Spectral theory of differential operators
Exercise set 1

Exercise 1 (Unbounded+continuous). In this exercise, by the sum $A+B$ of a linear operator A with a *continuous* operator B (both acting in a Hilbert space $\mathcal{H}$), we mean the operator defined by $A+B : u \mapsto Au + Bu$ on the domain $D(A+B) = D(A)$.

1. Assume that A is closable. Show that $A+B$ is closable with $\overline{A+B} = \overline{A} + B$.
2. Assume, in addition, that A is densely defined. Show that $(A+B)^* = A^* + B^*$.

Exercise 2 (Maximality). Let A and B be self-adjoint operators in a Hilbert space $\mathcal{H}$ such that $D(A) \subset D(B)$ and $Au = Bu$ for all $u \in D(A)$. Show that $D(A) = D(B)$. (This property is called the *maximality* of self-adjoint operators.)

Exercise 3 (Unitary equivalence).

1. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. Recall that a linear operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ is called *unitary* if it is bijective and $\|Uf\| = \|f\|$ for all $f \in \mathcal{H}_1$ (which is equivalent to $U^* = U^{-1}$).

   Let A be a linear operator in $\mathcal{H}_1$, B be a linear operator in $\mathcal{H}_2$. Assume that there exists a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $UD(A) = D(B)$ and $UAU^{-1}f = Bf$ for all $f \in D(B)$: such A and B are called *unitary equivalent*, one uses the writing $B = UAU^{-1}$.

   Let A and B be as above. Show:

   (a) if A is closable then B is closable too, and in that case $\overline{B} = U\overline{A}U^{-1}$.

   (b) if A is closable and densely defined, then also B is closable and densely defined and $B^* = UA^*U^{-1}$.

   (c) If A is closed/symmetric/self-adjoint, then also B has the respective property.

   (Remark that in all questions the roles of A and B can be interchanged.)

2. Let (λ_n) be an arbitrary sequence of complex numbers, $n \in \mathbb{N}$. In the Hilbert space $\ell^2(\mathbb{N})$ consider the following linear operator S:

$$D(S) = \big\{(x_n) : \text{there exists } N \text{ such that } x_n = 0 \text{ for all } n > N\big\},$$
$$S(x_n) = (\lambda_n x_n).$$

   Describe $\overline{S}$ and S^*

3. Let $\mathcal{H}$ be a separable Hilbert space and (e_n) be an orthonormal basis in $\mathcal{H}$. Consider the linear operator T with

$$D(T) := \text{the set of the finite linear combinations of } e_n$$

and assume that there exist $\lambda_n \in \mathbb{C}$ such that $Te_n = \lambda_n e_n$ for all n.

   (a) Describe $\overline{T}$ and T^*.

   (b) Let all λ_n be real. Show that T is essentially self-adjoint.

Exercise 4 (Harmonic oscillator in 1D). Consider the following differential expressions on $\mathbb{R}$:

$$L^+ := -\frac{d}{dx} + x, \quad L^- := \frac{d}{dx} + x, \quad H := -\frac{d^2}{dx^2} + x^2.$$

For the moment we consider them as linear maps on $C^\infty(\mathbb{R})$,

$$(L^+ f)(x) = -f'(x) + xf(x) \text{ etc.}$$

1. Show the identities $H = L^+L^- + I$ and $L^+(H + 2I) = HL^+$, with I being the identity map.

2. Consider the function $\phi_1 : x \mapsto e^{-x^2/2}$. Show that ϕ_1 is an eigenfunction of H and find the corresponding eigenvalue λ_1.

3. For $n \geq 2$ define recursively $\phi_n := L^+\phi_{n-1}$. Show that all ϕ_n are eigenfunctions of H and find the corresponding eigenvalues λ_n.

Now consider $\mathcal{H} := L^2(\mathbb{R})$ and the linear operator S:

$$S : f \mapsto Hf, \quad D(S) := C_c^\infty(\mathbb{R}).$$

4. Is S closable? symmetric?

5. Let f be a finite linear combination of ϕ_n. Show that $f \in D(\overline{S})$.

   Hint: Let $\chi \in C_c^\infty(\mathbb{R})$ with $\chi(x) = 1$ for $|x| \leq 1$ and $\chi(x) = 0$ for $|x| \geq 2$. Consider the functions $\chi_N : x \mapsto \chi\left(\frac{x}{N}\right)$ and $f_N := \chi_N f$ with large N.

6. Show that (ϕ_n) are mutually orthogonal in $\mathcal{H}$.

7. Let $f \in \mathcal{H}$ with $f \perp \phi_n$ for all n.

   (a) Show that f is orthogonal to all functions of the form $x^n e^{-x^2/2}$ with $n \in \mathbb{N}_0$.

   (b) Show that the function

   $$F : \mathbb{C} \ni z \mapsto \int_{\mathbb{R}} f(x) e^{-x^2/2} e^{-izx}\, \mathrm{d}x \in \mathbb{C}$$

   is holomorph and compute $F^{(n)}(0)$ for all $n \in \mathbb{N}_0$.

(c) Deduce that $f = 0$.

(d) Deduce that there exists an orthonormal basis of $\mathcal{H}$ consisting of eigenfunctions of $\overline{S}$.

8. Show that S is essentially self-adjoint.

Exercise 5. Consider the operator M_f from the lecture: $\Omega \subset \mathbb{R}^d$ is an open set, $\mathcal{H} := L^2(\Omega)$, pick $f \in C^0(\Omega)$, then

$$M_f : u \mapsto fu \text{ for } u \in D(M_f) = \big\{u \in L^2(\Omega) : fu \in L^2(\Omega)\big\}.$$

Give a detailed proof for $M_f^* = M_{\overline{f}}$.

Exercise 6. Let $\mathcal{H} := L^2(0,1)$. For $\lambda \in \mathbb{C}$ consider the linear operator

$$T : f \mapsto if', \quad D(T) := \big\{f \in C^\infty([0,1]) : f(1) = \lambda f(0)\big\}.$$

1. For which λ is T symmetric?

2. For which λ is T closable?

Exercise 7. Consider

$$\Omega = \big\{(x_1, x_2) : x_2 > 0\big\} \subset \mathbb{R}^2, \quad P = \Delta.$$

Choose $\chi \in C_c^\infty(\mathbb{R}^2)$ with $\chi(x) = 1$ for $|x| < 1$ and consider the function

$$u : \Omega \ni x \mapsto \chi(x) \log|x| \in \mathbb{C}.$$

Show that $u \in D(P_{\max})$ but $u \notin H^2(\Omega)$.

Exercise 8.

1. Let $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $\varphi \in C_c^\infty(\mathbb{R}^d)$. Recall why the convolution $f * \varphi$ is well-defined and belongs to $C^\infty(\mathbb{R}^d)$.

2. Let $k \in \mathbb{N}$ and $f \in H^k(\mathbb{R}^n)$.

   (a) Let $\varphi \in C_c^\infty(\mathbb{R}^n)$. Show that $f * \varphi \in H^k(\mathbb{R}^n)$.

   (b) Let ρ_δ be as in the lectures. Show that $f * \rho_\delta$ converges to f in $H^k(\mathbb{R}^n)$ for $\delta \to 0^+$.

   (c) Let $\chi \in C_c^\infty(\mathbb{R}^n)$ such that $\chi(x) = 1$ for $|x| \le 1$ and $\chi(x) = 0$ for $|x| \ge 2$. For $N > 0$ define $\chi_N : x \mapsto \chi\big(\frac{x}{N}\big)$. Show that $\chi_N f$ converges to f in $H^k(\mathbb{R}^n)$ for $\varepsilon \to 0^+$.

3. Show that $C_c^\infty(\mathbb{R}^n)$ is dense in $H^k(\mathbb{R}^n)$ for any $k \in \mathbb{N}$.

Exercise 9 (Sobolev embedding theorem). Let $k, d \in \mathbb{N}$ and $m \in \mathbb{N}_0$ with $k > m + \frac{d}{2}$.

1. Show: there exists $c > 0$ such that $\|\partial^\alpha \varphi\|_\infty \leq c\|\varphi\|_{H^k(\mathbb{R}^d)}$ for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq m$ and all $\varphi \in C_c^\infty(\mathbb{R}^d)$.

   Hint: Write the Fourier inversion formula for $\partial^\alpha \varphi$, multiply the subintegral function by $1 \equiv \langle \xi \rangle^{-k} \langle \xi \rangle^k$ and use the Cauchy-Schwarz inequality.

2. Equip

$$C^m_{L^\infty}(\mathbb{R}^d) := \left\{ u \in C^\infty(\mathbb{R}^d) : \partial^\alpha u \in L^\infty(\mathbb{R}^d) \text{ for all } \alpha \in \mathbb{N}_0^d \text{ with } |\alpha| \leq m \right\}$$

   with the norm $\|u\|_{m,\infty} := \sum_{|\alpha| \leq m} \|\partial^\alpha u\|_\infty$.

   Show that $H^k(\mathbb{R}^d) \subset C^m_{L^\infty}(\mathbb{R}^d)$ and that the embedding is continuous.

Exercise 10 (Sobolev spaces $\boldsymbol{H_0^k}$). For a non-empty open set $\Omega \subset \mathbb{R}^n$ and $k \in \mathbb{N}$ define

$$H_0^k(\Omega) := \text{the closure of } C_c^\infty(\Omega) \text{ in } H^k(\Omega).$$

Let $\Omega \subset \widetilde{\Omega} \subset \mathbb{R}^n$ be non-empty open sets. For a function u defined on Ω we denote by $\widetilde{u}$ its extension by zero to $\widetilde{\Omega}$.

Show: if $u \in H_0^k(\Omega)$, then $\widetilde{u} \in H^k(\widetilde{\Omega})$ with $\|\widetilde{u}\|_{H^k(\widetilde{\Omega})} = \|u\|_{H^k(\Omega)}$.

Spectral theory of differential operators
Exercise set 2

Exercise 11 (Sesquilinear forms and bounded operators). Let t be a closed sesquilinear form in $\mathcal{H}$ and T be the operator generated by t. Furthermore, let $B = B^* \in \mathcal{B}(\mathcal{H})$. Show:

1. the sesquilinear form

$$t_B : (u,v) \mapsto t(u,v) + \langle u, Bv\rangle_{\mathcal{H}}, \quad D(t_B) = D(t),$$

   is closed,

2. the operator T_B generated by t_B is

$$T_B : u \mapsto Tu + Bu, \quad D(T_B) = D(T).$$

Exercise 12 (Direct sums of forms and operators). Let t_j be closed sesquilinear forms in Hilbert spaces $\mathcal{H}_j$ and T_j be the associated operators in $\mathcal{H}_j$, $j \in \{1,2\}$. Recall that $\mathcal{H} := \mathcal{H}_1 \times \mathcal{H}_2$ is a Hilbert space for the scalar product

$$\big\langle (u_1,u_2),(v_1,v_2)\big\rangle_{\mathcal{H}_1\times\mathcal{H}_2} := \langle u_1, v_1\rangle_{\mathcal{H}_1} + \langle u_2, v_2\rangle_{\mathcal{H}_2}.$$

1. Show that the sesquilinear form t in $\mathcal{H}$,

$$D(t) = D(t_1) \times D(t_2), \quad t\big((u_1,u_2),(v_1,v_2)\big) = t_1(u_1,v_1) + t_2(u_2,v_2)$$

   is closed. We write $t = t_1 \oplus t_2$ and say that t is the *direct sum* of t_1 and t_2.

2. Show that the operator T generated by t is the direct sum, $T = T_1 \oplus T_2$, which is defined by

$$D(T) = D(T_1) \times D(T_2), \quad T(u_1,u_2) = (T_1u_1, T_2u_2).$$

Exercise 13 (Sesquilinear forms and unitary equivalence).

1. Let $\Theta : \mathcal{H}' \to \mathcal{H}$ be a unitary operator between Hilbert spaces $\mathcal{H}'$ and $\mathcal{H}$. Let t be a closed sesquilinear form in $\mathcal{H}$ and T be the operator in $\mathcal{H}$ generated by t. Define a sesquilinear form t' in $\mathcal{H}'$ by

$$D(t') = \Theta^{-1}D(t), \quad t'(u,v) = t(\Theta u, \Theta v).$$

   Show that t' is closed and that the operator T' in $\mathcal{H}'$ generated by t' is unitarily equivalent to T.

2. Let $\Omega, \Omega' \subset \mathbb{R}^d$ be open subsets and $\Phi : \Omega \to \Omega'$ be a C^∞-diffeomorphism. Show that the weak derivatives on Ω and Ω' satisfy the usual composition rule

$$\nabla(u \circ \Phi) = \big((\nabla u)\circ\Phi\big)D\Phi$$

   (if one writes ∇u as a line).

3. Let $\Omega, \Omega' \subset \mathbb{R}^d$ be open subsets such that $\Omega' = \Phi(\Omega)$ for some isometry $\Phi : \mathbb{R}^d \to \mathbb{R}^d$. Show that the Dirichlet/Neumann Laplacian in Ω' is unitarily equivalent to the Dirichlet/Neumann Laplacian in Ω.

   Hint: Any isometry Φ acts as $\Phi : x \mapsto Ax + b$ with a unitary matrix A and $b \in \mathbb{R}^d$. Consider the map

   $$\Theta : L^2(\Omega') \to L^2(\Omega), \quad \Theta u = u \circ \Phi,$$

   and use the first two parts of this exercise.

4. Is there any link between the Dirichlet/Neumann Laplacians in Ω and $\lambda\Omega$ with arbitrary $\lambda > 0$?

Exercise 14 (Lower semiboundedness in one dimension).

1. Check if the operator T,

   $$D(T) = C_c^\infty(0, \infty), \quad Tf = -if',$$

   is semibounded from below in $\mathcal{H} = L^2(0, \infty)$.

   Hint: consider $f : x \mapsto \chi(x)e^{ikx}$ with suitable $k \in \mathbb{R}$ and $\chi \in C_c^\infty(0, \infty)$.

2. Show the inequality

   $$\|f\|_\infty^2 \le \varepsilon \int_{\mathbb{R}} |f'|^2 \, \mathrm{d}x + \frac{1}{\varepsilon} \int_{\mathbb{R}} |f|^2 \, \mathrm{d}x \text{ for all } f \in H^1(\mathbb{R}) \text{ and } \varepsilon > 0.$$

   Hint: One can start with $|f(x)|^2 = \displaystyle\int_{-\infty}^{x} (|f|^2)'$ for $f \in C_c^\infty(\mathbb{R})$.

3. Let $V \in L^2(\mathbb{R})$ be real-valued. Show that the operator

   $$T : f \mapsto -f'' + Vf, \quad D(T) = C_c^\infty(\mathbb{R}),$$

   is semibounded from below in $\mathcal{H} = L^2(\mathbb{R})$.

4. Show that for any $f \in C_c^\infty(0, \infty)$ one has the *Hardy inequality*

   $$\int_0^\infty |f'(x)|^2 \, \mathrm{d}x \ge \int_0^\infty \frac{|f(x)|^2}{4x^2} \, \mathrm{d}x.$$

   Hint: represent $f(x) = \sqrt{x}\, g(x)$.

5. Let $V \in L^2(0, \infty)$ be real-valued and $\alpha \in \mathbb{R}$. Show that the operator T,

   $$D(T) = C_c^\infty(0, \infty), \quad \big(Tf\big)(x) = -f''(x) + \Big(\frac{\alpha}{x} + V(x)\Big) f(x)$$

   is semibounded from below in $\mathcal{H} = L^2(0, \infty)$.

Exercise 15 (Lower semiboundedness in higher dimensions). We will use the following assertion without proof: If $X \subset \mathbb{R}^d$ is closed and $f : X \to \mathbb{R}$ is a bounded continuous function, then f can be extended to a bounded continuous function on the whole of $\mathbb{R}^d$. (The assertion holds in a much more general setting of topological spaces and is known as Tietze extension theorem.)

Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with C^1 boundary and $n : \partial\Omega \to \mathbb{R}^d$ be the outer unit normal on $\partial\Omega$. Show:

1. n can be extended to a bounded continuous function $N : \mathbb{R}^d \to \mathbb{R}^d$.
2. there exists a bounded C^∞ function $\widetilde{N} : \mathbb{R}^d \to \mathbb{R}^d$ with $\|\widetilde{N} - N\|_\infty < \frac{1}{2}$.
3. there holds $\widetilde{N} \cdot n \geq \frac{1}{2}$ on $\partial\Omega$.
4. for any $u \in C^\infty(\overline{\Omega})$ there holds
$$\int_{\partial\Omega} |u|^2 \widetilde{N} \cdot n \, \mathrm{d}s = \int_\Omega \big[(\overline{u}\nabla u + u\overline{\nabla u}) \cdot \widetilde{N} + |u|^2 \operatorname{div} \widetilde{N}\big] \, \mathrm{d}x.$$
5. for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that for any $u \in C^\infty(\overline{\Omega})$ there holds
$$\int_{\partial\Omega} |u|^2 \, \mathrm{d}s \leq \varepsilon \int_\Omega |\nabla u|^2 \, \mathrm{d}x + C_\varepsilon \int_\Omega |u|^2 \, \mathrm{d}x.$$
6. for any bounded measurable function $\alpha : \partial\Omega \to \mathbb{R}$ the operator T
$$T : u \mapsto -\Delta u, \quad D(T) = \big\{u \in C^\infty(\overline{\Omega}) : \partial_n u = \alpha u \text{ on } \partial\Omega\big\}$$
is semibounded from below in $\mathcal{H} = L^2(\Omega)$.

   Remark: the boundary condition $\partial_n u = \alpha u$ is called *Robin boundary condition*.

   There exists an alternative terminology (sometimes considered as obsolete but still in use): the Dirichlet/Neumann/Robin boundary conditions are referred to as the first/second/third type boundary conditions.

Spectral theory of differential operators
Exercise set 3

Exercise 16 (Spectrum, direct sums, matrix operators).

1. Let T_j be linear operators in Hilbert spaces $\mathcal{H}_j$, $j \in \{1, 2\}$. Show:
$$\operatorname{spec}(T_1 \oplus T_2) = \operatorname{spec} T_1 \cup \operatorname{spec} T_2, \quad \operatorname{spec}_{\mathrm{p}}(T_1 \oplus T_2) = \operatorname{spec}_{\mathrm{p}} T_1 \cup \operatorname{spec}_{\mathrm{p}} T_2.$$

2. Let $\Omega \subset \mathbb{R}^d$ be a non-empty open set and let $L : \Omega \to M_2(\mathbb{C})$ be a continuous 2×2 matrix function such that $L(x)^* = L(x)$ for all $x \in \Omega$. Define an operator A in $\mathcal{H} = L^2(\Omega, \mathbb{C}^2)$ (L^2-functions with values in $\mathbb{C}^2$) by
$$Af(x) = L(x)f(x), \quad D(A) = \Big\{ f \in \mathcal{H} : \int_\Omega \|L(x)f(x)\|^2_{\mathbb{C}^2}\, \mathrm{d}x < +\infty \Big\}.$$
   (a) Show that A is self-adjoint.
   (b) Let $\lambda_1(x) \le \lambda_2(x)$ be the eigenvalues of $L(x)$. Show:
   $$\operatorname{spec} A = \overline{\operatorname{ran} \lambda_1} \cup \overline{\operatorname{ran} \lambda_2}$$
   and find a similar representation for $\operatorname{spec}_{\mathrm{p}} A$.

   Hint: For each $x \in \Omega$, let $\xi_1(x)$ and $\xi_2(x)$ be suitably chosen eigenvectors of $L(x)$. Consider the map
$$U : \mathcal{H} \to \mathcal{H}, \quad Uf(x) := \begin{pmatrix} \langle \xi_1(x), f(x) \rangle_{\mathbb{C}^2} \\ \langle \xi_2(x), f(x) \rangle_{\mathbb{C}^2} \end{pmatrix}$$
   and the operator $B = UAU^{-1}$.

3. Consider the operator T in $\mathcal{H} = l^2(\mathbb{Z})$ given by
$$Tf(n) = f(n-1) + f(n+1) + V(n)f(n), \quad V(n) = \begin{cases} 4, & \text{if } n \text{ is even}, \\ -2, & \text{if } n \text{ is odd}. \end{cases}$$
   Compute the spectrum of T.

   Hint: Consider the operators
$$U : l^2(\mathbb{Z}) \to l^2(\mathbb{Z}, \mathbb{C}^2), \quad Uf(n) := \begin{pmatrix} f(2n) \\ f(2n+1) \end{pmatrix}, \quad n \in \mathbb{Z},$$
$$F : \ell^2(\mathbb{Z}, \mathbb{C}^2) \to L^2\big((0, 2\pi), \mathbb{C}^2\big), \quad (Fg)(\theta) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} g(n) e^{in\theta},$$
$$S := UTU^{-1}, \quad \widehat{S} := FSF^{-1}.$$

Exercise 17 (Sufficient condition for $[0,\infty) \subset \operatorname{spec} T$).

1. Let $\Omega \subset \mathbb{R}^d$ be an open set and T be a linear operator in $\mathcal{H} := L^2(\Omega)$. Assume that there exists an open subset $\Omega' \subset \Omega$ with the following properties:

   - $C_c^\infty(\Omega') \subset D(T)$,
   - for any $u \in C_c^\infty(\Omega')$ one has $Tu = -\Delta u$,
   - for any $R > 0$ there is a ball of radius R contained in Ω' (open sets with this property are sometimes called *quasiconical*).

   For any $n \in \mathbb{N}$ let $r_n \in \Omega'$ such that $B_n(r_n) \subset \Omega'$. Pick $\chi \in C_c^\infty(\mathbb{R}^d)$ with $\operatorname{supp} \chi \subset B_1(0)$ and $\chi = 1$ in $B_{\frac{1}{2}}(0)$.

   Let $k \in \mathbb{R}$. Define $u_n \in C_c^\infty(\Omega')$ by
   $$u_n(x) = \chi\Big(\frac{x - r_n}{n}\Big) e^{ikx_1}.$$

   (a) Show that $\|u_n\|^2 \geq c n^d$ for some $c > 0$ independent of n,

   (b) Show that $\|(T - k^2)u_n\|^2 = O(n^{d-1})$ as $n \to \infty$. Remark: one can control L^2-norms by controlling the $\|\cdot\|_\infty$-norm and the size of the support.

   (c) Show that $[0,\infty) \subset \operatorname{spec} T$.

2. Compute the spectra of the Dirichlet and Neumann Laplacians on $(0,\infty)$.

Exercise 18 (Dirichlet/Neumann Laplacians on intervals/rectangles). Let $\ell \in (0,\infty)$.

1. Show that the eigenvalues of the Dirichlet Laplacian on $(0,\ell)$ are simple and given by $\pi^2 n^2/\ell^2$, $n \in \mathbb{N}$,

2. Show that for any $\varphi \in C_c^\infty(0,\ell)$ one has
   $$\int_0^\ell \big|\varphi'(x)\big|^2 \,\mathrm{d}x \geq \frac{\pi^2}{\ell^2} \int_0^\ell \big|\varphi(x)\big|^2 \,\mathrm{d}x.$$

3. Show that the eigenvalues of the Neumann Laplacian on $(0,\ell)$ are simple and given by $\pi^2 n^2/\ell^2$, $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

4. Let $\ell_1, \ell_2 \in (0,\infty)$. Compute the spectra of the Dirichlet and Neumann Laplacians on $(0,\ell_1) \times (0,\ell_2)$.

Exercise 19 (Application of the trace formula for Hilbert-Schmidt operators). Let us recall some constructions from the theory of ordinary differential equations (Green functions for boundary value problems).

Let $a_0, a_1 : [a,b] \to \mathbb{C}$ be continuous functions and $Ly := y'' + a_1 y + a_0 y$. Let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}$ and $R_1 y := \alpha_1 y(a) + \alpha_2 y'(a)$, $R_2 y := \beta_1 y(b) + \beta_2 y'(b)$. Assume that the only solution to $Ly = 0$ with $R_1 y = R_2 y = 0$ is the zero function.

Let y_1 be a non-zero solution of $Ly = 0$ with $R_1 y = 0$ and y_2 be a non-zero solution to $Ly = 0$ with $R_2 y = 0$. Consider $W := y_1 y_2' - y_1' y_2$ (Wronski determinant) and

$$G(x,s) = \begin{cases} \dfrac{y_1(x) y_2(s)}{W(s)}, & x < s, \\ \dfrac{y_1(s) y_2(x)}{W(s)}, & x > s, \end{cases}$$

then for any $f \in C^0([a,b])$ the function

$$y(x) := \int_a^b G(x,s) f(s) \, \mathrm{d}s$$

is the unique solution to $Ly = f$ with $R_1 y = R_2 y = 0$.

Now let T be the Dirichlet Laplacian on the interval $(0,1)$.

1. Show that T^{-1} is a Hilbert-Schmidt operator, deduce that it is an integral operator and compute its integral kernel.

2. Compute the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^4}.$$

Exercise 20 (Perturbations of operators with compact resolvents).
Let $U \in L^2_{\text{loc}}(\mathbb{R})$ be real-valued, lower semibounded, $\lim_{|x| \to +\infty} U(x) = +\infty$. In addition, let $W \in L^2_{\text{loc}}(\mathbb{R}) \cap L^1(\mathbb{R})$ be real-valued and $V := U + W$. Show that the operator

$$T = -\frac{d^2}{dx^2} + V$$

(defined through the Friedrichs extension) has compact resolvent.

Hint: Exercise 14 may be useful.

Exercise 21 ($-\Delta + V$ with compact resolvent but $V(x) \not\longrightarrow +\infty$ for $|x| \to +\infty$).

1. Let $V, W \in L^2_{\text{loc}}(\mathbb{R}^d)$ be real-valued, lower semibounded, with $V \le W$. Show: if $H^1_V(\mathbb{R}^d)$ is compactly embedded in $L^2(\mathbb{R}^d)$, then also $H^1_W(\mathbb{R}^d)$ is compactly embedded in $L^2(\mathbb{R}^d)$.

2. Let $a > 0$.

   (a) Compute the spectrum of the operator

   $$T_a := -\frac{d^2}{dx^2} + a^2 x^2$$

   defined through the Friedrichs extension in $L^2(\mathbb{R})$.

   Hint: The case $a = 1$ is already known (harmonic oscillator). Consider the unitary transform $U_a : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, $(U_a f)(x) = \sqrt[4]{a} f(\sqrt{a} x)$, and the operators $U_a^{-1} T_a U_a$.

(b) Deduce that for any $\varphi \in C_c^\infty(\mathbb{R})$ there holds

$$\int_{\mathbb{R}} \big(|\varphi'(x)|^2 + a^2x^2|\varphi(x)|^2\big)\,\mathrm{d}x \geq a \int_{\mathbb{R}} |\varphi(x)|^2\,\mathrm{d}x.$$

3. Deduce that for any $\varphi \in C_c^\infty(\mathbb{R}^2)$ there holds

$$\int_{\mathbb{R}^2} \Big(\big|\nabla\varphi(x,y)\big|^2 + x^2y^2\big|\varphi(x,y)\big|^2\Big)\,\mathrm{d}x\,\mathrm{d}y \\ \geq \frac{1}{2}\int_{\mathbb{R}^2} \Big(\big|\nabla\varphi(x,y)\big|^2 + \big(|x|+|y|\big)\big|\varphi(x,y)\big|^2\Big)\,\mathrm{d}x\,\mathrm{d}y.$$

Hint: if y is fixed, then the function $x \mapsto \varphi(x,y)$ belongs to $C_c^\infty(\mathbb{R})$

4. Deduce that the two-dimensional Schrödinger operator $T = -\Delta + x^2y^2$ has compact resolvent.

Exercise 22 (Dirichlet Laplacians with compact resolvents in unbounded domains).

1. Write the points $x \in \mathbb{R}^d$ as $x = (x', x_d)$ with $x' \in \mathbb{R}^{d-1}$ and $x_d \in \mathbb{R}$.

   Let $\Omega \subset \mathbb{R}^d$ be an open set which is bounded in the x'-direction, i.e. for some $r > 0$ one has $\Omega \subset \big\{(x', x_d) : |x'| < r\big\}$ (i.e. Ω

   Let $v : \mathbb{R} \to (0,\infty)$ be continuous with $\lim_{|t|\to\infty} v(t) = +\infty$. Equip

$$\widetilde{H}_v^1(\Omega) := \{u \in H_0^1(\Omega) : \int_\Omega v(x_d)|u(x)|^2\,\mathrm{d}x < \infty\}$$

   with the norm
$$\|u\|_v^2 := \|u\|_{H^1(\Omega)}^2 + \int_\Omega v(x_d)|u(x)|^2\,\mathrm{d}x.$$

   Show that $\widetilde{H}_v^1(\Omega)$ is compactly embedded into $L^2(\Omega)$.

2. Let $f : \mathbb{R} \to (0,\infty)$ be a continuous function with $\lim_{|x|\to\infty} f(x) = 0$. Consider the two-dimensional domain

$$\Omega := \big\{(x,y) : 0 < y < f(x)\big\} \subset \mathbb{R}^2.$$

   (a kind of strip whose width tends to zero at infinity).

   (a) Show that for any $\varphi \in C_c^\infty(\Omega)$ there holds

$$\int_\Omega |\nabla\varphi(x,y)|^2\,\mathrm{d}x \geq \frac{1}{2}\int_\Omega \Big(\big|\nabla\varphi(x,y)\big|^2 + \frac{\pi^2}{f(x)^2}\big|\varphi(x,y)\big|^2\Big)\,\mathrm{d}x\,\mathrm{d}x.$$

   Hint: for each fixed x the function $y \mapsto \varphi(x,y)$ is in $C_c^\infty\big(0, f(x)\big)$.

   (b) Deduce that the Dirichlet Laplacian in Ω has compact resolvent.

Spectral theory of differential operators

Exercise set 4

Exercise 23 (Abstract Schrödinger equation). Let A be a self-adjoint operator in a separable Hilbert space $\mathcal{H}$. Given $t \in \mathbb{R}$ we define e^{-itA} to be $f_t(A)$ for the function $f_t : \mathbb{R} \ni x \mapsto e^{-itx} \in \mathbb{C}$. Show:

1. for each $t \in \mathbb{R}$ the operator e^{-itA} is unitary,
2. $e^{-i(t+s)A} = e^{-itA}e^{-isA}$ for all $t, s \in \mathbb{R}$,
3. for any $v \in \mathcal{H}$ and $t \in \mathbb{R}$ there holds $e^{-itA}v = \lim_{s \to t} e^{-isA}v$,
4. $e^{itA}D(A) \subset D(A)$ and $Ae^{-itA} = e^{-itA}A$ on $D(A)$ for any $t \in \mathbb{R}$.

For $v \in D(A)$ consider the initial value problem

$$iu'(t) = Au(t) \text{ for all } t \in \mathbb{R}, \quad u(0) = v, \tag{1}$$

to be satisfied by a differentiable function $u : \mathbb{R} \ni t \mapsto u(t) \in \mathcal{H}$ such that $u(t) \in D(A)$ for any $t \in \mathbb{R}$. Show:

5. if u is a solution of (1), then $\|u\|$ is constant.
6. the function $u : \mathbb{R} \ni t \mapsto e^{-itA}v \in \mathcal{H}$ is a solution of (1).
7. this solution is unique.

Exercise 24 (Domains). Let T be a self-adjoint operator in a separable Hilbert space $\mathcal{H}$ and let X, μ, h be as in the spectral theorem.

1. For $n \in \mathbb{N}$ with $n \geq 2$ define $D_n(T) := \{x \in D(T) : \ Tx \in D_{n-1}(T)\}$, where we set $D_1(T) := D(T)$.
   (a) Show that $D_n(T)$ is dense in $\mathcal{H}$.
   (b) Let T_n be the restriction of T on $D_n(T)$. Show that T_n is essentially self-adjoint.
2. For any Borel function $f : \mathbb{R} \to \mathbb{C}$ define $f(T) := \Theta M_{f \circ h} \Theta^{-1}$. Show: if T is semibounded from below, then $Q(T) = D(\sqrt{|T|})$. Recall that the form domain $Q(T)$ was defined in the chapter dealing with the Friedrichs extension.

Exercise 25 (Abstract wave equation). Let A be a self-adjoint operator in a separable Hilbert space $\mathcal{H}$ such that $A \geq 0$ and $\ker A = \{0\}$. We say that a function $u : \mathbb{R} \to \mathcal{H}$ is a solution of the wave equation

$$u''(t) + Au(t) = 0, \tag{2}$$

if $u \in C^2(\mathbb{R}, \mathcal{H})$ and the inclusion $u(t) \in D(A)$ and the equality (2) hold for any $t \in \mathbb{R}$.

For $t \in \mathbb{R}$ we define $C_t, S_t : \mathbb{R} \to \mathbb{R}$ by

$$C_t(x) = \cos(t\sqrt{x}) \text{ and } \quad S_t(x) = \frac{\sin(t\sqrt{x})}{\sqrt{x}} \text{ for } x > 0, \quad C_t(x) = S_t(x) = 0 \text{ for } x \le 0.$$

Let $u_0 \in D(A)$ and $u_1 \in D(\sqrt{A})$ and define $\varphi, \psi : \mathbb{R} \to \mathcal{H}$ by

$$\varphi(t) = C_t(A)u_0, \quad \psi(t) = S_t(A)u_1.$$

1. Show that $\varphi(t)$ and $\psi(t)$ belong to $D(A)$ for any $t \in \mathbb{R}$.
2. Show that $\varphi \in C^1(\mathbb{R}, \mathcal{H})$ and that $\varphi'(t) = -AS_t(A)u_0$ for any $t \in \mathbb{R}$.
3. Show that $\psi \in C^1(\mathbb{R}, \mathcal{H})$ and that $\psi'(t) = C_t(A)u_1$ for any $t \in \mathbb{R}$.
4. Show that both φ and ψ are solutions of (2).

Now we would like to show that $u(t) = \varphi(t) + \psi(t)$ is the unique solution to (2) satisfying the initial conditions $u(0) = u_0$ and $u'(0) = u_1$. Let w be any solution satisfying the same initial conditions. Set $v(t) := u(t) - w(t)$, $t \in \mathbb{R}$.

5. Show the equality

$$\frac{d}{dt}\big\langle v(t), Av(t)\big\rangle = \big\langle v'(t), Av(t)\big\rangle + \big\langle Av(t), v'(t)\big\rangle.$$

   Hint: use the classical definition of the derivative.
6. Show that the value $E(t) = \big\langle v'(t), v'(t)\big\rangle + \big\langle v(t), Av(t)\big\rangle$ is independent of t.
7. Show that $v(t) = 0$ for all $t \in \mathbb{R}$.

Let A :=the free Laplacian in $\mathcal{H} := L^2(\mathbb{R})$.

8. Show that for $f \in C_c^\infty(\mathbb{R})$ one has

$$C_t(A)f(x) = \frac{f(x+t) + f(x-t)}{2}, \quad S_t(A)f(x) = \frac{1}{2}\int_{x-t}^{x+t} f(s)\,\mathrm{d}s, \quad x \in \mathbb{R}.$$

Exercise 26 (Essential self-adjointness for semibounded operators). Let T be a densely defined symmetric operator in a Hilbert space $\mathcal{H}$ with $T \ge 0$. Let $a > 0$.

1. Show that for any $x \in D(T)$ there holds

$$\|Tx\|^2 + a^2\|x\|^2 \le \|(T+a)x\|^2 \le 2\big(\|Tx\|^2 + a^2\|x\|^2\big).$$

2. Show that $\overline{\operatorname{ran}(T+a)} = \operatorname{ran}(\overline{T}+a)$.

3. Show that the following three assertions are equivalent:

   (a) T is essentially self-adjoint,

   (b) $\ker(T^*+a) = \{0\}$,

   (c) $\operatorname{ran}(T+a)$ is dense in $\mathcal{H}$.

Exercise 27 (Kato-Rellich theorem). We are going to complete the proof of the Kato-Rellich theorem.

Let A be a self-adjoint operator in a separable Hilbert space $\mathcal{H}$ and B be a symmetric operator in $\mathcal{H}$ which is A-bounded with relative bound < 1.

1. Let $\mathcal{D} \subset D(A)$ be a subspace on which A is essentially self-adjoint. Show that $A+B$ is also essentially self-adjoint on $\mathcal{D}$.

2. Now assume additionally that A is semibounded from below.

   (a) Show that $\|B(A+\lambda)^{-1}\| < 1$ for all sufficiently large $\lambda > 0$.

   (b) Deduce that $A+B$ is semibounded from below.

Exercise 28. Let $V \in L^\infty_{\text{loc}}(\mathbb{R}^d)$ be real-valued and consider the associated multiplication operator M_V in $\mathcal{H} = L^2(\mathbb{R}^d)$.

1. Show that the spectrum of M_V is purely essential.

2. Show that M_V is essentially self-adjoint on $C_c^\infty(\mathbb{R}^d)$.

Exercise 29.

1. Let T be the free Laplacian in $\mathcal{H} := L^2(\mathbb{R}^d)$.

   (a) Show that ∂_j is infinitesimally small with respect to T.

   (b) Show that ∂_j is not T-compact.
   Hint: compute the spectrum of $T + i\partial_j$.

   (c) Let $a \in C_c^\infty(\mathbb{R}^d)$. Show that $a\partial_j$ is T-compact.
   Hint: Use compact embeddings of H_0^1 in L^2 on bounded domains.

   (d) Let $a \in C^\infty(\mathbb{R}^d)$ such that $\lim_{|x|\to\infty} a(x) = 0$. Show that $a\partial_j$ is T-compact.

2. Let $A \in C^\infty(\mathbb{R}^d, \mathbb{R}^d)$ such that A and ∇A are bounded. Consider the operator $T_A := (i\nabla + A)^2$ on $D(T_A) = C_c^\infty(\mathbb{R}^d)$,

$$T_A : u \mapsto \sum_{j=1}^d (i\partial_j + A_j)^2 u, \quad (i\partial_j + A_j)u := i\partial_j u + A_j u.$$

   Such operators are usually called *magnetic Schrödinger operators.*

(a) Show that T_A is essentially self-adjoint and determine the domain of its closure. We denote the closure again by T_A.

(b) Assume that $\lim_{|x|\to\infty} |\nabla A(x)| + |A(x)| = 0$. Compute the essential spectrum of T_A, then the whole spectrum of T_A.

Exercise 30 (Existence of several eigenvalues).

1. Let T be a lower semibounded self-adjoint operator in a Hilbert space $\mathcal{H}$. Assume that the essential spectrum of T is non-empty and denote

$$\Sigma := \inf \operatorname{spec}_{\text{ess}} T.$$

Furthermore, assume that there exist N linearly independent vectors $f_1, \dots, f_N$ in $D(T)$ such that all eigenvalues of the $N \times N$ matrix

$$\Big(\big\langle f_j, (T - \Sigma) f_k \big\rangle\Big)_{j,k=1}^{N}$$

are strictly negative. Show that T has at least N eigenvalues in $(-\infty, \Sigma)$.

2. Consider the following operator T in $\mathcal{H} = L^2(\mathbb{R})$:

$$T = \frac{d^4}{dx^4} + 2\frac{d^2}{dx^2} + 1, \quad D(T) = H^4(\mathbb{R}).$$

(a) Show that T is self-adjoint and compute its spectrum. Hint: Use the Fourier transform.

(b) Let $V \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ be real-valued. Show that the operator

$$S := T + V, \quad D(S) = H^4(\mathbb{R}),$$

is self-adjoint and compute its essential spectrum.

(c) Let $\mathcal{F}$ be the Fourier transform in $L^2(\mathbb{R})$ and $\widehat{V} := \mathcal{F}V$. Give an explicit expression for the operator $\widehat{S} := \mathcal{F} S \mathcal{F}^{-1}$ and describe its domain.

(d) Let $\varphi \in C_c^\infty(\mathbb{R})$ with $\varphi \geq 0$ and $\|\varphi\|_{L^1(\mathbb{R})} = 1$. For $\varepsilon > 0$ and $q \in \mathbb{R}$ consider the following functions:

$$\varphi_{q,\varepsilon} : \mathbb{R} \ni \xi \mapsto \frac{1}{\varepsilon} \varphi\Big(\frac{\xi - q}{\varepsilon}\Big).$$

Show that these functions belong to $D(\widehat{S})$ and that

$$\lim_{\varepsilon \to 0+} \big\langle \varphi_{q,\varepsilon}, \widehat{S} \varphi_{r,\varepsilon} \big\rangle = \widehat{V}(q - r) \quad \text{for } q, r = \pm 1.$$

(e) Assume that $\widehat{V}(0) < 0$ and $\big|\widehat{V}(2)\big| < \big|\widehat{V}(0)\big|$. Show that the operator S has at least two negative eigenvalues.

Spectral theory of differential operators
Exercise set 5

Exercise 31. Let $\alpha \in \mathbb{R}$. Consider the following sesquilinear form t in $L^2(\mathbb{R})$:

$$t(u,u) = \int_{\mathbb{R}} \big|u'(x)\big|^2 \,\mathrm{d}x + \alpha\big|u(0)\big|^2, \quad D(t) = H^1(\mathbb{R}).$$

1. Show that t is closed. (Hint: Exercise 14.)

Denote

- $T :=$ the self-adjoint operator generated by t,
- $S :=$ the restriction of T on $C_c^\infty(\mathbb{R} \setminus \{0\})$,
- $T_0 :=$ the free Laplacian on $\mathbb{R}$,
- $S_0 :=$ the restriction of T_0 on $C_c^\infty(\mathbb{R} \setminus \{0\})$,

2. Show that $S = S_0$.
3. Let $\lambda \in \mathbb{C}$. Show that $\ker(S^* - \lambda)$ is contained in $C^\infty\big((-\infty, 0]\big) \cap C^\infty\big([0,\infty)\big)$ and is finite-dimensional.
4. Deduce that $(T+i)^{-1} - (T_0+i)^{-1}$ is a compact operator.
5. Compute the essential spectrum of T.
6. Compute the discrete spectrum of T.

Exercise 32 (Bottom of the spectrum). Let T be a lower semibounded self-adjoint operator and t be its closed sesquilinear form.

1. Show that the following two conditions are equivalent:
   (a) $u \in \ker\big(T - \Lambda_1(T)\big)$,
   (b) $u \in D(t)$ and $t(u,u) = \Lambda_1(T)\|u\|^2$.
2. Let T be the Dirichlet Laplacian on an open set Ω. Show: if $\inf \operatorname{spec} T$ is an eigenvalue, then it is strictly positive.

Exercise 33 (Poincaré-Wirthinger inequality).

1. Let T be a lower sembounded self-adjoint operator and t be its closed sesquilinear form. Assume that $\Lambda_1(T)$ is an isolated point of $\operatorname{spec} T$ and denote by P the orthogonal projector on $\ker\big(T - \Lambda_1(T)\big)$. Show that for any $u \in D(t)$ one has the inequality

$$t(u,u) \geq \Lambda_1(T)\|Pu\|^2 + \Lambda_2(T)\big\|(I-P)u\big\|^2.$$

2. Let $\Omega \subset \mathbb{R}^d$ be a bounded connected open set with Lipschitz boundary and T be the Neumann Laplacian in Ω. Show that for any $u \in H^1(\Omega)$ one has

$$\int_\Omega |\nabla u(x)|^2 \, \mathrm{d}x \geq E_2(T) \int_\Omega \left| u(x) - \frac{1}{|\Omega|} \int_\Omega u(y) \, \mathrm{d}y \right|^2 \mathrm{d}x.$$

Exercise 34 (0 is always in the Neumann spectrum).

Let $\Omega \subset \mathbb{R}^d$ be an arbitrary open set and T be the Neumann Laplacian in Ω. We want to show that $0 \in \operatorname{spec} T$.

For $n \in \mathbb{N}$ denote $\Omega_n := \Omega \cap \{x \in \mathbb{R}^d : |x| < n\}$.

1. Show that for some $n_k \to +\infty$ one has

$$\frac{|\Omega_{n_k}| - |\Omega_{n_k - 1}|}{|\Omega_{n_k - 1}|} \xrightarrow{k \to \infty} 0.$$

2. Let $\chi : \mathbb{R} \to \mathbb{R}$ be a C^∞-function with $\chi(t) = 1$ for $t < 0$ and $\chi(t) = 0$ for $t \geq 1$. Consider the functions

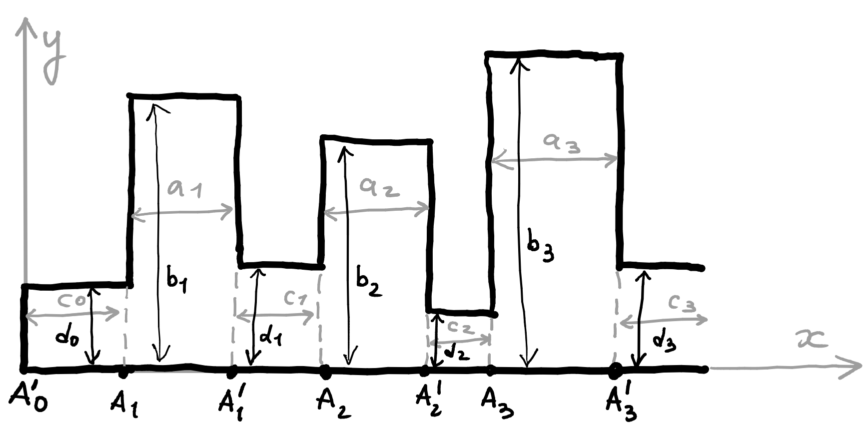
$$\varphi_n : \Omega \to \mathbb{R}, \quad \varphi_n(x) = \chi\big(|x| - (n-1)\big), \quad n \in \mathbb{N}.$$

Show that there exist $K > 0$ and $N \in \mathbb{N}$ such that

$$\frac{\displaystyle\int_\Omega |\nabla \varphi_n|^2 \, \mathrm{d}x}{\displaystyle\int_\Omega |\varphi_n|^2 \, \mathrm{d}x} \leq K \, \frac{|\Omega_n| - |\Omega_{n-1}|}{|\Omega_{n-1}|} \text{ for any } n \geq N.$$

3. Deduce that $0 \in \operatorname{spec} T$.

Exercise 35 (Neumann Laplacians: rooms and passages). Let $\Omega \subset \mathbb{R}^2$ be an open set that can be decomposed in infinitely many rectangles as shown on the picture:



Namely let $a_j, b_j, c_j, d_j > 0$. Define

$$A_k := c_0 + \sum_{j=1}^{k-1} (a_j + c_j), \quad k \in \mathbb{N}, \qquad A'_k := A_{k+1} - c_k, \quad k \in \mathbb{N}_0, \qquad L := \lim_{k\to\infty} A_k.$$

Consider the function $h : (0, L) \to (0, \infty)$,

$$h(x) := \begin{cases} d_j, & A'_j < x \le A_{j+1} \text{ for some } j \in \mathbb{N}_0, \\ b_j, & A_j < x \le A'_j \text{ for some } j \in \mathbb{N}, \end{cases}$$

and the open set

$$\Omega := \big\{(x, y) : 0 < x < L,\ 0 < y < h(x)\big\}.$$

Pick any C^∞ function $\chi : \mathbb{R} \to \mathbb{R}$ with $\chi(t) = 0$ for $t < -\frac{1}{2}$ and $\chi(t) = 1$ for $t \ge 0$ and consider the functions φ_n on Ω defined by

$$\varphi_n(x, y) = \chi\Big(\frac{x - A_n}{c_{n-1}}\Big)\chi\Big(\frac{A'_n - x}{c_n}\Big), \quad n \in \mathbb{N}.$$

1. Show that φ_n have disjoint supports.
2. Show: there exists a constant $K > 0$ such that

$$\frac{\displaystyle\int_\Omega \big|\nabla\varphi_n(x, y)\big|^2 \,\mathrm{d}x\,\mathrm{d}y}{\displaystyle\int_\Omega \big|\varphi_n(x, y)\big|^2 \,\mathrm{d}x\,\mathrm{d}y} \le K \frac{\dfrac{d_{n-1}}{c_{n-1}} + \dfrac{d_n}{c_n}}{a_n b_n} \text{ for any } n \in \mathbb{N}.$$

3. Use this computation to construct a bounded open set Ω such that the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is not compact and the Neumann Laplacian in Ω has non-empty essential spectrum..

Exercise 36 (Continuity of Dirichlet eigenvalues with respect to domain).

1. Let $d \ge 2$ and $\Omega \subset \mathbb{R}^d$ be a bounded open set. For $\lambda > 0$ define

$$\Omega_\lambda := \big\{(\lambda x_1, x_2, \dots, x_d) : (x_1, \dots, x_d) \in \Omega\big\}.$$

   Let $n \in \mathbb{N}$ be fixed. Show that the n-th eigenvalue of the Dirichlet Laplacian in Ω_λ is continuous with respect to λ.

2. Let $\Omega_j, \Omega \subset \mathbb{R}^d$ be bounded open sets such that

$$\Omega_j \subset \Omega_{j+1} \text{ for all } j \in \mathbb{N}, \qquad \Omega = \bigcup_{j=1}^\infty \Omega_j.$$

   Let $n \in \mathbb{N}$ be fixed. Show that the n-th Dirichlet eigenvalue of Ω_j converges to the n-th Dirichlet eigenvalue of Ω as $j \to \infty$.

Exercise 37 (Weyl asymptotics for Schrödinger operators). For any function $F : \mathbb{R}^2 \to \mathbb{R}$ we define its negative part $F_- := \max\{0, -F\}$.

Let $V : \mathbb{R}^2 \to \mathbb{R}$ be real-valued, continuous, such that $V \geq 0$ outside a compact set. Consider the parameter-dependent Schrödinger operator

$$T = -\Delta + \lambda V \text{ in } L^2(\mathbb{R}^2), \quad \lambda > 0.$$

and denote

$$\mathcal{N}(\lambda) := \text{the number of negative eigenvalues of } T$$

(which is finite as shown in the lectures). We are going to show that

$$\lim_{\lambda \to +\infty} \frac{\mathcal{N}(\lambda)}{\lambda} = \frac{1}{4\pi} \int_{\mathbb{R}^2} V_-(x)\,\mathrm{d}x. \tag{3}$$

Choose $R > 0$ such that $V(x) \geq 0$ for all $x \notin (-R, R) \times (-R, R)$. Let $n \in \mathbb{N}$. For $m = (m_1, m_2) \in (1, \dots, n) \times (1, \dots, n)$ consider the open squares

$$S_{n,m} = \Big(-R + 2R\,\frac{m_1 - 1}{n}, -R + 2R\,\frac{m_1}{n} \Big) \times \Big(-R + 2R\,\frac{m_2 - 1}{n}, -R + 2R\,\frac{m_2}{n} \Big),$$

$$\text{and denote } S_n := \bigcup_{m_1, m_2 = 1}^{n} S_{n,m}, \quad \widetilde{S}_n := \mathbb{R}^2 \setminus \overline{S_n}.$$

Introduce $U_n^\pm : \mathbb{R}^2 \to \mathbb{R}$ by:

$$U_n^-(x) = \begin{cases} U_{n,m}^- := \inf_{x \in S_{n,m}} V, & x \in S_{n,m} \text{ with some } m, \\ 0, & x \notin S_n, \end{cases}$$

$$U_n^+(x) = \begin{cases} U_{n,m}^+ := \sup_{S_{n,m}} V, & x \in S_{n,m} \text{ with some } m, \\ 0, & x \notin S_n, \end{cases}$$

and denote by

- $T_n^+ :=$ the self-adjoint operator in $L^2(S_n)$ given by the sesquilinear form

$$t_n^+(u, u) = \int_{S_n} \big|\nabla u(x)\big|^2\,\mathrm{d}x + \lambda \int_{S_n} U_n^+ \big|u(x)\big|^2\,\mathrm{d}x, \quad D(t_n^+) = H_0^1(S_n)$$

- $T_n^- :=$ the self-adjoint operator in $L^2(\mathbb{R}^2)$ given by the sesquilinear form

$$t_n^-(u, u) = \int_{S_n \cup \widetilde{S}_n} \big|\nabla u(x)\big|^2\,\mathrm{d}x + \lambda \int_{\mathbb{R}^2} U_n^- \big|u(x)\big|^2\,\mathrm{d}x, \quad D(t_n^-) = H^1(S_n \cup \widetilde{S}_n).$$

1. Show that $T_n^\pm$ can be represented as direct sums of operators $A_{n,m}^\pm$ in $L^2(S_{n,m})$ and $\widetilde{A}_n$ in $L^2(\widetilde{S}_n)$ whose spectra can be computed explicitly.

2. Let $\mathcal{N}_n^\pm(\lambda)$ be the number of negative eigenvalues of $T_n^\pm$. Show that both numbers are finite and that

$$\mathcal{N}_n^+(\lambda) \leq \mathcal{N}(\lambda) \leq \mathcal{N}_n^-(\lambda) \text{ for all } n \in \mathbb{N} \text{ and } \lambda > 0$$

3. Show that
$$\lim_{\lambda\to+\infty}\frac{\mathcal{N}_n^{\pm}(\lambda)}{\lambda}=\frac{1}{4\pi}\int_{\mathbb{R}^2}\big(U_n^{\pm}\big)_-(x)\,\mathrm{d}x.$$

4. Let $\varepsilon>0$. Show: one can find $n_\varepsilon\in\mathbb{N}$ such that
$$\left|\int_{\mathbb{R}^2}\big(U_n^{\pm}\big)_-(x)\,\mathrm{d}x-\int_{\mathbb{R}^2}V_-(x)\,\mathrm{d}x\right|<\varepsilon \text{ for all } n\geq n_\varepsilon..$$

5. Show the relation (3).

Exercise 38 (Rapidly decaying potentials produce finitely many eigenvalues). Let $d\geq 3$ and $V\in L^\infty(\mathbb{R}^d)$ real-valued with
$$V(x)=o\Big(\frac{1}{|x|^2}\Big) \text{ for } |x|\to\infty.$$

Consider the Schrödinger operator $T=-\Delta+V$ in $L^2(\mathbb{R}^d)$.

1. Compute the essential spectrum of T.

Let H be the Hardy potential,
$$H:\ \mathbb{R}^d\ni x\mapsto\frac{(d-2)^2}{4|x|^2}\in\mathbb{R}.$$

2. Show: for some $a\in(0,1)$ one has $V\geq -aH+W$, where W is a bounded real-valued potential vanishing outside a compact set.

3. Show that $T\geq -(1-a)\Delta+W$.

4. Deduce that T has at most finitely many negative eigenvalues.

Exercise 39 (Dirichlet Laplacians in infinite cylinders).
Let $\omega\subset\mathbb{R}^d$ be a bounded open set and
$$\Omega:=\omega\times\mathbb{R}\subset\mathbb{R}^{d+1}.$$

We denote the points of $x\in\mathbb{R}^{d+1}$ as $x=(x',y)$ with $x'\in\mathbb{R}^d$ and $y\in\mathbb{R}$. Denote by T_ω and T_Ω the Dirichlet Laplacians in ω and Ω respectively and denote
$$\Lambda:=E_1(T_\omega).$$

1. Show that $T_\Omega\geq\Lambda$.

2. Show: if $u\in D(T_\omega)$ and $\varphi\in C_c^\infty(\mathbb{R})$, then the function $v:(x',y)\mapsto u(x')\varphi(y)$ belongs to $D(T_\Omega)$, and compute $T_\Omega v$.

3. Let u be an eigenfunction of T_ω for the first eigenvalue. Furthermore, let $\chi \in C_c^\infty(\mathbb{R})$ with $\chi(t) = 1$ for $|t| \le 1$ and $\chi(t) = 0$ for $|t| \ge 2$. Let $k \ge 0$. Show that the functions

$$v_n : (x', y) \mapsto u(x') e^{iky} \chi\Big(\frac{y}{n}\Big)$$

form a Weyl sequence for T_Ω and $\Lambda + k^2$.

4. Show that $\operatorname{spec} T_\Omega = [\Lambda, \infty)$.

Let $V \in C^0(\overline{\Omega})$ be real-valued with $V(x) \to 0$ as $|x| \to \infty$.

5. Recall why $T_\Omega + V$ is a well-defined self-adjoint operator, and show that its essential spectrum is $[\Lambda, \infty)$.

   Hint: Take the above functions v_n and consider $w_n : (x, y) \mapsto v_n(x, y - 3n)$. One may also use Persson's theorem.

6. Assume <u>in addition</u> that

   - there exists $W \in L^1(\mathbb{R})$ with $\big|V(x', y)\big| \le W(y)$ for all $(x', y) \in \Omega$,
   - $V \le 0$,
   - there exists a non-empty interval $(a, b) \subset \mathbb{R}$ such that $V(x', y) < 0$ for all $(x', y) \in \omega \times (a, b)$.

   Show that $T_\Omega + V$ has at least one eigenvalue in $(-\infty, \Lambda)$.

Exercise 40 (Dirichlet Laplacians in half-infinite cylinders and perturbations). Let $\omega \subset \mathbb{R}^d$ be a bounded open set and

$$\Omega := \omega \times (0, \infty) \subset \mathbb{R}^{d+1}.$$

We denote the points of $x \in \mathbb{R}^{d+1}$ as $x = (x', y)$ with $x' \in \mathbb{R}^d$ and $y \in \mathbb{R}$. Denote

$$\Lambda := E_1(T_\omega)$$

and let T be the Dirichlet Laplacian in Ω. Let $V \in C^0(\overline{\Omega})$ be real-valued with $V(x) \to 0$ as $|x| \to \infty$.

1. Show that $\operatorname{spec} T = [\Lambda, \infty)$.

2. Show that $\operatorname{spec}_{\text{ess}}(T + V) = [\Lambda, \infty)$.

   Hint: one may proceed very similarly to Exercise 39.

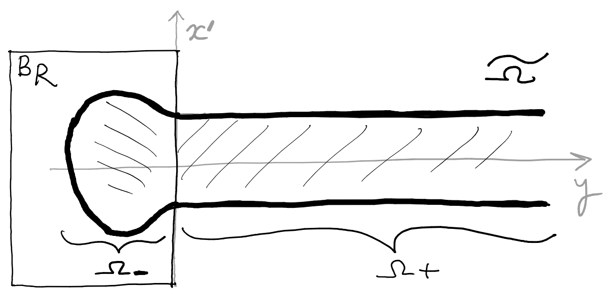
3. Assume that $V(x) = o(|x|^{-2})$ as $|x| \to \infty$. Show: there exists $\lambda_0 > 0$ such that one has $\operatorname{spec}(T + \lambda V) = [\Lambda, \infty)$ for all $\lambda \in (-\lambda_0, \lambda_0)$.

   Hint: one may use the one-dimensional Hardy inequality (Exercise 14).

Now let $\widetilde{\Omega} \subset \mathbb{R}^{d+1}$ be an open set such that:

- $\Omega_+ := \widetilde{\Omega} \cap \big\{(x', y) : \; y > 0\big\} = \Omega$,
- $\Omega_- := \widetilde{\Omega} \cap \big\{(x', y) : \; y < 0\big\}$ is bounded,

in other words, $\widetilde{\Omega}$ is obtained by attaching a bounded open set to the left end of Ω. Denote by $\widetilde{T}$ the Dirichlet Laplacian in $\widetilde{\Omega}$.



4. Show that $\operatorname{spec}_{\text{ess}} \widetilde{T} = [\Lambda, \infty)$.

For open $U \subset \widetilde{\Omega}$ denote

$$\widetilde{C}_c^\infty(U) = \Big\{u : U \to \mathbb{C} : \; u \text{ can be extended to a function in } C_c^\infty(\widetilde{\Omega})\Big\}$$

and consider the sesquilinear forms $t_\pm$ in $L^2(\Omega_\pm)$ given by

$$t_\pm(u, u) = \int_{\Omega_\pm} |\nabla u|^2 \, \mathrm{d}x, \quad D(t_\pm) = \widetilde{C}_c^\infty(\Omega_\pm).$$

5. Show that both $t_\pm$ are closable.

We denote their closures again by $t_\pm$ and the associated self-adjoint operators in $L^2(\Omega_\pm)$ by $T_\pm$.

6. Show that T_- has compact resolvent.

   Hint: Let $R > 0$ such that $\Omega_- \subset (-R, R)^d \times (-R, 0) =: B_R$. Show that the embedding $D(t_-) \hookrightarrow H^1(B_R)$ is continuous.

7. Show that $\operatorname{spec} T_+ = [\Lambda, \infty)$.

8. Show that $\widetilde{T}$ has at most finitely many eigenvalues in $(-\infty, \Lambda)$.

   Hint: Compare $\widetilde{T}$ with $T_- \oplus T_+$.

9. Propose an explicit example of $\widetilde{\Omega}$ of the above type such that $\widetilde{T}$ actually has at least one eigenvalue in $(-\infty, \Lambda)$.