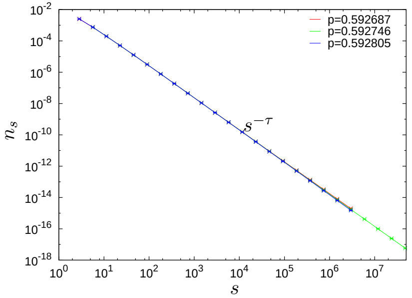
percolation

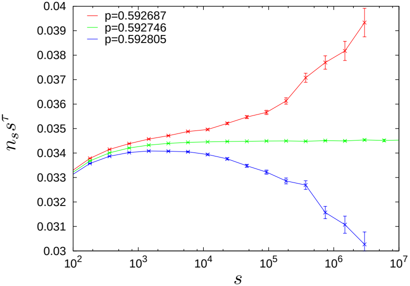


Ising model

K. Binder and W. Kob, *Glassy Materials and Disordered Solids: An Introduction to Their Statistical Mechanics*

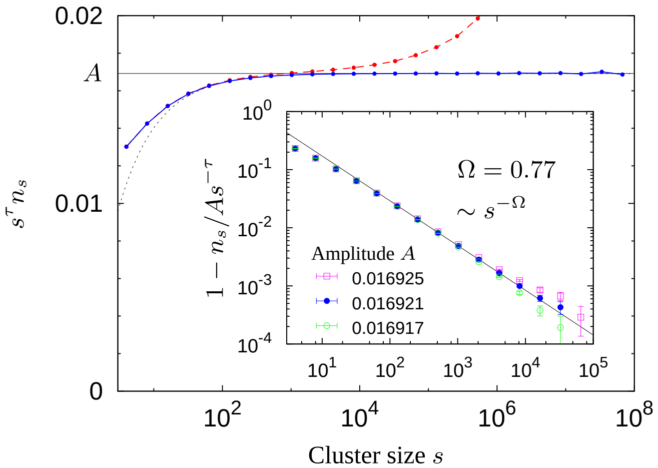
Fisher exponent





Simulation based on Hoshen-Kopelman algorithm

- periodic boundary conditions reduce finite size correction
- box length $L = 45,000$
- realizations 195,000
- critical density $p_c = 0.5927460$

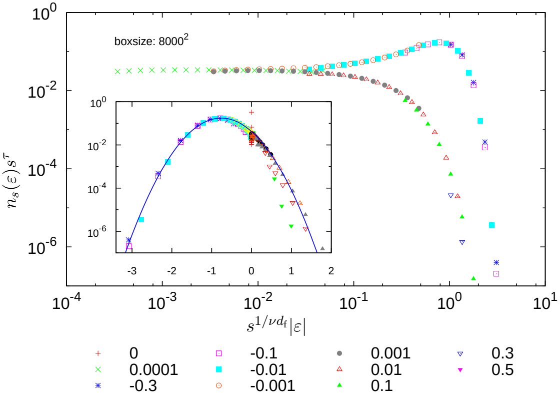


A. Kammerer, F. Höfling, and T. Franosch, EPL **84** (2008) 66002

power-law corrections

$$n_s(p_c) = As^{-\tau}(1 + Bs^{-\Omega} + \dots) \qquad s \to \infty$$

new universal correction exponent $\Omega = 0.77$



Scaling behavior

- distance
  $\varepsilon = (p - p_c)/p_c$
- all cluster alike
- compare size
  $R_s \sim s^{1/d_f}$
  with correlation
  length $\xi \sim |\varepsilon|^{-\nu}$

$$n_s(\varepsilon) = s^{-\tau}\hat{n}(\varepsilon s^{1/\nu d_f})$$

$\hat{n}$: scaling function, excellent data collapse

Transport on percolating systems

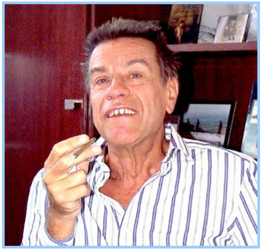
- random walker on occupied sites
  - ant in the labyrinth (de Gennes)
- fractal geometry causes anomalous transport
- mean-square displacement
  - all cluster average

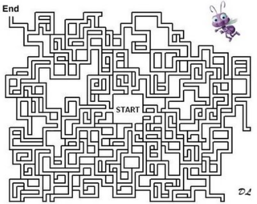
$$\delta r^2(t) \sim t^{2/z}$$

  - infinite cluster

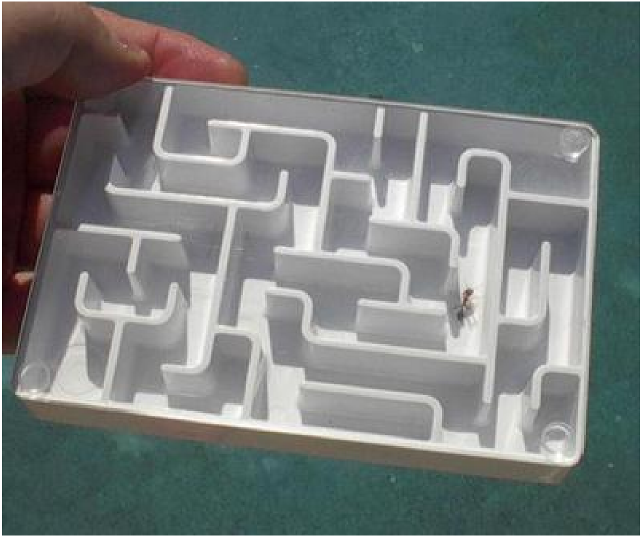
$$\delta r_\infty^2(t) \sim t^{2/d_w}$$

→ subdiffusive
→ up to where the system is homogeneous



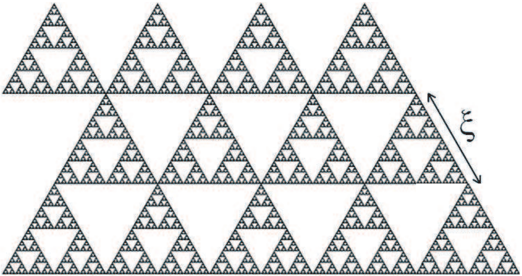


Ant Farmer John



www.AntFarmerJohn.com

Crossover to homogeneous System



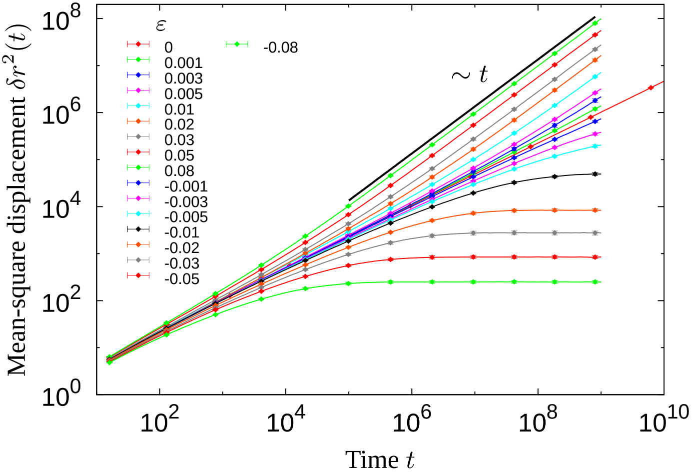
- crossover to diffusion at scale ξ

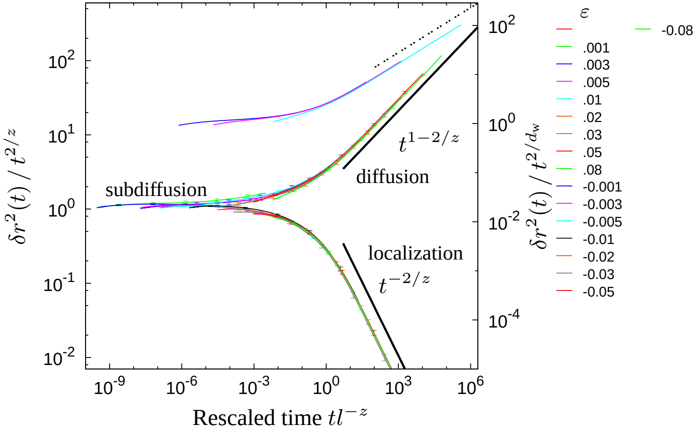
$$\delta r^2_{\text{Sierpiński}} \sim \begin{cases} t^{2/d_w} & \text{anomalous for } t \ll t_\xi \\ t & \text{diffusive for } t \gg t_\xi \end{cases}$$

- crossover time $t_\xi \sim \xi^{d_w/2}$
- walk dimension for Sierpiński $d_w = \log 5 / \log 2$

D. ben Avraham and S. Havlin

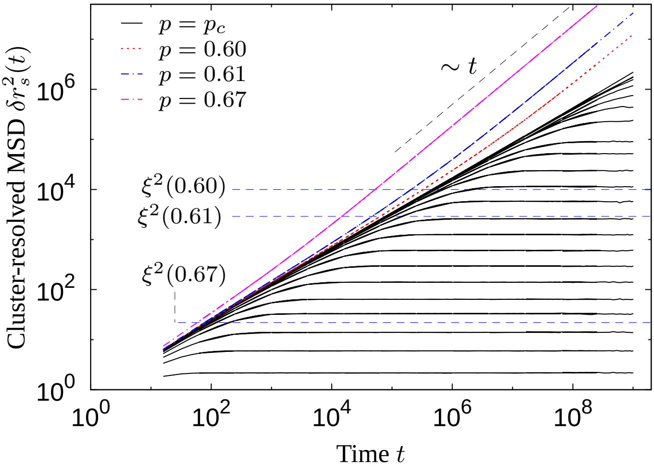
mean-square displacement





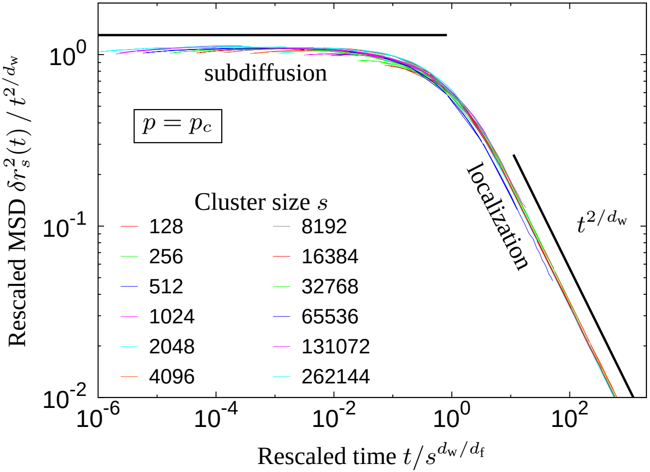
Scaling: $\delta r^2(t;\varepsilon) = t^{2/z}\delta\hat{r}^2_{\pm}(t\ell^{-z})$

cluster resolved transport



$$\delta r_s^2(t \to \infty) = R_s^2 \sim s^{2/d_f} \qquad \delta r_\infty^2(t) \sim t^{2/d_w}$$

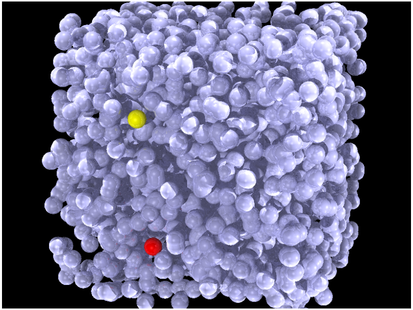
cluster resolved transport



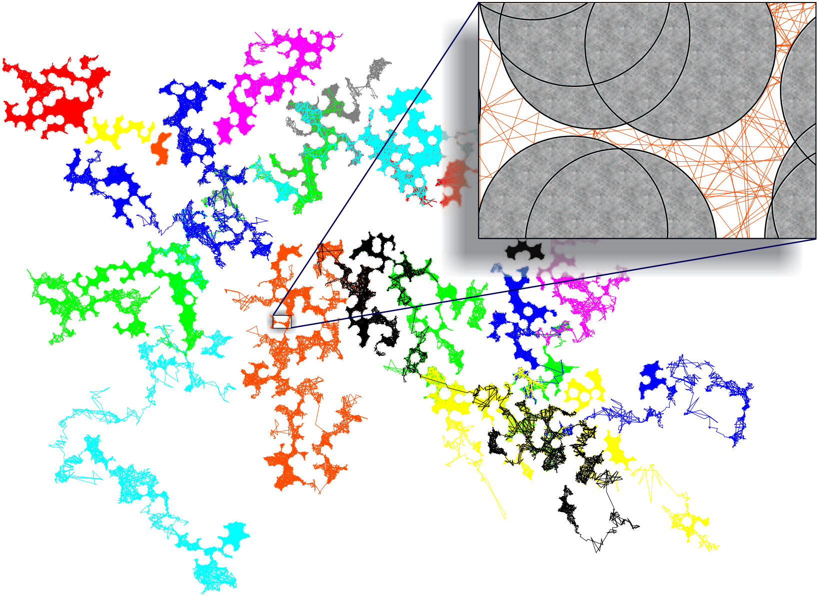
Scaling: $\delta r_s^2(t) = t^{2/d_\mathrm{w}} \delta \hat{r}_\pm^2(t s^{-d_\mathrm{w}/d_\mathrm{f}})$

Lorentz Model

- classical gas of non-interacting, structureless particles
- randomly distributed, fixed obstacles:
  → overlapping hard spheres
  Swiss Cheese model

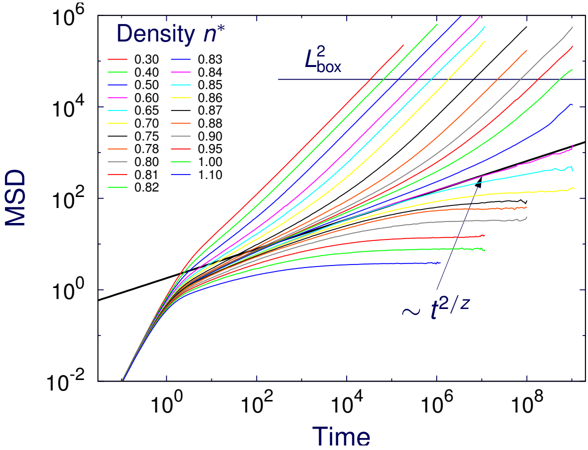


- ballistic motion, elastic scattering
  or Brownian motion
- relevant for transport in disordered media
- single control parameter: reduced obstacle density $n^* = n\sigma^3$ $(d = 3)$



Mean-Square Displacement

$$\delta r^2(t) = \left\langle |\mathbf{R}(t) - \mathbf{R}(0)|^2 \right\rangle$$ (three dimensional system)

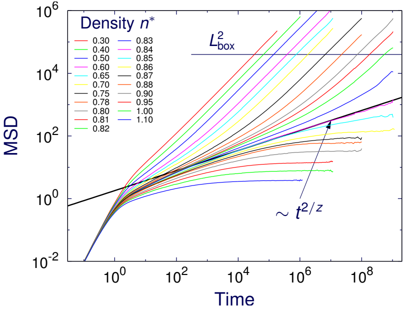


F. Höfling, T. Franosch, E. Frey, PRL **96**, 165901 (2006)

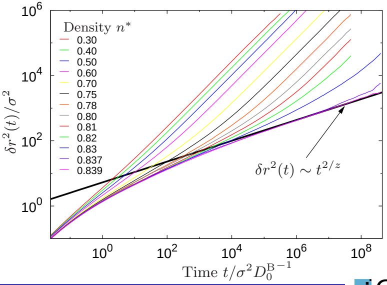
Mean-Square Displacement

- two regimes for $t \to \infty$
  - $n^* < n_c^*$ → Diffusion $\delta r^2(t) \simeq 6Dt$
  - $n^* > n_c^*$ → Localization $\delta r^2(t) \simeq \ell^2$
- close to n_c^*: intermediate time window until $\delta r^2(t) \approx \ell^2$
  → subdiffusive motion, $\delta r^2(t) \sim t^{2/z}$
- at $n^* = 0.84 \approx n_c^*$:
  - anomalous diffusion five time decades
  - dynamic exponent $z \approx 6.25$

ballistic

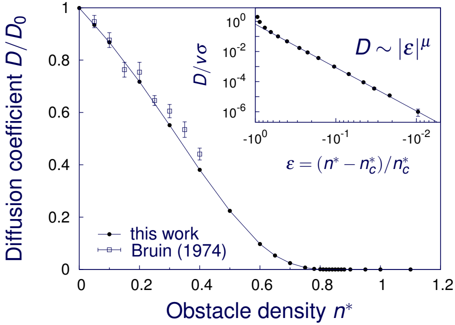


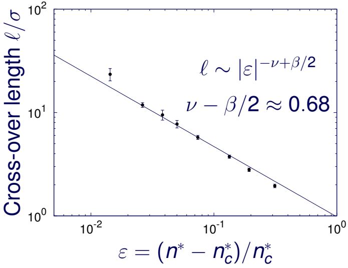
Brownian



Diffusion Coefficient

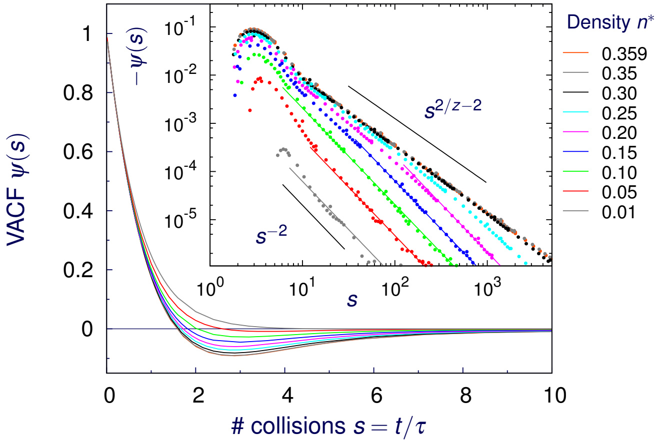
- D vanishes as $D \sim |\varepsilon|^{\mu}$, exponent $\mu \approx 2.88$
- ℓ diverges as $\ell \sim |\varepsilon|^{-0.68}$
- critical density: $n_c^* = 0.839(4)$, $\phi_c = 0.9702(5)$





$D_0 = v\sigma/3\pi n\sigma^3$ (Boltzmann)

VACF – two dimensions



F. Höfling, T. Franosch, Phys. Rev. Lett. **98**, 140601 (2007)

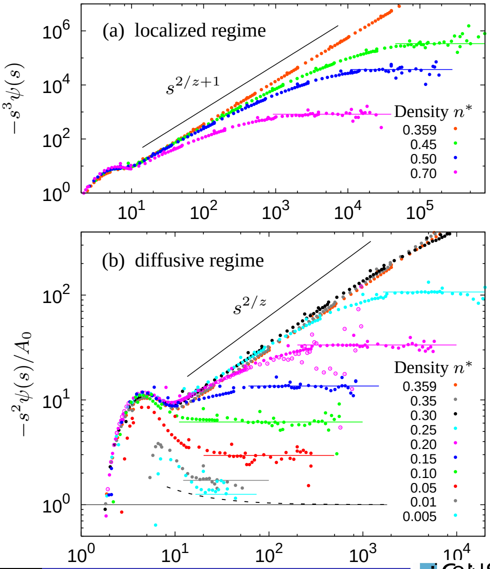
- Noise level 10^{-7} (!), power-law over several decades
- density-dependent exponents or crossover scenario?

VACF – rectification

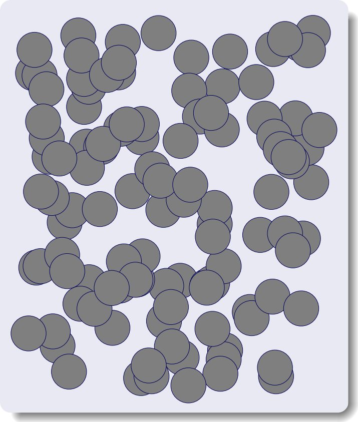
- crossover scenario
- cancellation at intermediate density, $n^* \approx 0.1$
  Alder and Alley (1978)
- long-time tails in the localized regime due to power-law distributed exit rates from cul-de-sac

$$\psi(t) \sim t^{-3} \quad (d = 2)$$

Machta and Moore (1985)



Mapping to a Percolating Network

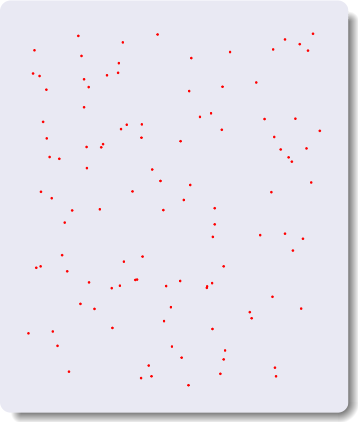


- Voronoi tessellation of the obstacle centers
- gap width → bond strength W
- bonds crossing the obstacles have zero strength
- random resistor network with fractal distribution of weak bonds,

$$\rho(W) \sim W^{-\alpha} \text{ for } W \to 0$$

Elam, Kerstein, and Rehr (1984)

Mapping to a Percolating Network



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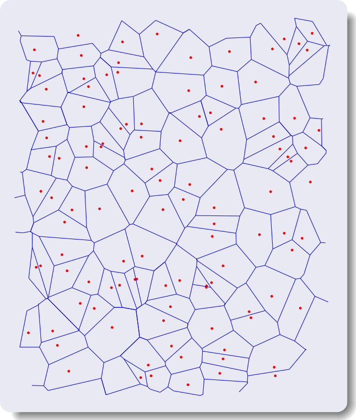

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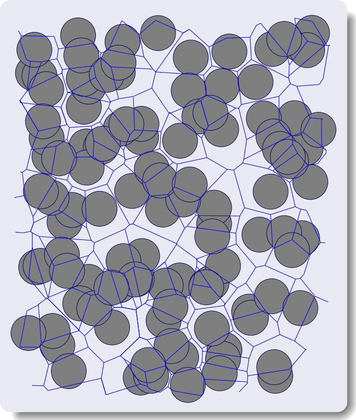

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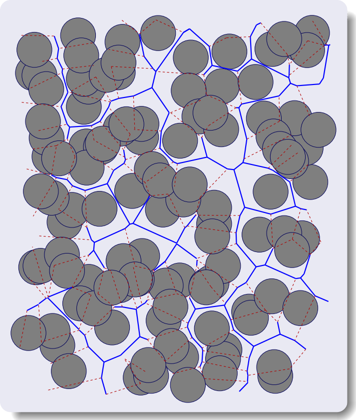

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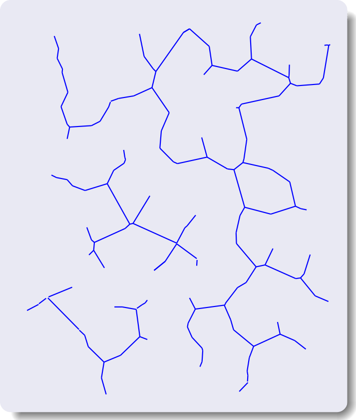

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Elam, Kerstein, and Rehr (1984)

Results of Continuum Percolation Theory

- critical exponents:
  $$P \sim |n - n_c|^{\beta}, \qquad \xi \sim |n - n_c|^{-\nu}, \qquad D \sim |n - n_c|^{\mu}$$
- mean-cluster radius: $\ell \sim |n - n_c|^{-\nu+\beta/2}$
- scaling relation: $z - 2 = \mu/(\nu - \beta/2)$
- geometric exponents ν and β are universal for lattice and continuum percolation Elam, Kerstein, and Rehr (1984)
- dynamical exponents as z and μ not
  weak conductances dominate or irrelevant
  Halperin, Feng, and Sen (1985)
- hyperscaling relation:

$$\mu = (d-2)\nu + 1/(1-\alpha) \; > \; \mu^{\text{lat}}, \qquad (\alpha \text{ sufficiently large})$$

Straley (1982); Stenull and Janssen (2001)

Results of Continuum Percolation Theory

- critical exponents:
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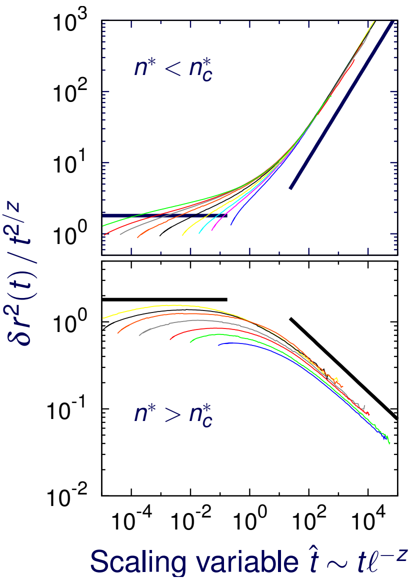
- dynamical exponents as z and μ not
  weak conductances dominate or irrelevant
  Halperin, Feng, and Sen (1985)
- hyperscaling relation (3D Lorentz model):
  $$\mu = \nu + 2 \qquad (\alpha = \tfrac{1}{2})$$
  Machta and Moore (1985)

Testing the Dynamic Scaling Ansatz



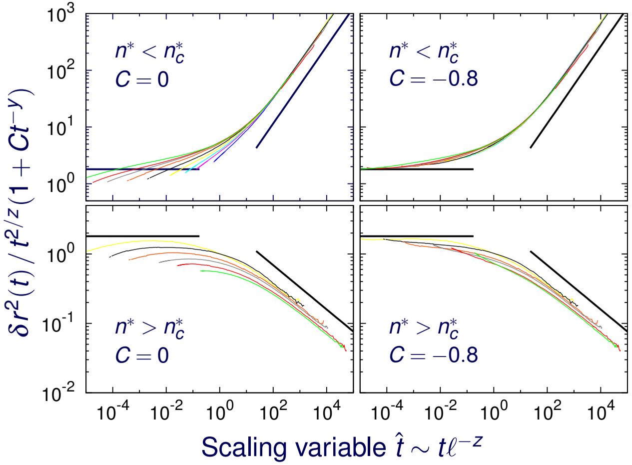
$$\delta r^2(t;\varepsilon) = t^{2/z}\delta\hat{r}^2_\pm(\hat{t})$$

- excellent data collapse in the diffusive regime
- rapid convergence towards large-$\hat{t}$ asymptotes
- small $\hat{t}$: asymptotic convergence as $n^* \to n_c^*$
- corrections to scaling relevant for $\hat{t} \ll 1$
- apparent density-dependent exponents

Corrections to Scaling

corrections to scaling approximately

$$\delta r^2(t;\varepsilon) = t^{2/z}\delta\hat{r}^2_{\pm}(\hat{t})(1 + Ct^{-y})$$



- new universal correction exponent y
- data at $n^* = 0.84$: $0.15 \lesssim y \lesssim 0.4$
- scaling plots for $y = 0.34$ and $C = -0.8$

Corrections to scaling

- Corrections to scaling for the cluster distribution at criticality

$$n_s(\varepsilon = 0) = s^{-d/d_\mathrm{f}-1}[A + Bs^{-\Omega}]$$

- Extensive Monte Carlo Simulation for Lattice Percolation
  $\Omega = 0.64 \pm 0.02$
- Extended scaling hypothesis

$$y = \frac{d_\mathrm{f}}{d_w}\Omega = \frac{\nu d - \beta}{z(\nu - \beta/2)}\Omega$$

→ $y = 0.34$

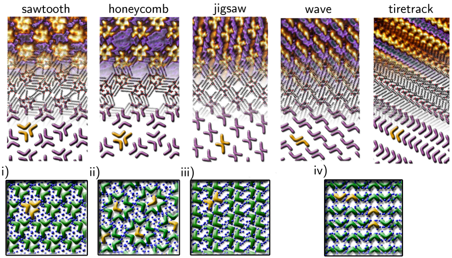
Anomalous Transport

- Subdiffusion quite common
  - heterogeneous media, strong size disparities
  - obstructed motion
- Lorentz model generic model for anomalous transport
  - Origin: fractal nature of the clusters
  - continuum percolation, random resistor network
  - mean-square displacement, non-Gaussian parameter
  - also: VACF (long-time tails), non-fickian diffusion, van Hove correlation function, intermediate scattering function,
- exponents and scaling
  - universality fixes exponents
  - large crossover regimes
    → apparent density-dependent exponents
  - analogy to molecular crowding

Outlook

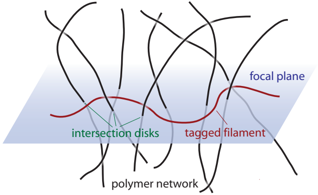
Further Research interests

- molecular self-assembly



C. Rohr, M. Balbas-Gambra,
K. Gruber, E. Constable, E. Frey,
T. Franosch, B. Hermann

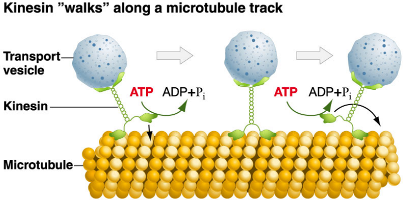
- entangled dynamics of a stiff rod

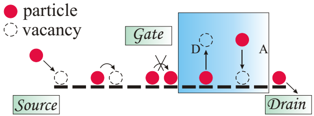


F. Höfling, E. Frey, and T. Franosch, PRL **101**, 120605(2008)
T. Munk, F. Höfling, E. Frey, and T.Franosch, EPL **85**, 30003 (2009)

Further Interests – Driven Transport

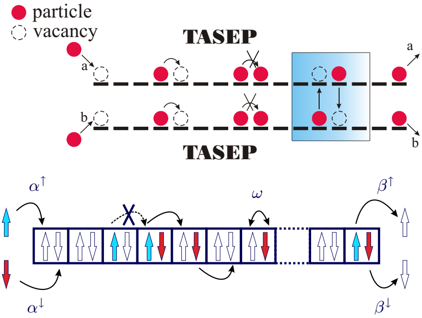
Molecular Motors





A. Parmeggiani, T. Franosch, E. Frey,
PRL **90**, 086601 (2003)

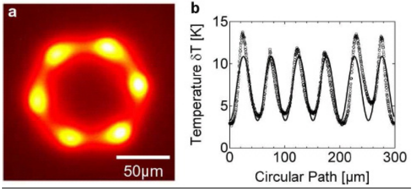
Classical Spin Transport

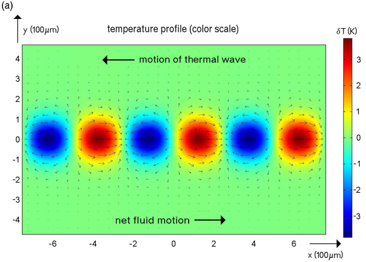


T. Reichenbach, T. Franosch, E. Frey,
PRL **97**, 050603(2006).

Further Interests – Hydrodynamics at Microscales

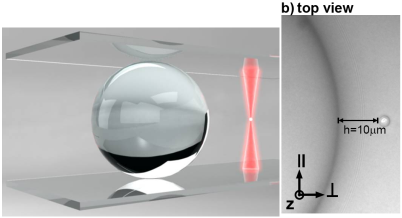
Light-induced fluid flow

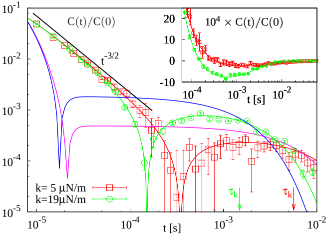




F.M. Weinert, J.A. Kraus, T. Franosch, D. Braun, PRL (2008)

Brownian motion





S. Jeney, B. Lukić, J.A. Kraus, T. Franosch, L. Fórro, PRL (2008)

Collaborations – Acknowledgement

- Anomalous transport
  - Felix Höfling, Tobias Munk, Axel Kammerer (LMU München)
  - Joachim Rädler, Margaret Horton, Doris Heinrich (LMU München)
  - Thomas Voigtmann, Jürgen Horbach, Matthias Sperl, Andreas Mayer (DLR Cologne)
- Nonequilibrium phase transitions
  - Andrea Parmeggiani (Montpellier 2), Paolo Pierobon (Insitut Curie), Tobias Reichenbach, Mauro Mobilia, Anna Melbinger (LMU München)
- Order phenomena and self-assembly
  - Andreja Šarlah (University of Ljubljana)
  - Bianca Hermann, Marta Balbás Gambra LMU München
- Fluid dynamics on micro- and nano-scales
  - Franz Weinert, Dieter Braun, Jonas Kraus LMU München
  - Branimir Lukić, Sylvia Jeney EPFL

Lehrstuhl Erwin Frey LMU München

Appendix

Exponents in percolation

Two dimensional lattice percolation

- fractal dimension $d_f = 91/48 \approx 1.90$
- correlation length $\nu = 4/3$
- infinite cluster $\beta = 5/36$
- Fisher exponent $\tau = 187/91$

Acknowledgement – Collaboration

- Non-equilibrium Transport
  - Andrea Parmeggiani (CNRS-Université Montpellier 2)
  - Paolo Pierobon, Tobias Reichenbach, Mauro Mobilia, Anna Melbinger (LMU München)
- Rods & Needles
  - Felix Höfling (Hahn-Meitner Institute, LMU München)
  - Tobias Munk (LMU München)
- Ordering in molecular crystals
  - Andreja Šarlah (University of Ljubljana), Clemens Bechinger (University of Stuttgart)
  - Bianca Hermann, Marta Balbás Gambra LMU München (new)
- Fluidics at the microscale (new)
  - Dieter Braun, Jonas Kraus LMU München, Sylvia Jeney EPFL

and for collaboration and continuous support

Erwin Frey LMU München

Van Hove self-correlation function

$$G(\mathbf{r}, t) := \left\langle \delta(\mathbf{R}(t) - \mathbf{R}(0) - \mathbf{r}) \right\rangle$$

- probability distribution of the particle positions
- diffusive peak vanishes
- sharp peak at a definite distance remains

vanHove-0,75

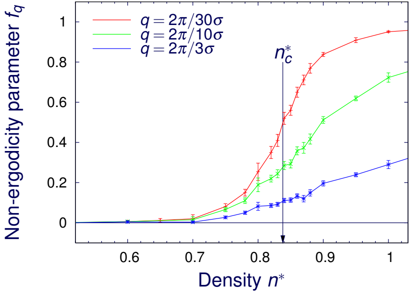
consider long-time limit:

- $G(\mathbf{r}, t \to \infty) \equiv 0$ → all particles can diffuse away
- finite peak → localization of some particles
- coexistence of localized and diffusing particles below n_c^*, phase space is decomposed into finite and infinite subsets

Non-ergodicity Parameters f_q

$$\Phi^s_{\mathbf{q}}(t) := \left\langle e^{i\mathbf{q}\cdot(\mathbf{R}(t)-\mathbf{R}(0))} \right\rangle$$

- Fourier transform of $G(\mathbf{r},t)$
- incoherent inelastic scattering function $\Phi^s_{\mathbf{q}}(t)$
- non-ergodicity parameter: $f_q := \Phi^s_q(t\to\infty)$



- $f_q > 0$ at *all* densities
  → phase space is always decomposed
- transition affects f_q in next-to-leading order only:
  $$f_q \sim const + |\varepsilon|^\beta$$

Kertész and Metzger (1983)

Non-ergodicity Parameters f_q

$$\Phi^s_{\mathbf{q}}(t) := \left\langle e^{i\mathbf{q}\cdot(\mathbf{R}(t)-\mathbf{R}(0))} \right\rangle$$

- Fourier transform of $G(\mathbf{r},t)$
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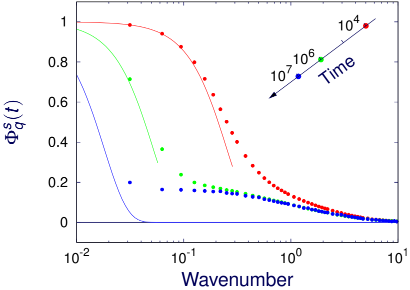
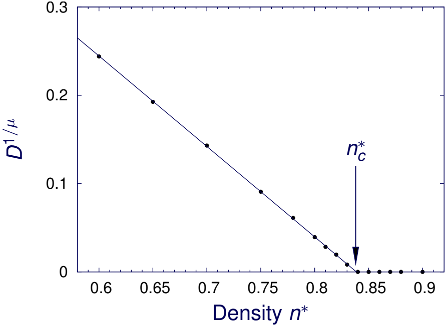
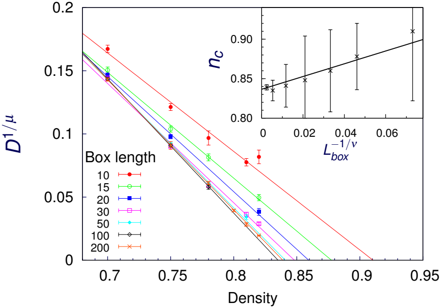


- Gaussian approximation:

  $\Phi^s_q(t) \approx e^{-Dq^2 t}$ ➔ f_q should vanish in diffusive systems
- in presence of non-Gaussian corrections: valid only for $q \ll 1/\sqrt{Dt}$ ➔ breaks down as $t \to \infty$

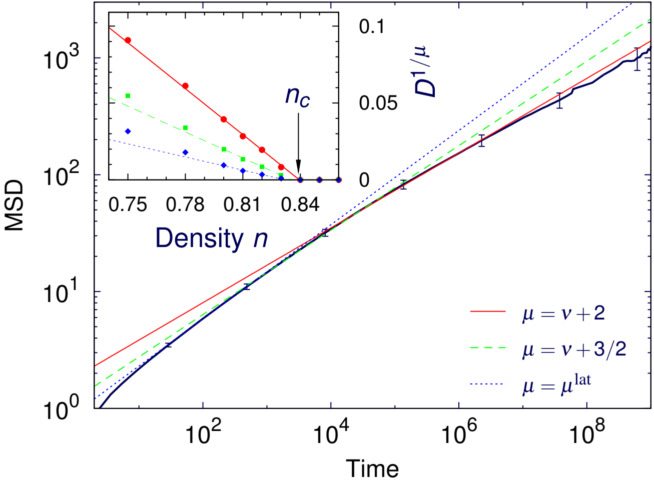
Fitting n_c^*





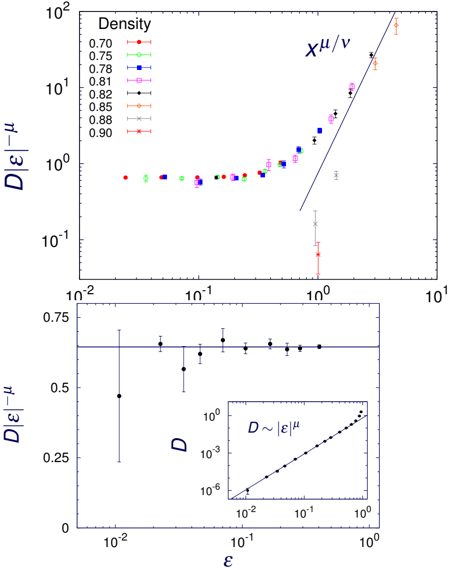
μ	n_c^*	Δn_c^*
2.87	0.8388	0.0041
2.88	0.8390	0.0040
2.89	0.8392	0.0040

Exponents



- $\mu = (d-2)\nu + 1/(1-\alpha)$
- compatible only with Machta and Moore
- $\alpha = 1/2$

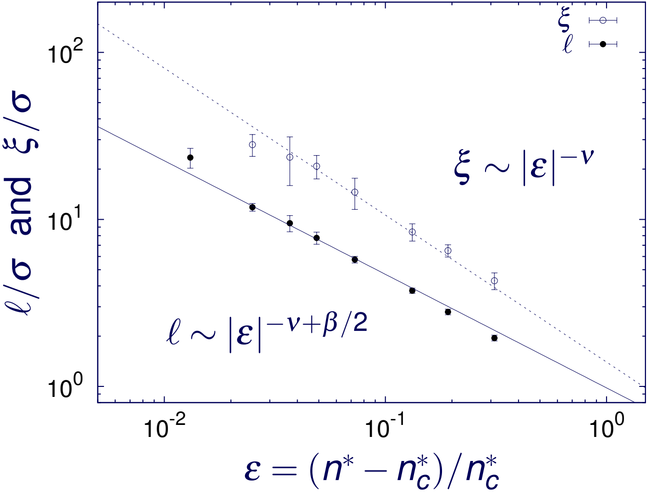
Finite Size Effects



- Finite-size scaling prediction

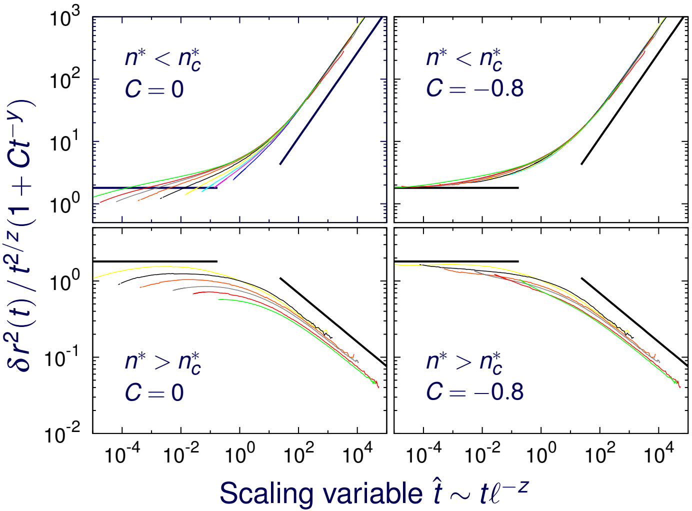
$$D(\varepsilon; L) = |\varepsilon|^{\mu}\hat{D}^{\pm}(\xi/L)$$

  - large boxes
    $D(\varepsilon < 0; L \gg \xi) \sim |\varepsilon|^{\mu}$
  - small boxes
    $D(\varepsilon; L \ll \xi) \sim L^{-\mu/\nu}$



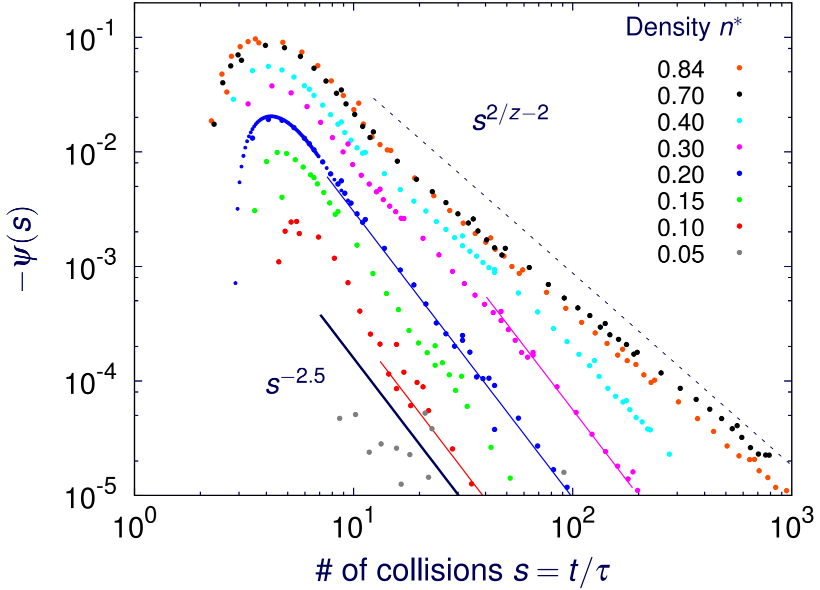
- Divergence of the correlation length
- ξ extracted from non-gaussian parameter

Corrections to Scaling



Correction-to-scaling exponent $y = 0.34$

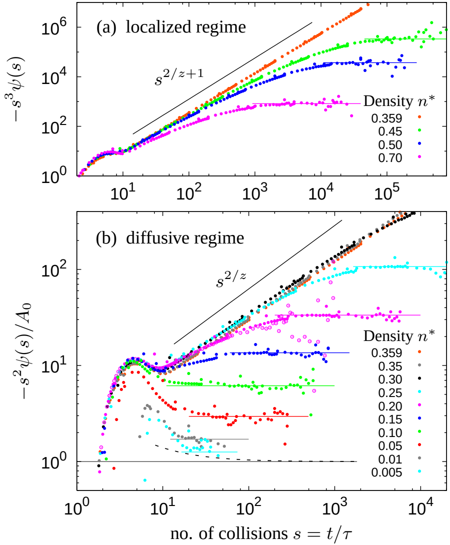
VACF in 3d



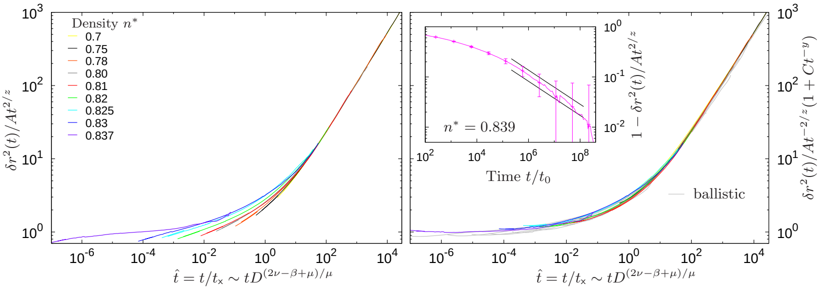
Density n^*
0.84
0.70
0.40
0.30
0.20
0.15
0.10
0.05
$s^{2/z-2}$
$s^{-2.5}$
$-\psi(s)$
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{0}
10^{1}
10^{2}
10^{3}
# of collisions $s = t/\tau$

Long-time tails

- rectification, sensitive test
- crossover scenario
  Götze, Leutheusser, and Yip (1981)
- cancellation effects
- growth close to n_c
- low-density: difficult
- predicted tail in localized regime $\psi(t) \propto t^{-3}$
  Machta and Moore (1985)
- prediction for super-Burnett, non-gaussian parameter
- 3d ..

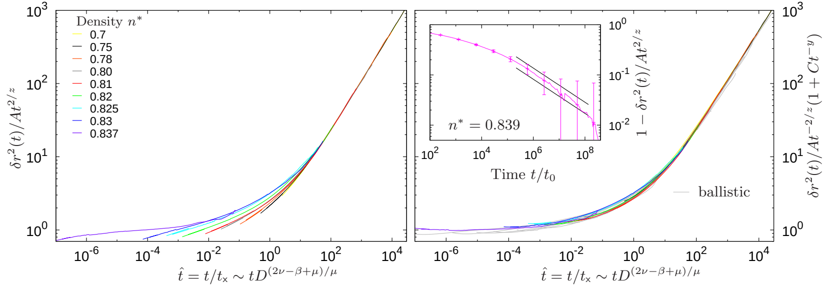


Testing the Dynamic Scaling Ansatz



- leading order scaling
  $\delta r^2(t;\varepsilon) = t^{2/z}\delta\hat{r}^2_{\pm}(\hat{t})$
- excellent data collapse in the diffusive regime
- small $\hat{t}$: asymptotic convergence as $n^* \to n^*_c$

Testing the Dynamic Scaling Ansatz



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- excellent data collapse in the diffusive regime
- small $\hat{t}$: asymptotic convergence as $n^* \to n^*_c$

- extend scaling by irrelevant coupling:
  $$\delta r^2(t;\varepsilon) = t^{2/z}\delta\hat{r}^2_\pm(\hat{t})\left[1 + t^{-y}\Delta_\pm(\hat{t})\right]$$
- new universal correction exponent y
- at criticality ($\hat{t} = 0$): $\delta r^2(t;\varepsilon) \propto t^{2/z}(1 + Ct^{-y})$
- approximate correction function $\Delta_\pm(\hat{t}) = C$
- scaling plots for $y = 0.34$

Non-Gaussian Parameter

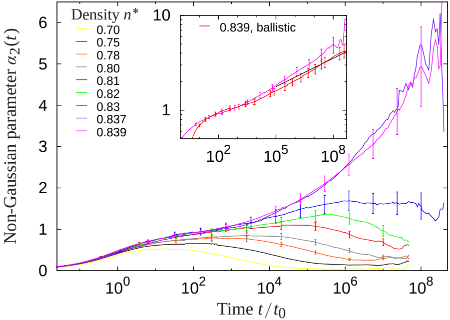
- mean-quartic displacement, $\delta r^4(t;\varepsilon) = \int \mathrm{d}\mathbf{r}\, r^4 G(\mathbf{r},t;\varepsilon)$:

$$\delta r^4(t) \sim t^{4/\tilde{z}} \quad (\varepsilon = 0), \qquad \delta r^4(t\to\infty) \sim \begin{cases} \xi^2 \ell^2 & (\varepsilon > 0) \\ (Dt)^2 |\varepsilon|^{-\beta} & (\varepsilon < 0) \end{cases}$$

  - different exponent $\tilde{z} \approx 5.4 \neq z$

$$\alpha_2(t) := \tfrac{3}{5}\,\delta r^4(t)/[\delta r^2(t)]^2 - 1$$

- sensitive to heterogeneities
- diffusive regime: $\alpha_2(\infty) > 0$
- critical law:
  $\alpha_2(t) \sim t^{4/\tilde{z} - 4/z} \sim t^{0.097}$
  divergent



Brownian particles

Dynamic Scaling Ansatz

- van Hove self-correlation function $G(\mathbf{r},t) = \langle \delta(\mathbf{R}(t) - \mathbf{R}(0) - \mathbf{r}) \rangle$
  → Probability to travel distance $\mathbf{r}$ in time t

$$G(\mathbf{r},t;\varepsilon) = \xi^{-\beta/\nu - d} \mathscr{G}_{\pm}(\mathbf{r}/\xi, t\ell^{-z})$$

- two diverging length scales:
  - correlation length $\xi \sim |\varepsilon|^{-\nu}$ rescales geometry
  - cross-over length $\ell \sim |\varepsilon|^{-\nu+\beta/2}$ rescales time
  - three non-trivial exponents → no CTRW, fractional FP, ...
- scaling ansatz for the MSD from $\delta r^2(t;\varepsilon) = \int \mathrm{d}\mathbf{r}\, r^2 G(\mathbf{r},t;\varepsilon)$:

$$\delta r^2(t;\varepsilon) = t^{2/z} \delta\hat{r}^2_{\pm}(\hat{t}), \qquad \text{where} \quad \hat{t} \sim t\ell^{-z}$$

  - critical dynamics ($\hat{t} = 0$) recovered: $\delta r^2(t) \sim t^{2/z}$.

Corrections to Scaling

- extend scaling ansatz by including an irrelevant coupling:

$$\delta r^2(t;\varepsilon) = t^{2/z}\delta\hat{r}^2_\pm(\hat{t})\big(1 + t^{-y}\Delta_\pm(\hat{t})\big)$$

- new universal correction exponent y
- at criticality $(\hat{t} = 0)$: $\quad \delta r^2(t;\varepsilon) \propto t^{2/z}\big(1 + Ct^{-y}\big)$
- data at $n^* = 0.84$: $0.15 \lesssim y \lesssim 0.4$
- approximate correction function $\Delta_\pm(\hat{t}) = C$
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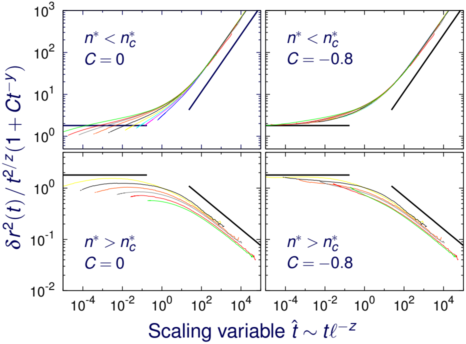

Lorentz Model

- Molecular Dynamics simulations
  - first accurate data in the relevant regime
  - mean-square displacement, non-Gaussian parameter
  - also: VACF (long-time tails), non-fickian diffusion, van Hove correlation function, intermediate scattering function,
- anomalous transport over several decades
  - Origin: fractal nature of the clusters
  - continuum percolation
  - universality class of random resistor networks
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    → apparent density-dependent exponents
  - significant corrections to scaling
  - analogy to molecular crowding