Critical Binder cumulant in two-dimensional Ising models

1. **Background**

   Ising model and Binder cumulant

2. **Some Previous Results**

   Universal and non-universal features of the critical Binder cumulant

3. **Present Results**

   - Isotropic nn Ising model with various boundary conditions and on different lattice types
   - Square lattice with anisotropic nn and nnn couplings
   - Critical Binder cumulant and Wulf shape

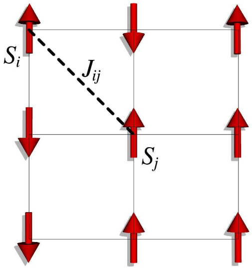
W.S. and L.N. Shchur, J. Phys. A 38, L739 (2005)

W.S., Eur. Phys. J. B 51, 223 (2006)

Ising model

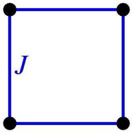
$$\mathcal{H} \;=\; -\sum J_{ij} S_i S_j$$

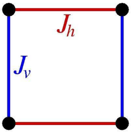
$$S_i, S_j \;=\; \pm 1$$

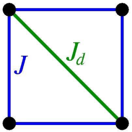


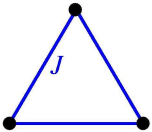
Square lattice

Triangular lattice









isotropic

anisotropic

rectangular symmetry

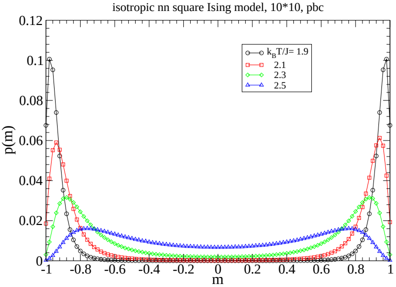
nnn interactions along one diagonal

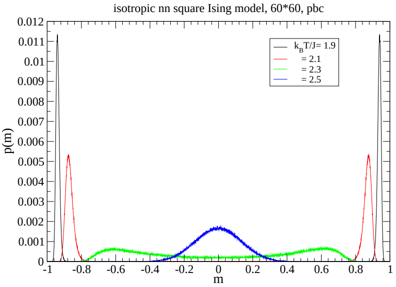
$\widehat{=}$ square lattice with $J_d = J$

Magnetization histograms

Distribution function of the magnetization $p(m) = \frac{1}{\mathcal{Z}} \sum_K e^{-\mathcal{H}(K)/k_B T}$

K: configurations with magnetization m





isotropic nn model, $L \times L$ sites, periodic boundary conditions, near critical temperature

$k_B T_c / J = 2.269...$

Observations: $T < T_c$: two-peak structure

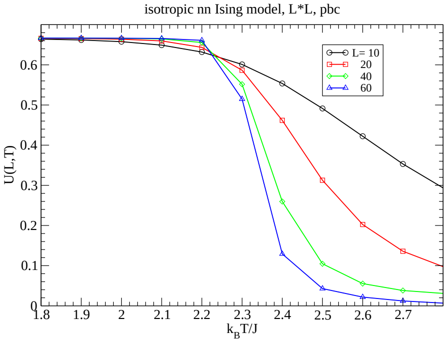
$T > T_c$: $p(m)$ approaching Gaussian function for larger L

Characterization of distributions by moments and/or cumulants:

Binder cumulant

Fourth-order cumulant of the distribution function of the magnetization

$$U \;=\; 1 \;-\; \frac{< M^4 >}{3 < M^2 >^2} \qquad \text{(Binder, 1981)}$$



Binder cumulant in the thermodynamic limit:

$$U(T) \;=\; \begin{cases} \frac{2}{3} & T < T_c \\ U^* & T = T_c \\ 0 & T > T_c \end{cases}$$

From crossing points of $U(L_1) = U(L_2)$, one may estimate conveniently T_c;

U^*: critical Binder cumulant (universal?)

$U^* = 0.61069...$ (Kamieniarz+Blöte, 1993)

isotropic nn Ising model, square shape, periodic boundary conditions

PREVIOUS results: Isotropic nn square lattice Ising model with square shape

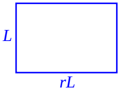
- U^* has been found to be,
  employing periodic bc, INDEPENDENT of:
  + spin value
  S=1/2, S=1 (Nicolaides and Bruce, 1988)
  + discrete or continuous nature of (Ising-type) spin variable
  Nicolaides and Bruce, 1988; Kamieniarz and Blöte, 1993

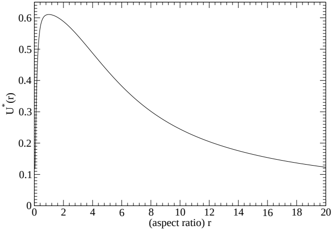
- On the other hand, for the isotropic nn square Ising model,
  U^* has been observed to DEPEND on
  + boundary conditions
  periodic, free, fixed,...boundaries
  K.Binder, D.W. Heermann, W.Janke, D.P.Landau, A. Milchev,...(scattered results)

Varying shape, lattice type, and anisotropy

With periodic bc, U^* has been found/argued to DEPEND on

- Shape of the lattice





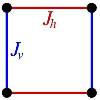
$U^*(r)$: isotropic, square nn Ising model with rectangular shape, aspect ratio r; exact calculations augmented by finite-size extrapolations

Kamieniarz and Blöte (K+B), 1993

- Lattice type(?)

  Slightly different values of U^* for isotropic nn Ising models on square lattice with square shape and on triangular lattice with rhombus shape (K+B, 1993)

- Anisotropy of nn couplings, J_v/J_h:



$U^* = U^*(J_v/J_h)$

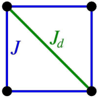
for square shapes, $r = 1$; with a mapping onto the isotropic nn Ising model with rectangular shape so that

$U^*(r = 1, J_v/J_h) = U^*(r, J_v/J_h = 1)$ where

$\sinh(2J_h/k_B T_c(J_v/J_h)) = r$,

(which follows from setting $r = \xi_v/\xi_h$; ξ correlation length) (K+B, 1993)

- Anisotropy of nnn couplings, J_d:



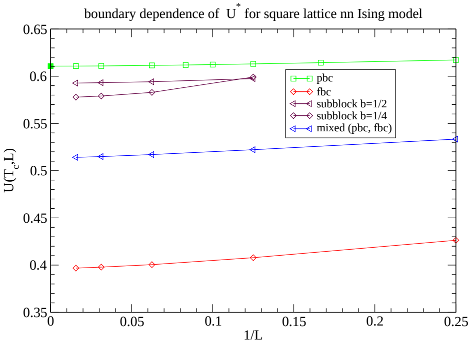
$U^* = U^*(J_d/J)$,

but there is no mapping

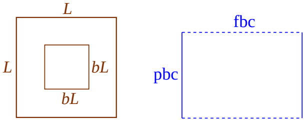
$U^*(J_d/J, r = 1) = U^*(0, r)$, which would keep rectangular symmetry (Chen+Dohm, 2004)

Our aim: Monte Carlo study on those (non)universal aspects of U^*

PRESENT results: isotropic nn Ising model on square lattice – various boundary conditions

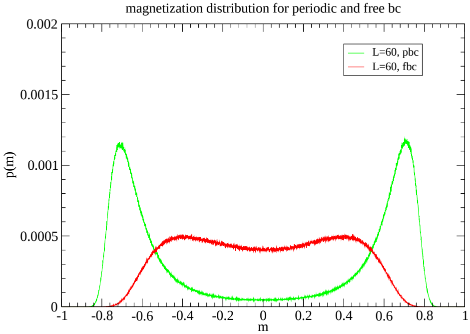


Cumulant at T_c for square lattice with free, periodic, mixed and other boundary conditions



Subblocks (Binder, 1981): squares of size $bL * bL$, embedded in square of $L * L$ sites; 'heat bath bc' when $b \longrightarrow 0$, with $U^* = 0.560 \pm 0.002$

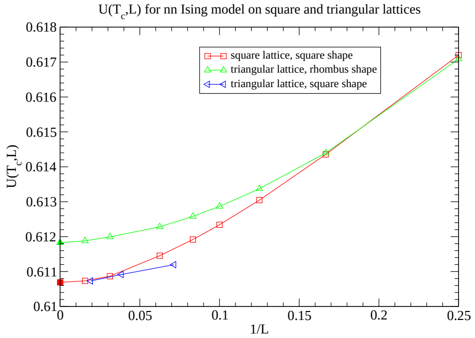
Mixed bc: pbc for two opposite sides, fbc for the other two opposite sites of squares of size L^2



Cumulant at T_c for square lattice, $L = 60$, with periodic and free boundary conditions

Note: less pronounced two-peak-distribution for free boundary conditions, and U^* is smaller than in the case of periodic boundary conditions

Isotropic nn Ising model on triangular and square lattices

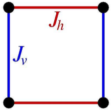


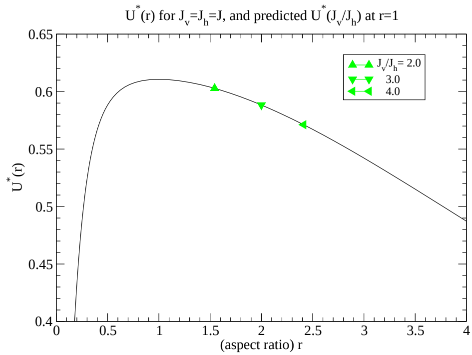
$U(T_c, L)$ for isotropic nn Ising model on square lattice (square shape), triangular lattice (rhombus shape), and triangular lattice (square shape)

W.S., E. P.J.B, 2006; Lübeck, W.S.+Hucht (in preparation)

Suggestion: For given shape, isotropic models lead to the same U^* (checked, in addition, for other rectangular shapes)

Square lattice with different nn (horizontal and vertical) couplings





PREDICTION:

$U^*(J_v = J_h, r)$=

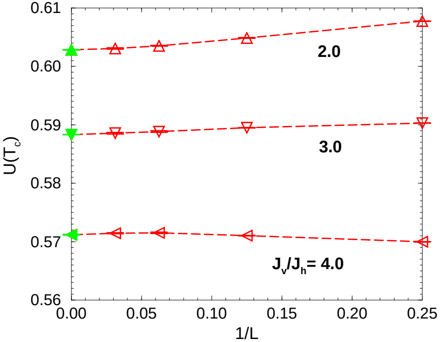
$U^*(J_v/J_h, r = 1)$ such that:

$\sinh\left(2J_h/k_B T_c\right) = r$

(K+B, 1993)

To be checked by MC simulations:

Checking the prediction

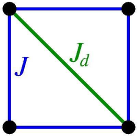


Cumulant $U(T_c, L)$ for aniso-tropic nn Ising model, L^2 spins $(r = 1)$, and various J_v/J_h, as compared to previous findings on $U^*(J_v/J_h = 1, r)$ in the isotropic case, presuming the mapping

Our findings confirm the predicted mapping

$U^*(J_v/J_h, r = 1) = U^*(J_v/J_h = 1, r)$ with $\sinh(2J_h/k_BT_c) = r$

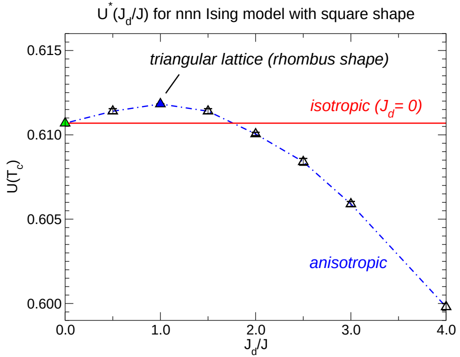
Square lattice with anisotropic nnn interactions



Statements of Chen and Dohm (2004):

- $U^*(J_d/J, r = 1)$ varies continuosly with J_d/J
- Keeping rectangular symmetry, there is no mapping of U^* onto the isotropic case such that $U^*(r = 1, J_d/J) = U^*(r, J_d/J = 0)$
- In general: Violation of 'two-scale-factor universality' due to anisotropy

Checking by simulations



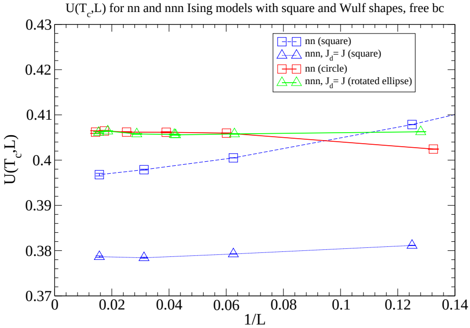
Simulated U^* for nnn Ising model on square lattice with r=1, including nn isotropic case, $J_d = 0$

Overshooting: No mapping with $U^*(J_d/J, r = 1) = U^*(J_d/J = 0, r)$

Note: U^* of square lattice nnn Ising model, $J_d = J$, with square shape is identical to that of the triangular lattice nn Ising model with rhombus shape (similarily for other ratios of J_d/J (Dohm, 2006)).

Question: Can dependence of U^* on anisotropy be transcribed, in general, into dependence on shape?

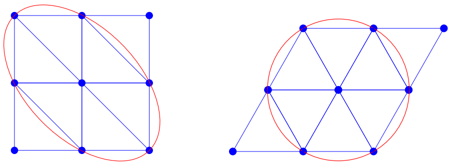
Critical Binder cumulant in (an)isotropic systems and Wulf shape



Wulf shapes at T_C with free bc : U^* for nn isotropic square Ising model with **circle** shape and nnn anisotropic, $J_d = J$, case with **rotated ellipse** shape, having L^2 sites.- For comparison: nn and nnn cases, free bc, with square shapes, L^2 sites.

Recall: The equilibrium Wulf shape, at T_c, of an Ising droplet results from the orientational dependence of the surface free energy at criticality, reflecting the interactions.

Note:



There are the same spins in the rotated ellipse for the anisotropic nnn, $J_d = J$, Ising model on the square lattice and in the circle for the nn Ising model on the triangular lattice;

thence U^*(nnn,sq,ellipse)=U^*(iso nn,tria,circle)= U^*(iso nn,sq,circle)

Question(suggestion):

Does U^* take a generic/unique value when one considers systems (free bc) with their Wulf shape at criticality ?

Summary

- The critical Binder cumulant U^* in 2d Ising models depends on boundary conditions, system shapes, anisotropy of interactions.
- For **isotropic** models, U^* depends on shape and boundary conditions, but not on details of interactions and lattice type.
- For given boundary condition, the dependence of U^* on ANISOTROPY may be mapped onto a dependence on the SHAPE: verification for the nn anisotropic case, keeping rectangular symmetry; evidence for the nnn anisotropic Ising model, considering rhombus (parallelogram) shapes.
- **Question**: Can a generic/unique value of U^* be obtained for Ising models with a shape following from the Wulf construction at criticality, using, e.g., free boundary conditions?