§ 7. Symbol spaces and oszillatory integrals

mit konst. Koeff.

Motivation: $D : \underset{\mathcal{D}(Y)}{\overset{\|}{C_0^\infty(Y)}} \to \underset{\mathcal{D}(X) \subset \mathcal{D}'(X)}{\overset{\|}{C_0^\infty(X)}}$ differential operator

$$\langle Du, v\rangle = \Big\langle \int_{\mathbb{R}^q} e^{i\langle \cdot,\xi\rangle} \sigma_D(\cancel{x},\xi)\,\hat{u}(\xi)\,d\xi, v\Big\rangle$$

$$= \int_{\mathbb{R}^q} \Big[\int_{\mathbb{R}^q} e^{i\langle x,\xi\rangle} \sigma_D(\cancel{x},\xi)\,\hat{u}(\xi)\,d\xi \Big] \cdot v(x)\,dx$$

= (writing out Fourier - integral for $\hat{u}$)

$$\int_{\mathbb{R}^q} \left(\int_{\mathbb{R}^q} \sigma_D(\cancel{x}\xi) \left(\int_{\mathbb{R}^q} e^{i\langle x-y,\xi\rangle} u(y)\,dy \right) d\xi \right) v(x)\,dx$$

$$\text{"} = \int_{\mathbb{R}^q\times\mathbb{R}^q} \Big[\underbrace{\int_{\mathbb{R}^q} e^{i\langle x-y,\xi\rangle} \overset{\sim |\xi|^{\alpha}}{\sigma_D(x,\xi)}\,d\xi}_{= k_D(x,y) \text{ Schwartz - kernel for } D} \Big] (v\otimes u)(x,y)\,dx\,dy \text{"}$$

Problems:

$$\boxed{\text{Integral for } k_D(x;y) \text{ does not converge}}$$

$\Rightarrow$ We now make such integrals rigorous ("oszill. integrals") and use it to understand Schwartz-kernels of diff. and pseudo-diff. operators.

Definition 7.1 (Symbol functions of Hörmander type (1,0))

$\Omega \subset \mathbb{R}^n$ open, $N \in \mathbb{Z}_+$ (often $N = n, n/2$) ; $m \in \mathbb{R}$ (or $\mathbb{C}$)

$$S^m(\Omega \times \mathbb{R}^N) := \{ a \in C^\infty(\Omega \times \mathbb{R}^N) \mid$$

$$\forall K \subset \Omega \text{ cpt } \forall \alpha \beta \; \exists C = C(\alpha, \beta, K) \; \forall (x,\xi) \in K \times \mathbb{R}^N :$$

$$(**) \qquad | \partial_x^\alpha \partial_\xi^\beta a(x,\xi) | \leq C \cdot (1 + \|\xi\|)^{m - |\beta|} \}$$

Definition 7.2 (Classical symbols)

$$CS^m(\Omega \times \mathbb{R}^N) := \{ a \in S^m(\Omega \times \mathbb{R}^N) \mid$$

- for each $j \in \mathbb{N}_0$ there exists $a_{m-j} \in C^\infty(\Omega \times \mathbb{R}^N)$ such that $a_{m-j}(x, \lambda\xi) = \lambda^{m-j} a_{m-j}(x,\xi)$ for $|\lambda| \geq 1$, $\|\xi\| \geq 1$.
- $\forall M \in \mathbb{N} : \left(a - \sum_{j=0}^{M} a_{m-j} \right) \in S^{m-M-1}(\Omega \times \mathbb{R}^N) \}$

$\text{ord}(a_m - a_{m-1}) \hat{=} m-2$

We write in short:

$$\boxed{a \sim \sum_{j \geq 0} a_{m-j}}$$

"asymptotic expansion of classical symbols"

Remarks 7.3

(a) The best constants $C(\alpha, \beta, K) [a \in S^m]$ in $(**)$ are seminorms on S^m w.r.t which S^m becomes a Frechet space.

(b) $m \leq m' \Rightarrow S^m \subseteq S^{m'}$

(ü) BUT: $CS^m \not\subseteq CS^{m'}$ i.A. The inclusion holds for classical symbols only if $(m' - m) \in \mathbb{Z}_+$.

(c) $CS^{-\infty} = S^{-\infty} := \bigcap_{m \in \mathbb{R} \text{ (or } \mathbb{Z})} S^m$

Note that a countable intersection of Frechet-spaces is again naturally a Frechet-space.

(d) Let $p(x,\xi) = \sum_{|\alpha| \le k} a_\alpha(x)\xi^\alpha$ with $a_\alpha \in C^\infty(\Omega)$

(be a symbol of a differential operator $\sum_{|\alpha|\le k} a_\alpha(x) D^\alpha$)

Then $p \in CS^K(\Omega \times \mathbb{R}^n)$ (CHECK (**))

(e) Let $a \in C^\infty(\Omega \times \mathbb{R}^N)$ be positively homogeneous ie $a(x,\lambda\xi) = \lambda^m a(x,\xi)$, $\lambda \ge 1$, $|\xi| \ge 1$

Then $a \in CS^m(\Omega \times \mathbb{R}^N)$ [ie $(a_{m-j}) \in CS^{m-j}$ in the Def 7.2].

(f) Previous definitions may be extended to cones $\Gamma \subset \mathbb{R}^N$ instead of $\mathbb{R}^N$ (this will be important later)

Topology of symbol spaces

Theorem 7.4 $\Omega \subset \mathbb{R}^n$ open, $m < m'$

On bounded subsets of $S^m(\Omega \times \mathbb{R}^N)$ the topology of $\underline{\underline{S^{m'}}}$ is the topology of pointwise convergence, ie

$(a_j) \subset S^m$ bded sequence that conv. to a ptwise

Then $a \in S^m$ (in particular smooth) and $a_j \xrightarrow[S^{m'}]{} a$.

Sketch of the proof: (detailed proof — excersize)

Let $K \subset \Omega$ be cpt. Since (a_j) is bded in S^m:

$$(*) \quad \forall \alpha\beta: \; |\partial_x^\alpha \partial_\xi^\beta a_j(x,\xi)| \le C(\alpha,\beta,K)\,(1+|\xi|)^{m-|\beta|}$$

ie all derivatives are uniformly bded on cpt subset of $\Omega \times \mathbb{R}^N$.

Arzela-Asoli $\Rightarrow \exists\, (a_{i_j})$, $a_{i_j} \xrightarrow[j\to\infty]{} a \in C^\infty(\Omega\times\mathbb{R}^N)$ in C^∞. a is also the pointwise limit of (a_j). In particular $(*)$ also holds for a and hence $a \in S^m(\Omega\times\mathbb{R}^N)$.

Remains to prove $a_j \xrightarrow[S^{m'}]{} a$. ☺ $\square$

Theorem 7.5 Let $m < m'$. In the topology of $S^{m'}$, the space $S^{-\infty}$ is dense in S^m.

Sketch of the proof: (detailed proof — excersize)

Let $a \in S^m(\Omega\times\mathbb{R}^N)$. We construct an approx. sequence:

$$\rho \in C_c^\infty(\mathbb{R}^N) \text{ with } \begin{array}{l} \rho(\xi) \equiv 1, \; |\xi| \le 1 \\ \rho(\xi) \equiv 0, \; |\xi| \ge 2 \end{array}$$

$$\text{Put } a_j(x,\xi) = \rho\left(\frac{\xi}{j}\right)\cdot a(x,\xi)$$

Clearly, $a_j \in S^{-\infty}$ since $a_j(x,\xi) = 0$ if $|\xi| \ge 2j$.

Remains to prove $a_j \xrightarrow[S^{m'}]{} a$. ☺ $\square$

Proposition 7.6 (Asymptotic summation lemma)
Let $m_1 > m_2 > \dots$ be real numbers, $\lim_{j\to\infty} m_j = -\infty$ and $a_j \in S^{m_j}$. Then there exists $a \in S^{m_1}$ st

$$a \sim \sum_{j=1}^{\infty} a_j$$

$$\text{ie } \forall N: a - \sum_{j=1}^{N} a_j \in S^{m_{N+1}}$$

Such a is unique modulo $S^{-\infty}$.

Proof: Fix semi-norms $p_{j,0} \leq p_{j,1} \leq \dots$ that generate Frechet-topology on S^{m_j}

By Thm 7.5 we may choose $b_j \in S^{-\infty}$ st

$$p_{\mu,\nu}(a_j - b_j) \leq 2^{-j} \text{ for } 0 \leq \mu, \nu \leq j-1$$

$$[\text{ note that } S^{m_{j-1}} \supseteq S^{m_j} \text{ as } m_j \longrightarrow -\infty]$$

$\Rightarrow \sum_{j=k}^{\infty}(a_j - b_j)$ converges is S^{m_k}, since for $r, s \geq \nu+1$ $\sum_{j=r}^{s} p_{k\nu}(a_j - b_j) \leq \sum_{j=r}^{\infty} 2^{-j} = 2^{-r+1}$

Define:

$$a := \sum_{j=1}^{\infty}(a_j - b_j) \in S^{m_1}$$

$$\text{Then } a - \sum_{j=1}^{N} a_j = \underbrace{-\sum_{j=1}^{N} b_j}_{\in S^{-\infty}} + \underbrace{\sum_{j=N+1}^{\infty} a_j - b_j}_{\in S^{m_{N+1}}}$$

This proves existence.
Uniqueness up to $S^{-\infty}$: if $a - b \in S^{m_j}$ for any $j \Rightarrow a - b \in S^{-\infty}$.

$\boxed{\checkmark}$

Oszillatory integrals

Definition 7.7 $\varphi \in C^\infty(\Omega \times \dot{\mathbb{R}}^N)$ $\{ \dot{\mathbb{R}}^N = \mathbb{R}^N \setminus \{0\} \}$ is called a phase function, if

- $\operatorname{Im} \varphi \geq 0$ (φ is $\mathbb{C}$-valued)
- $\varphi(x, \lambda\xi) = \lambda \varphi(x,\xi)$ for $\lambda > 0$, $\xi \in \dot{\mathbb{R}}^N$
- $d\varphi(x,\xi) \neq 0$ for $(x,\xi) \in \Omega \times \dot{\mathbb{R}}^N$

Examples:
- $\Omega = \mathbb{R}^n$, $N = n$; $\varphi(x,\xi) = \langle x, \xi \rangle$
- $\Omega = \mathbb{R}^n \times \mathbb{R}^n$, $N = n$; $\varphi(x,y,\xi) = \langle x-y, \xi \rangle$

Consider a phase fn φ and $a \in S^m(\Omega \times \mathbb{R}^N)$ with $\boxed{m < -N}$

oszillatory integral

$$\boxed{I(a,\varphi)\underbrace{(x)}_{\overset{\cap}{\Omega}} := \underbrace{\int}_{\mathbb{R}^N} e^{i\varphi(x,\xi)} a(x,\xi)\, d\xi}$$

$I(a,\varphi) \in C^\infty(\Omega)$ since for $K \subset \Omega$ cpt and $(x,\xi) \in K \times \mathbb{R}^N$ we have

$$|\partial_x^\alpha a(x,\xi)| \leq C(K,\alpha)(1+|\xi|)^m \quad (\overset{\overset{m<-N}{\downarrow}}{\in} L^1(\mathbb{R}^N))$$

$\Rightarrow$ integral above exists and can be differentiated under the integral. (note: $\partial_x^\alpha \varphi(x,\xi) = |\xi|\, \partial_x^\alpha \varphi(x, \frac{\xi}{|\xi|})$)

As a distribution $I(a,\varphi) \in \mathcal{D}'(\Omega)$:

$$\langle I(a,\varphi), u \rangle = \int_{\mathbb{R}^N} \int_{\Omega} e^{i\varphi(x,\xi)} a(x,\xi)\, u(x)\, dx\, d\xi, \quad u \in \mathcal{D}(\Omega)$$

Next week: We relax the requirement $m < -N$

(recap from previous week)

Smoothness of oszillatory integrals

$$I(a,\varphi)(x) := \int_{\mathbb{R}^N} e^{i\varphi(x,\xi)} a(x,\xi)\,d\xi \quad , x\in\Omega$$

where
- φ is a phase function ie
  - $\Im m\,\varphi \geq 0$
  - $\varphi(x,\lambda\xi) = \lambda\varphi(x,\xi)$ for $\lambda>0$, $\xi\in\dot{\mathbb{R}}^N$
  - $d\varphi(x,\xi)\neq 0$ for $(x,\xi)\in\Omega\times\dot{\mathbb{R}}^N$
- a is a symbol of order $m<-N$, ie
  - for $K\subset\Omega$ cpt and $(x,\xi)\in K\times\mathbb{R}^N$
    $|\partial_x^\alpha a(x,\xi)| \leq C(K,\alpha)(1+|\xi|)^m$

__Observe:__
$$e^{i\varphi(x,\xi)} = e^{i\,\mathrm{Re}\,\varphi}\cdot e^{-\Im m\,\varphi} = e^{i\,\mathrm{Re}\,\varphi}\cdot e^{-|\xi|\cdot\Im m\,\varphi(x,\underbrace{\xi/|\xi|}_{\in\,\mathbb{S}^{N-1}})}$$

$$\Rightarrow |\partial_x^\alpha e^{i\varphi(x,\xi)}| \leq \mathrm{Const}\cdot|\xi|^{|\alpha|}$$

Hence differentiating under the integral makes $|\xi|\to\infty$ behaviour worse. Hence:

__FACT:__ For $a\in S^m(\Omega\times\mathbb{R}^N)$ with $m<-N$

$$I(a,\varphi)\in C^{-N-m}(\Omega)$$

ie the order of differentiability depends on the order of the symbol. In particular: for $a\in S^{-\infty}(\Omega\times\mathbb{R}^N)$, $I(a,\varphi)\in C^\infty(\Omega)$ and will later correspond to smoothing operators.

Extension of oszillatory integrals to symbols of arbitrary order:

Theorem 7.8 Let $\varphi \in C^\infty(\Omega \times \dot{\mathbb{R}}^N)$ be a phase function.
Then $a \mapsto I(a,\varphi)$ has a unique cts extension

$$S^{+\infty}(\Omega\times\mathbb{R}^N) := \bigcup_{m\in\mathbb{R}} S^m(\Omega\times\mathbb{R}^N) \longrightarrow \mathcal{D}'(\Omega)$$

Proof: Uniqueness: Let $F : S^\infty \to \mathcal{D}'$ be cts and $F|_{S^{-\infty}} = 0$. For any $m \in \mathbb{R}$ choose $m' > m$. $F|_{S^{m'}}$ is cts and moreover by Thm 7.5, $S^{-\infty}$ is dense in S^m in the $S^{m'}$- topology and hence

$$F|_{S^m} \equiv 0 \quad \text{for all } m\in\mathbb{R}.$$

Existence: We need the following lemma: (proof later)

Lemma 7.9 For any given phase function φ there exist $(a_j) \subset S^0(\Omega\times\mathbb{R}^N)$, $(b_j) \subset S^{-1}(\Omega\times\mathbb{R}^N)$ and $c \in S^{-1}(\Omega\times\mathbb{R}^N)$ such that with

$$L := \underset{\underset{\boxed{N = \dim \mathbb{R}^N}}{\uparrow}}{\sum_{j=1}^{N}} a_j(x,\xi)\frac{\partial}{\partial \xi_j} + \underset{\underset{\boxed{n = \dim \mathbb{R}^n\ (\Omega\subset\mathbb{R}^n)}}{\uparrow}}{\sum_{j=1}^{n}} b_j(x,\xi)\frac{\partial}{\partial x_j} + c(x,\xi)$$

we have $L^t e^{i\varphi} = e^{i\varphi}$, where L^t is the adjoint of L wrt bilinearpairing $\int fg$. Furthermore $L(S^m) \subset S^{m-1}$.

Then we can prove existence as follows:
Let $a \in S^m(\Omega\times\mathbb{R}^N)$, $u \in \mathcal{D}(\Omega)$. Assume first $m < -N$.
Then by Lemma 7.9 we have for any $k \in \mathbb{Z}_+$:

$$\langle I(a,\varphi),u\rangle = \int_{\mathbb{R}^N}\int_{\Omega} \left\{ (L^t)^k e^{i\varphi(x,\xi)} \right\} a(x,\xi)\, u(x)\, dx\, d\xi$$

(Fubini)

$$= \int_{\mathbb{R}^N}\int_{\Omega} e^{i\varphi(x,\xi)} \underbrace{\left\{ L^k a(x,\xi) u(x) \right\}}_{\in S^{m-k}(\Omega\times\mathbb{R}^N)} dx\, d\xi$$

since for $a \in S^m$, also $a\cdot u \in S^m$ for $u \in \mathcal{D}(\Omega)$.
The RHS (last integral) makes sense for any $m-k<-N$.
Hence we define the extension for any $a \in S^m$:

$$\langle I(a,\varphi),u\rangle := \int_{\mathbb{R}^N}\int_{\Omega} e^{i\varphi}\left\{ L^k(a\cdot u)\right\} dx\, d\xi$$

with $k \in \mathbb{Z}_+$ such that $m-k<-N$.
Obviously, construction independent of k.

Continuity of the extension:

(ü) $$S^m(\Omega\times\mathbb{R}^N)\times\mathcal{D}(\Omega) \to S^{m-k}(\Omega\times\mathbb{R}^N)$$
$$(a,u) \longmapsto L^k(a\cdot u)$$

is continuous. Hence $a \mapsto I(a,\varphi)$ is a cts mapping $S^m \to \mathcal{D}'$ for any $m \in \mathbb{R}$. ▨

Remark 7.10 Continuity of $a \mapsto I(a,\varphi)$ means that

$$a_j \xrightarrow{S^{m'}} a \Rightarrow I(a,\varphi) = \lim_{j\to\infty} I(a_j,\varphi) \text{ as distrib-ns.}$$

Hence by Thm 7.5 with $a_j(x,\xi) = \rho(\xi/j)\cdot a(x,\xi)$, $a \in S^m$:

$$a_j \in S^{-\infty},\quad a_j \xrightarrow{S^{m'}} a$$ and hence for $u \in \mathcal{D}(\Omega)$:

$$\boxed{\langle I(a,\varphi),u\rangle = \underbrace{\lim_{j\to\infty}}\ \underbrace{\int_{\mathbb{R}^N}\int_{\Omega}} e^{i\varphi(x,\xi)}\rho(\xi/j)\, a(x,\xi)\, u(x)\, dx\, d\xi}$$

(this can be used to provide alternative proof of existence)

Proof of Lemma 7.9

Construction of L:

$$\Phi(x,\xi) := \sum_{j=1}^{n} \left| \frac{\partial\varphi}{\partial x_j}(x,\xi) \right|^2 + |\xi|^2 \cdot \sum_{j=1}^{N} \left| \frac{\partial\varphi}{\partial \xi_j}(x,\xi) \right|^2$$

Since $d\varphi$ is nowhere vanishing, $\Phi(x,\xi) \neq 0$ for $(x,\xi) \in \Omega \times \dot{\mathbb{R}}^N$

- $\Phi(x,\lambda\xi) = \lambda^2 \Phi(x,\xi)$, for $\lambda > 0$
- Problem: Φ is only defined on $\Omega \times \dot{\mathbb{R}}^N$. Hence choose a cutoff-function $\chi \in C_0^\infty(\mathbb{R}^N)$ st

$$\chi(\xi) = \begin{cases} 1, & |\xi| \leq 1 \\ 0, & |\xi| \geq 2 \end{cases}$$

Define:

$$L^t := \frac{1-\chi}{i\Phi} \left(\sum_{j=1}^{N} |\xi|^2 \overline{\frac{\partial\varphi}{\partial\xi_j}} \frac{\partial}{\partial\xi_j} + \sum_{j=1}^{n} \overline{\frac{\partial\varphi}{\partial x_j}} \frac{\partial}{\partial x_j} \right) + \chi.$$

$$=: \sum_{j=1}^{N} \tilde{a}_j \frac{\partial}{\partial\xi_j} + \sum_{j=1}^{n} \tilde{b}_j \frac{\partial}{\partial x_j} + \tilde{c}$$

with $(\tilde{a}_j) \subset S^0$, $(\tilde{b}_j)$ and $\tilde{c} \in S^{-1}$.

Clearly:

$$L^t e^{i\varphi} = \left(\frac{1-\chi}{i\cancel{\Phi}} i\cancel{\Phi} + \chi \right) e^{i\varphi} = e^{i\varphi}$$

$L(S^m) \subset S^{m-1}$ follows from the fact that

$$\frac{\partial}{\partial x_j} : S^m \to S^m, \quad \frac{\partial}{\partial\xi_j} : S^m \to S^{m-1}, \quad \begin{aligned} S^m \times S^{\tilde{m}} &\to S^{m+\tilde{m}} \\ (a,b) &\mapsto a \cdot b. \end{aligned}$$

$\square$

Exercise: Let $p \in S^m(\{0\} \times \mathbb{R}^n)$, naturally $p \in \mathcal{S}'(\mathbb{R}^n)$.
The oszillatory integral $I(p, -\langle x, \xi \rangle)$

$$= \int e^{-i\langle x, \xi\rangle} p(\xi)\, d\xi$$

is then the Fourier transform of the tempered distribution $p \in \mathcal{S}'(\mathbb{R}^n)$.

Singular support of $I(a, \varphi) \in \mathcal{D}'(\Omega)$:

Definition 7.11 For a phase function $\varphi : \Omega \times \dot{\mathbb{R}}^N \to \mathbb{C}$ put

$$d_\xi \varphi := \sum_{j=1}^{N} \frac{\partial \varphi}{\partial \xi_j} d\xi_j$$

$$C_\varphi := \{ (x,\xi) \mid d_\xi \varphi \big|_{(x,\xi)} = 0 \}$$

"conic nbds"

Remarks 7.12:

1) $\varphi(x,y,\xi) = \langle x-y, \xi \rangle$. Then $d_\xi \varphi = \sum_{j=1}^{N} (x_j - y_j)\, d\xi_j$

$$C_\varphi = \{ (x,x,\xi) \mid x \in \Omega, \xi \in \dot{\mathbb{R}}^N \}$$
$$= (\text{diagonal of } \Omega \times \Omega) \times \dot{\mathbb{R}}^N$$

2) C_φ is conic in the following sense:
if $(x,\xi) \in C_\varphi$, and $\lambda > 0$, then $(x, \lambda\xi) \in C_\varphi$.

Proposition 7.13 ~~(without proof)~~

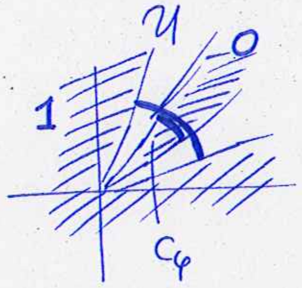
1) For $a \in S^m(\Omega \times \mathbb{R}^N)$, singsupp $I(a,\varphi) \subset \pi_\Omega(C_\varphi)$

2) If $a \in S^m$ vanishes in a conic nbd $\mathcal{U}$ of C_φ, then $I(a,\varphi) \in C^\infty(\Omega)$.

$= \{ x \in \Omega \mid \exists\, \xi \in \dot{\mathbb{R}}^N$ with $(x,\xi) \in C_\varphi \}$

[$\mathcal{U}$ conic nbd of C_φ if $\mathcal{U}$ itself is conic and $\overline{C_\varphi} \subset \mathcal{U}$, $\overline{C_\varphi}$ – closure in $(\Omega \times \dot{\mathbb{R}}^N)$.]

<u>Proof:</u> We prove (2) first



Choose $\tilde{\rho} \in C^\infty(\Omega \times \mathbb{S}^{N-1})$ such that

- $\tilde{\rho} \restriction (\mathcal{U} \cap (\Omega \times \mathbb{S}^{N-1}))^c \equiv 1$
- $\tilde{\rho} \restriction (C_\varphi \cap (\Omega \times \mathbb{S}^{N-1})) \equiv 0$

$\rightarrow$ extend homogeneously to $\Omega \times \dot{\mathbb{R}}^N$ by $\rho(x, \lambda\xi) = \lambda^0 \rho(x,\xi)$

$\rightarrow$ smoothen out to $\tilde{\rho} \in C^\infty(\Omega \times \mathbb{R}^N)$ at $\xi = 0$.

$\Rightarrow$ For $a \in S^m(\Omega \times \mathbb{R}^N)$ with $a|_{\mathcal{U}} = 0$

we have $\rho \cdot a = a$.

Put: $$L^t := \rho(x,\xi) \cdot \left\{ \frac{1-\chi(\xi)}{i\Phi(x,\xi)} \sum_{j=1}^N |\xi|^2 \overline{\frac{\partial\varphi}{\partial\xi_j}} \frac{\partial}{\partial\xi_j} + \chi(\xi) \right\}$$

where $$\Phi(x,\xi) = |\xi|^2 \cdot \sum_{j=1}^N \left| \frac{\partial\varphi}{\partial\xi_j}(x,\xi) \right|^2$$

and $\chi(\xi) = \begin{cases} 1, & |\xi| \leq 1 \\ 0, & |\xi| \geq 2 \end{cases}$ (cf. Lemma 7.9)

Then
- $L^t e^{i\varphi} = \rho e^{i\varphi}$
- $(L^t e^{i\varphi}) \cdot a = \rho \cdot e^{i\varphi} \cdot a = e^{i\varphi} \cdot a$
- $L(S^m) \subset S^{m-1}$ as before.

Now $$\underset{(x)}{\underset{\wedge}{I(a,\varphi)}} = \int_{\mathbb{R}^N} e^{i\varphi(x,\xi)} \underbrace{(L^k a)(x,\xi)}_{\in S^{m-k}} \, d\xi$$

is smooth in $x \in \Omega$, since we <u>may</u> take $k \in \mathbb{N}$ infinitely large.

(▨)

We now prove (1): Claim is equivalent to saying that for $u \in C_0^\infty(\Omega \setminus \pi_\bullet(C_\varphi))$, $I(a,\varphi)[u]$ acts on u as an integration against a smooth fn.

Remark on the construction of L^t in Prop 7.13

$$L^t = \rho(x,\xi)\cdot\left\{ \frac{1-\chi(\xi)}{i\Phi(x,\xi)}\cdot\sum_{j=1}^{N}\overline{\frac{\partial\varphi}{\partial\xi_j}}\cdot\frac{\partial}{\partial\xi_j} + \chi(\xi)\right\}$$

where

- $\Phi(x,\xi) = \sum_{j=1}^{N}\left|\frac{\partial\varphi}{\partial\xi_j}(x,\xi)\right|^2$
- $\rho\cdot a = a$ and $\rho \equiv 0$ over $C_\varphi \cap \{|\xi| \geq 1\}$

Question: Why ρ-factor here?
(recall, there was no factor like this in Lemma 7.9)

Answer: For $(x,\xi)\in C_\varphi : (d_\xi\varphi)(x,\xi) = 0$

$\Rightarrow \forall_j : \frac{\partial\varphi}{\partial\xi_j}(x,\xi) = 0$

$\Rightarrow \Phi(x,\xi) = 0$. However, since Φ is quadratic in $\frac{\partial\varphi}{\partial\xi_j}$'s, the coeff. of L^t

$$\frac{1-\chi(\xi)}{i\Phi(x,\xi)}\cdot\overline{\frac{\partial\varphi}{\partial\xi_j}}(x,\xi)$$

may still be singular at $(x,\xi)\in C_\varphi \cap \{|\xi|\geq 1\}$.
Hence $\rho(x,\xi)$-factor is needed to make L^t-coeff. smooth:

$$\rho\cdot\frac{1-\chi}{i\Phi}\cdot\overline{\frac{\partial\varphi}{\partial\xi_j}} \in C^\infty(\Omega\times\mathbb{R}^N)$$

and only then is L^t a true differential operator //