Spectral theory and asymptotic methods

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Contents

Notation

Here we list some conventions used throughout the text.

The symbol $\mathbb{N}$ denotes the sets of the natural numbers starting from 1.

If (M,μ) is a measure space and $f : M \to \mathbb{C}$ is a measurable function, then we denote

$$\operatorname{ess}_\mu \operatorname{ran} f := \Big\{ z \in \mathbb{C} : \, \mu\big\{ m \in M : \big|z - f(m)\big| < \varepsilon \big\} > 0 \text{ for all } \varepsilon > 0 \Big\},$$

$$\operatorname{ess}_\mu \sup |f| := \inf \Big\{ a \in \mathbb{R} : \, \mu\big\{ m \in M : \big|f(m)\big| > a \big\} = 0 \Big\}.$$

If the measure μ is uniquely determined by the context, then the index μ will be sometimes omitted.

In what follows the phrase "Hilbert space" should be understood as "separable complex Hilbert space". Most propositions also work in the non-separable case if reformulated in a suitable way. If the symbol "$\mathcal{H}$" appears without explanations, it denotes a certain Hilbert space. If $\mathcal{H}$ is a Hilbert space and $x, y \in \mathcal{H}$, then by $\langle x, y \rangle$ we denote the scalar product of x and y. If there is more than one Hilbet space in play, we use the more detailed notation $\langle x, y \rangle_{\mathcal{H}}$. We assume that the scalar product is linear with respect to the *second* argument and as anti-linear with respect to the first one, i.e. that for all $\alpha \in \mathbb{C}$ we have $\langle x, \alpha y \rangle = \langle \overline{\alpha} x, y \rangle = \alpha \langle x, y \rangle$. This means, for example, that the scalar product in the standard space $L^2(\mathbb{R})$ is defined by

$$\langle f, g \rangle = \int_{\mathbb{R}} \overline{f(x)} g(x) \, dx.$$

If A is a finite or countable set, we denote by $\ell^2(A)$ the vector space of the functions $x : A \to \mathbb{C}$ with

$$\sum_{a \in A} \big|\xi(a)\big|^2 < \infty,$$

and this is a Hilbert space with the scalar product

$$\langle x, y \rangle = \sum_{a \in A} \overline{x(a)} y(a).$$

If $\mathcal{H}$ and $\mathcal{G}$ are Hilbert spaces, then by $\mathcal{L}(\mathcal{H}, \mathcal{G})$ and $\mathcal{K}(\mathcal{H}, \mathcal{G})$ we denotes the spaces of the linear operators and the one of the compact operators from $\mathcal{H}$ and $\mathcal{G}$, respectively. Furtheremore, $\mathcal{L}(\mathcal{H}) := \mathcal{L}(\mathcal{H}, \mathcal{H})$ and $\mathcal{K}(\mathcal{H}) := \mathcal{K}(\mathcal{H}, \mathcal{H})$.

If $\Omega \subset \mathbb{R}^d$ is an open set and $k \in \mathbb{N}$, then $H^k(\Omega)$ denotes the kth Sobolev space, i.e. the space of L^2 functions whose weak partial derivatives up to order k are also in $L^2(\Omega)$, see Section 1.2, and by $H^k_0(\Omega)$ we denote the completion of $C^\infty_c(\Omega)$ with respect to the norm of $H^k(\Omega)$. The symbol $C^k(\Omega)$ denotes the space of functions on Ω whose partial derivatives up to order k are continuous; i.e. the set of the continuous functions is denoted as $C^0(\Omega)$. This should not be confused with $C_0(\mathbb{R}^d)$ which is the set of the continuous functions f on $\mathbb{R}^d$ vanishing at infinity: $\lim_{|x|\to\infty} f(x) = 0$. The subindex $_{\text{comp}}$ or $_c$ means that we only consider the functions with compact supports in the respective space (i.e. the functions vanishing outside a compact set). E.g. $H^1_{\text{comp}}(\mathbb{R}^d)$ is the set of the functions from $H^1(\mathbb{R}^d)$ having compact supports.

Recommended books

During the preparation of the notes I used a part of the text by Bernard Helffer which is available online:

- B. Helffer: *Spectral theory and applications. An elementary introductory course.* Available at `http://www.math.u-psud.fr/~helffer/`.

An extended version of the above text was published as a book:

- B. Helffer: *Spectral theory and its applications.* Cambridge Studies in Applied Mathematics, 2012.

Other recommended books are

- G. Teschl: *Mathematical methods in quantum mechanics. With applications to Schrödinger operators.* AMS, 2009. Available from `http://www.mat.univie.ac.at/~gerald/`
- B. Helffer: *Semi-Classical Analysis for the Schrödinger Operator and Applications. Lecture Notes in Mathematics.* Springer, 1988 (mostly for more advanced material related to the asymptotic analysis).

Additional references on particular topics will be given during the course.

At many points we will be obliged to use some facts on distributions and Sobolev spaces. I tried to include some elementary facts in these notes and I hope that it will be sufficient. An excellent introduction to the theory of distributions (which contains all necessary information on the Sobolev spaces) can also be found in these lecture notes by Patrick Gérard:

`https://www.imo.universite-paris-saclay.fr/~pgerard/Distributions2019_Chap1,2,3.pdf`

1 Unbounded operators

1.1 Closed and adjoint operators

A *linear operator* T in $\mathcal{H}$ is a linear map from a subspace (the domain of T) $D(T) \subset \mathcal{H}$ to $\mathcal{H}$. The range of T is the set $\operatorname{ran} T := \{Tx : x \in D(T)\}$. We say that a linear operator T is *bounded* if the quantity

$$\mu(T) := \sup_{\substack{x \in D(T) \\ x \neq 0}} \frac{\|Tx\|}{\|x\|}$$

is finite. In what follows, the word combination "an unbounded operator" should be understood as "an operator which is not assumed to be bounded". If $D(T) = \mathcal{H}$ and T is bounded, we arrive at the notion of a continuous linear operator in $\mathcal{H}$; the space of such operators is denoted by $\mathcal{L}(\mathcal{H})$. This is a Banach space equipped with the norm $\|T\| := \mu(T)$.

During the whole course, by introducing a linear operator we always assume that its domain is dense, if the contrary is not stated explicitly.

If T is a bounded operator in $\mathcal{H}$, it can be uniquely extended to a continuous linear operator. Let us discuss a similar idea for unbounded operators.

The *graph* of a linear operator T in $\mathcal{H}$ is the set

$$\operatorname{gr} T := \big\{(x, Tx) : x \in D(T)\big\} \subset \mathcal{H} \times \mathcal{H}.$$

For two linear operators T_1 and T_2 in $\mathcal{H}$ we write $T_1 \subset T_2$ if $\operatorname{gr} T_1 \subset \operatorname{gr} T_2$. I.e. $T_1 \subset T_2$ means that $D(T_1) \subset D(T_2)$ and that $T_2 x = T_1 x$ for all $x \in D(T_1)$; the operator T_2 is then called an extension of T_1 and T_1 is called a restriction of T_2.

Definition 1.1 (Closed operator, closable operator).

- A linear operator T in $\mathcal{H}$ is called *closed* if its graph is a closed subspace in $\mathcal{H} \times \mathcal{H}$.
- A linear operator T in $\mathcal{H}$ is called *closable*, if the closure $\overline{\operatorname{gr} T}$ of the graph of T in $\mathcal{H} \times \mathcal{H}$ is still the graph of a certain operator $\overline{T}$. This operator $\overline{T}$ with $\operatorname{gr} \overline{T} = \overline{\operatorname{gr} T}$ is called the *closure* of T. □

The following proposition is obvious:

Proposition 1.2. *A linear operator T in $\mathcal{H}$ is closed if and only if the three conditions*

- $x_n \in D(T)$,
- x_n *converge to* x *in* $\mathcal{H}$,
- Tx_n *converge to* y *in* $\mathcal{H}$

imply the inclusion $x \in D(T)$ and the equality $y = Tx$.

Definition 1.3 (Graph norm). Let T be a linear operator in $\mathcal{H}$. Define on $D(T)$ a new scalar product by $\langle x, y\rangle_T = \langle x, y\rangle + \langle Tx, Ty\rangle$. The associated norm $\|x\|_T := \sqrt{\langle x, x\rangle_T} = \sqrt{\|x\|^2 + \|Tx\|^2}$ is called the *graph norm* for T. □

The following assertion is also evident.

Proposition 1.4. *Let T be a linear operator in $\mathcal{H}$.*

- *T is closed iff $D(T)$ is complete in the graph norm.*
- *If T is closable, then $D(\overline{T})$ is exactly the completion of $D(T)$ with respect to the graph norm.*

Informally, one could say that $D(\overline{T})$ consists of those x for which there is a unique candidate for $\overline{T}x$ if one tries to extend T by density. I.e., a vector $x \in \mathcal{H}$ belongs to $D(\overline{T})$ iff:

- there exists a sequence $(x_n) \subset D(T)$ converging to x,
- their exists the limit of Tx_n,
- this limit is the same for any sequence x_n satisfying the previous two properties.

Let us consider some simple examples. More sophisticated examples involving differential operators will be discussed later in Section 1.2.

Example 1.5 (Bounded linear operators are closed). By the closed graph theorem, a linear operator T in $\mathcal{H}$ with $D(T) = \mathcal{H}$ is closed if and only if it is bounded. In this course we consider mostly unbounded closed operators.

Example 1.6 (Multiplication operator). Take $\mathcal{H} = L^2(\mathbb{R}^d)$ and pick $f \in L^\infty_{\text{loc}}(\mathbb{R}^d)$. Introduce a linear operator M_f in $\mathcal{H}$ as follows:

$$D(M_f) = \{u \in L^2(\mathbb{R}^d) : fu \in L^2(\mathbb{R}^d)\} \text{ and } \quad M_f u = fu \text{ for } u \in D(M_f).$$

It can be easily seen that $D(M_f)$ equipped with the graph norm coincides with the weighted space $L^2\big(\mathbb{R}^d, (1 + |f|^2)dx\big)$, which is complete. This shows that M_f is closed.

On the other hand, denote by T the restriction of M_f to the functions with compact supports. The functions with compact supports are dense in $L^2\big(\mathbb{R}^d, (1 + |f|^2)dx\big)$, hence, the closure $\overline{T}$ of T is exactly M_f. It also follows that that T is not closed. □

Example 1.7 (Non-closable operator). Take $\mathcal{H} = L^2(\mathbb{R})$ and pick a $g \in \mathcal{H}$ with $g \neq 0$. Consider the operator L defined on $D(L) = C^0(\mathbb{R}) \cap L^2(\mathbb{R})$ by $Lf = f(0)g$. Assume that there exists the closure $\overline{L}$ and let $f \in D(\overline{L})$. One can find two sequences $(f_n), (g_n)$ in $D(L)$ such that both converge in the L^2 norm to f but such that $f_n(0) = 0$ and $g_n(0) = 1$ for all n. Then $Lf_n = 0$, $Lg_n = g$ for all n, and both sequences Lf_n and Lg_n converge, but to different limits. This contradicts the closedness of $\overline{L}$. Hence L is not closed. □

Recall that for $T \in \mathcal{L}(\mathcal{H})$ its adjoint T^* is defined by the relation

$$\langle x, Ty \rangle = \langle T^*x, y \rangle \text{ for all } x, y \in \mathcal{H}.$$

The proof of the existence comes from the Riesz representation theorem: for each $x \in \mathcal{H}$ the map $\mathcal{H} \ni y \mapsto \langle x, Ty \rangle \in \mathbb{C}$ is a continuous linear functional, which means that there exists a unique vector, denoted by T^*x with $\langle x, Ty \rangle = \langle T^*x, y \rangle$ for all $y \in \mathcal{H}$. One can then show that the map $x \mapsto T^*x$ is linear, and by estimating the scalar product one shows that T^* is also continuous. Let us use the same idea for unbounded operators.

Definition 1.8 (Adjoint operator). If T be a linear operator in $\mathcal{H}$, then its *adjoint* T^* is defined as follows. The domain $D(T^*)$ consists of the vectors $u \in \mathcal{H}$ for which the map $D(T) \ni v \mapsto \langle u, Tv \rangle \in \mathbb{C}$ is bounded with respect to the $\mathcal{H}$-norm. For such u there exists, by the Riesz theorem, a unique vector denoted by T^*u such that $\langle u, Tv \rangle = \langle T^*u, v \rangle$ for all $v \in D(T)$. □

We note that the implicit assumption $\overline{D(T)} = \mathcal{H}$ is important here: if it is not satisfied, then the value T^*u is not uniquely determined, one can add to T^*u an arbitrary vector from $D(T)^\perp$.

Let us give a geometric interpretation of the adjoint operator. Consider a unitary linear operator

$$J : \mathcal{H} \times \mathcal{H} \to \mathcal{H} \times \mathcal{H}, \quad J(x, y) = (y, -x)$$

and note that J commutes with the operator of the orthogonal complement in $\mathcal{H}\times\mathcal{H}$, i.e. $J(V)^\perp = J(V^\perp)$ for any $V \subset \mathcal{H} \times \mathcal{H}$. This will be used several times during the course.

Proposition 1.9 (Geometric interpretation of the adjoint). *Let T be a linear operator in $\mathcal{H}$. The following two assertions are equivalent:*

- *$u \in D(T^*)$ and $f = T^*u$,*
- *$\big\langle (u, T^*u), J(v, Tv) \big\rangle_{\mathcal{H}\times\mathcal{H}} = 0$ for all $v \in D(T)$.*

In other words,

$$\operatorname{gr} T^* = J(\operatorname{gr} T)^\perp. \tag{1.1}$$

As a simple application we obtain

Proposition 1.10. *One has $(\overline{T}\,)^* = T^*$, and T^* is a closed operator.*

Proof. Follows from (1.1): the orthogonal complement is always closed, and $J(\operatorname{gr} T)^\perp = J(\overline{\operatorname{gr} T})^\perp$. □

Up to now we do not know if the domain of the adjoint contains non-zero vectors. This is discussed in the following proposition.

Proposition 1.11 (Domain of the adjoint). *Let T be a closable operator $\mathcal{H}$, then:*

(i) $D(T^*)$ is a dense subspace of $\mathcal{H}$,

(ii) $T^{**} := (T^*)^* = \overline{T}$.

Proof. The item (ii) easily follows from (i) and (1.1): one applies the same operations again and remark that $J^2 = -1$ and that taking twice the orthogonal complement results in taking the closure.

Now let us prove the item (i). Let a vector $w \in \mathcal{H}$ be orthogonal to $D(T^*)$: $\langle u, w\rangle = 0$ for all $u \in D(T^*)$. Then one has $\langle J(u, T^*u), (0, w)\rangle_{\mathcal{H}\times\mathcal{H}} \equiv \langle u, w\rangle + \langle T^*u, 0\rangle = 0$ for all $u \in D(T^*)$, which means that $(0, w) \in J(\operatorname{gr} T^*)^\perp = \overline{\operatorname{gr} T}$. As the operator T is closable, the set $\overline{\operatorname{gr} T}$ must be a graph, which means that $w = 0$. □

Let us look at some examples.

Example 1.12 (Adjoint for bounded operators). The general definition of the adjoint operator is compatible with the one for continuous linear operators.

Example 1.13. As an exercise, one can show that for the multiplication operator M_f from example 1.6 one has $(M_f)^* = M_{\overline{f}}$. □

The following definition introduces two classes of linear operator that will be studied throughout the course.

Definition 1.14 (Symmetric, self-adjoint, essentially self-adjoint operators). We say that a linear operator T in $\mathcal{H}$ is *symmetric* (or *Hermitian*) if

$$\langle u, Tv\rangle = \langle Tu, v\rangle \quad \text{for all } u, v \in D(T),$$

or, equivalently, if $T \subset T^*$. Furthermore:

- T is called *self-adjoint* if $T = T^*$,
- T is called *essentially self-adjoint* if $\overline{T}$ is self-adjoint. □

An important feature of symmetric operators is their closability:

Proposition 1.15. *Symmetric operators are closable.*

Proof. Indeed for a symmetric operator T we have $\operatorname{gr} T \subset \operatorname{gr} T^*$ and, due to the closedness of T^*, $\overline{\operatorname{gr} T} \subset \operatorname{gr} T^*$. □

Example 1.16 (Bounded symmetric operators are self-adjoint). Note that for $T \in \mathcal{L}(\mathcal{H})$ the fact to be symmetric is equivalent to the fact to be self-adjoint, but it is not the case for unbounded operators! □

Example 1.17 (Self-adjoint multiplication operators). As follows from example 1.13, the multiplication operator M_f from example 1.6 is self-adjoint iff $f(x) \in \mathbb{R}$ for a.e. $x \in \mathbb{R}^d$. $\square$

A large class of self-adjoint operators comes from the following proposition.

Proposition 1.18. *Let T be an* injective *self-adjoint operator, then its inverse is also self-adjoint.*

Proof. We show first that $D(T^{-1}) := \operatorname{ran} T$ is dense in $\mathcal{H}$. Let $u \perp \operatorname{ran} T$, then $\langle u, Tv\rangle = 0$ for all $v \in D(T)$. This can be rewritten as $\langle u, Tv\rangle = \langle 0, v\rangle$ for all $v \in D(T)$, which shows that $u \in D(T^*)$ and $T^* u = 0$. As $T^* = T$, we have $u \in D(T)$ and $Tu = 0$. As T in injective, one has $u = 0$

Now consider the operator $S : \mathcal{H} \times \mathcal{H} \to \mathcal{H} \times \mathcal{H}$ given by $S(x, y) = (y, x)$. One has then $\operatorname{gr} T^{-1} = S(\operatorname{gr} T)$. We conclude the proof by noting that S commutes with J and with the operation of the orthogonal complement in $\mathcal{H} \times \mathcal{H}$. $\square$

Exercise 1. Let H_1 and H_2 be Hilbert spaces. Let A be a linear operator in H_1, B be a linear operator in H_2. Assume that there exists a unitary operator $U : H_2 \to H_1$ such that $D(A) = UD(B)$ and that $U^*AUf = Bf$ for all $f \in D(B)$; such A and B are called *unitary equivalent*.

(a) Let two operators A and B be unitarily equivalent. Show that A is closed/symmetric/self-adjoint iff B has the respective property.

(b) Let (λ_n) be an arbitrary sequence of complex numbers, $n \in \mathbb{N}$. In the Hilbert space $\ell^2(\mathbb{N})$ consider the operator S:

$$D(S) = \big\{(x_n) : \text{there exists } N \text{ such that } x_n = 0 \text{ for } n > N\big\}, \quad S(x_n) = (\lambda_n x_n).$$

Describe the closure of S.

(c) Now let H be a separable Hilbert space and T be a linear operator in H with the following property there exists an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of H with $e_n \in D(T)$ and $Te_n = \lambda_n e_n$ for all $n \in \mathbb{N}$, where λ_n are some complex numbers.

1. Describe the closure $\overline{T}$ of T. Hint: one may use (a) and (b).
2. Describe the adjoint T^* of T.
3. Let all λ_n be real. Show that the operator $\overline{T}$ is self-adjoint.

Exercise 2. Let A and B be self-adjoint operators in a Hilbert space H such that $D(A) \subset D(B)$ and $Au = Bu$ for all $u \in D(A)$. Show that $D(A) = D(B)$. (This property is called the *maximality* of self-adjoint operators.)

Exercise 3. In *this exercise*, by the sum $A + B$ of a linear operator B with a *continuous* operator B; both acting in a Hilbert space H, we mean the operator S defined by $D(S) = D(A)$, $Su := Au + Bu$. We note that defining the sum of two operators becomes a non-trivial task if unbounded operators are involved.)

(a) Let A be a closed and B be continuous. Show that $A + B$ is closed.

(b) Assume, in addition, that A is densely defined. Show that $(A + B)^* = A^* + B^*$.

1.2 Differential operators and Sobolev spaces

The study of closability and adjointness issues for differential operators is non-trivial and it leads to the consideration of distributions and Sobolev spaces. The full theory of Sobolev spaces is rather involved (and usually they are discussed in a special course on distributions; currently there is such a course proposed by Prof. Daniel Grieser), so we just present some key points without detailed proofs. Later we will work with many operators whose domains are not known explicitly, and it is important to understand if they are closed/self-adjoint or not.

Let $\Omega \subset \mathbb{R}^d$ be an open set and $\alpha \in \mathbb{N}^d$ a multi-index. Let $f \in L^1_{\text{loc}}(\Omega)$ and $g \in L^1_{\text{loc}}(\Omega)$ such that for any $\varphi \in C_c^\infty(\Omega)$ one has the equality

$$\int_\Omega g\varphi = (-1)^{|\alpha|} \int_\Omega f \partial^\alpha \varphi, \tag{1.2}$$

where $C_c^\infty(\Omega)$ is the space of C^∞-functions on Ω vanishing outside a compact set. If a function g with the above property exists, then it is unique, and one says that g is the *weak/distributional ∂^α-derivative of f in Ω*, which will be written as $g = \widetilde{\partial^\alpha} f$. If f is of class $C^{|\alpha|}$, then one has the equality $\widetilde{\partial^\alpha} f = \partial^\alpha f$, as the equality (1.2) is obtained by applying a partial integration $|\alpha|$ times, but $\widetilde{\partial^\alpha}$ can be applied to a larger class of functions. The construction extends to differential expressions with constants coefficients: if

$$P := \sum_{|\alpha| \le m} c_\alpha \partial^\alpha, \quad c_\alpha \in \mathbb{C},$$

and $f \in L^1_{\text{loc}}(\Omega)$, then $g = \widetilde{P} f$ if and only if $g \in L^1_{\text{loc}}(\Omega)$ such that

$$\int_\Omega g\varphi = \sum_{|\alpha| \le m} c_\alpha (-1)^{|\alpha|} \int_\Omega f \partial^\alpha \varphi$$

for any $\varphi \in C_c^\infty(\Omega)$. If $f \in C^m$, then one simply has $\widetilde{P} f = Pf = \sum_{|\alpha| \le m} c_\alpha \partial^\alpha f$.

With the above conventions, let us consider the following linear operator T in the Hilbert space $\mathcal{H} = L^2(\Omega)$:

$$Tu = Pu, \quad D(T) = C_c^\infty(\Omega).$$

Using the definition of the adjoint operator one easily shows that

$$T^* u = \widetilde{P'} u, \quad D(T^*) = \big\{ u \in L^2(\Omega) : \widetilde{P'} u \in L^2(\Omega) \big\},$$

where P' is the formal adjoint of P, i.e.

$$P' = \sum_{|\alpha| \le m} \overline{c_\alpha} (-1)^{|\alpha|} \partial^\alpha.$$

(The formal adjoint has the property that $\langle \varphi, P\psi \rangle = \langle P'\varphi, \psi \rangle$ for any $\varphi, \psi \in C_c^\infty(\Omega)$.) The operator T^* is automatically closed (Proposition 1.10). The differential expression P will be called *formally self-adjoint* if $P = P'$, i.e. $c_\alpha = (-1)^{|\alpha|} \overline{c_\alpha}$

for all α. The most important examples for us are

$$P = -i\partial_{x_j}, \quad P = -\Delta \equiv -\sum_{j=1}^{d} \partial_{x_j}^2.$$

For the rest of the section we assume that P is formally self-adjoint.

Then one easily sees that $T \subset T^*$, i.e. that T is symmetric and, hence, closable. The closure of T is usually called the *minimal operator* generated by the differential expression P and is denoted $P_{\min}$. The operator T^* is called the *maximal operator* generated by the differential expression P and is denoted by $P_{\max}$.

It is natural to ask if one has $P_{\min} = P_{\max}$. If the equality holds, then $\overline{T} = T^*$, hence, T is essentially self-adjoint, while $T^* = P_{\max}$ is self-adjoint. If this above equality does not hold, then $\overline{T}$ is just symmetric, but is not self-adjoint. Checking $P_{\min} = P_{\max}$ is a difficult question as, in general, it depends on the geometry of Ω or, more precisely of the regularity properties of its boundary. It is not our objective to study the general case, but we are going to look at some specific examples.

Example 1.19. Let $\Omega = \mathbb{R}^d$ and $P = -i\partial_1$. We are going to show that $P_{\min} = P_{\max}$ and they are self-adjoint in $L^2(\mathbb{R}^d)$. In view of what is already said, we just need to show that $C_c^\infty(\mathbb{R}^d)$ is dense in $D(P_{\max})$ in the graph norm.

Remark first that the weak derivative $\widetilde{\partial}_1$ still satifies the Leibniz rule. Namely, let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ with $g := \widetilde{\partial}_1 f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. In addition, let $\chi \in C^\infty(\mathbb{R}^d)$. For any $\varphi \in C_c^\infty(\mathbb{R}^d)$ one has $\chi\varphi \in C_c^\infty(\mathbb{R}^d)$, hence,

$$\int_{\mathbb{R}^d} g\chi\varphi = -\int_{\mathbb{R}^d} f\partial_1(\chi\varphi) = -\int_{\mathbb{R}^d} f\partial_1\chi \cdot \varphi - \int_{\mathbb{R}^d} f\chi \cdot \partial_1\varphi.$$

This can be rewritten as

$$\int_{\mathbb{R}^d} (\widetilde{\partial}_1 f\chi + f\partial_1\chi)\varphi = -\int_{\mathbb{R}^d} \chi f \cdot \partial_1\varphi, \quad \varphi \in C_c^\infty(\mathbb{R}^d),$$

which literally means that $\widetilde{\partial}_1(\chi f) = \partial_1\chi f + \widetilde{\partial}_1 f \cdot \chi$.

Now let $\chi \in C_c^\infty(\mathbb{R}^d)$ with and $0 \le \chi \le 1$ and such that $\chi(x) = 1$ for $|x| \le 1$, and for $n \in \mathbb{N}$ define $\chi_n(x) := \chi(x/n)$. Let us take $u \in D(P_{\max})$, i.e.

$$u \in L^2(\mathbb{R}^d) \text{ with } \widetilde{\partial}_1 u \in L^2(\mathbb{R}^d),$$

and denote $u_n := \chi_n u$. One easily sees (dominated convergence) that $u_n \to u$ in $L^2(\mathbb{R}^d)$ as n becomes large. At the same time,

$$\widetilde{\partial}_1 u_n(x) = \frac{1}{n}\partial_1\chi(x)u(x) + \widetilde{\partial}_1 u(x) \cdot \chi_n(x).$$

Each summand on the right-hand side is L^2, hence, $u_n \in D(P_{\max})$. Moreover, the first summand tends to zero in $L^2(\mathbb{R}^d)$, while the second one converges to $\widetilde{\partial}_1 u$

(dominated convergence). As each u_n has compact support and belongs to $D(P_{\max})$, it follows that the set

$$D_c(P_{\max}) = \{u \in D(P_{\max}) : \; u \text{ has compact support}\}$$

is dense in $D(P_{\max})$ in the graph norm, and it remains to check that each function from $D_c(P_{\max})$ can be approximated in the graph norm by functions from $C_c^\infty(\mathbb{R}^d)$. This is a standard regularization procedure. Namely, let $u \in D_c(P_{\max})$. Pick $\rho \in C_c^\infty(\mathbb{R}^d)$ with $\int \rho = 1$ and for $\varepsilon > 0$ set

$$u_\varepsilon(x) = \frac{1}{\varepsilon^d} \int_{\mathbb{R}^d} u(y) \rho\Big(\frac{x-y}{\varepsilon}\Big) dx.$$

It is easily seen that $u_\varepsilon \in C_c^\infty(\mathbb{R}^d)$, and some computation shows that $u_\varepsilon \to u$ and $\partial_1 u_\varepsilon \to \widetilde{\partial}_{x_1} u$ in $L^2(\mathbb{R}^d)$ (We will omit these technical details. An interested reader may try to give a complete proof of these statements.) This concludes the proof. $\square$

In order to continue we will need some basics on Sobolev spaces. For $k \in \mathbb{N}$ the kth Sobolev space $H^k(\Omega)$ is defined as

$$H^k(\Omega) = \big\{ f \in L^2(\Omega) : \widetilde{\partial^\alpha} f \in L^2(\Omega) \text{ for all } |\alpha| \le k \big\},$$

which becomes a Hilbert space if equipped with the norm

$$\|f\|^2_{H^k(\Omega)} = \sum_{|\alpha| \le k} \|\widetilde{\partial^\alpha} f\|^2_{L^2(\Omega)}.$$

It can be shown that for bounded domains Ω with sufficiently regular boundaries, the space $H^k(\Omega)$ can be defined as the completion of $C^\infty(\overline{\Omega})$ with respect the the above H^k-norm. In other words, $C^\infty(\overline{\Omega})$ is a dense suspace of $H^k(\Omega)$ for any $k \in \mathbb{N}$.

It is a remarkable fact that in the absence of boundaries there is an alternative description of the Sobolev spaces. Namely, the Sobolev spaces $H^k(\mathbb{R}^d)$ can be characterized using the Fourier transform, which we will briefly address now. Recall that the Fourier transform $\widehat{f}$ of a function $f \in L^1(\mathbb{R}^d)$ is given by

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x) e^{-i\xi \cdot x} \, dx, \quad \chi \in \mathbb{R}^d,$$

and if $\widehat{f} \in L^1$, then one ha sthe a.e. equality

$$f(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \widehat{f}(\xi) e^{i\xi \cdot x} \, d\xi, \quad x \in \mathbb{R}^d. \tag{1.3}$$

It is known that $L^2 \cap L^1 \ni f \mapsto \widehat{f}$ extends by density to a unitary operator $\mathcal{F} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$, and it is common to write $\widehat{f}$ instead of $\mathcal{F}f$ even if $f \notin L^1$. One has the following easy observation:

Proposition 1.20. *Let $f \in L^2(\mathbb{R}^d)$, then $\widetilde{P}f \in L^2(\mathbb{R}^d)$ if and only if $p(\xi)\widehat{f} \in L^2(\mathbb{R}^d)$, where*

$$p(\xi) = \sum_{|\alpha| \le m} c_\alpha (i\xi)^\alpha.$$

Moreover, in this case one has $\widetilde{P}f = g$, where g is the unique L^2-function with $\widehat{g} = p(\xi)\widehat{f}$.

Proof. We prefer to give the proof for the case $P = -i\partial_1$ only (the general case is left as an exercise: one may follow the same constructions but with a more involved notation). Recall that for any $\varphi \in C_c^\infty(\mathbb{R}^d)$ one has $\widehat{\partial_1\varphi}(\xi) = i\xi\widehat{\varphi}(\xi)$.

Let $f \in L^2(\mathbb{R}^d)$ with $g := -i\widetilde{\partial}_1 f \in L^2(\mathbb{R}^d)$. According to the definition of weak derivatives, for any $\varphi \in C_c^\infty(\mathbb{R}^d)$ one has

$$\int_{\mathbb{R}^d} (-i\overline{\partial_1\varphi} f) = -\int_{\mathbb{R}^d} \overline{\varphi} g,$$

which can be rewritten as $\langle i\partial_1\varphi, f\rangle = \langle \varphi, g\rangle$ with $\langle \cdot, \cdot\rangle$ being the L^2 scalar product. Due to the unitarity of the Fourier transform this implies $\langle \widehat{i\partial_1\varphi}, \widehat{f}\rangle = -\langle \widehat{\varphi}, \widehat{g}\rangle$ and then

$$\int_{\mathbb{R}^d} \xi_1 \overline{\widehat{\varphi}(\xi)} \widehat{f}(\xi) d\xi = \int_{\mathbb{R}^d} \overline{\widehat{\varphi}(\xi)} \widehat{g}(\xi) d\xi.$$

As $C_c^\infty(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$, it follows that the Fourier transforms of all functions of $C_c^\infty(\mathbb{R}^d)$ form a dense subspace in $L^2(\mathbb{R}^d)$, and then $\xi_1\widehat{f} = \widehat{g} \in L^2(\mathbb{R}^d)$.

Now assume that $f \in L^2(\mathbb{R}^d)$ with $\xi_1\widehat{f} \in L^2(\mathbb{R}^d)$, then there exists a unique $g \in L^2(\mathbb{R}^d)$ with $\widehat{g} = \xi_1\widehat{f}$. Then for any $\varphi \in C_c^\infty(\mathbb{R}^d)$ one has

$$\int_{\mathbb{R}^d} \overline{\widehat{\varphi}(\xi)} \xi_1 \widehat{f}(\xi) d\xi = \int_{\mathbb{R}^d} \overline{\widehat{\varphi}(\xi)} \widehat{g}(\xi) d\xi,$$

which can be regoruped into $\langle \widehat{i\partial_1\varphi}, \widehat{f}\rangle = -\langle \widehat{\varphi}, \widehat{g}\rangle$. Using again the unitarity of the Fourier transform we obtain $\langle i\partial_1\varphi, f\rangle = \langle \varphi, g\rangle$ and then

$$\int_{\mathbb{R}^d} (-i\overline{\partial_1\varphi} f) = -\int_{\mathbb{R}^d} \overline{\varphi} g \text{ for any } \varphi \in C_c^\infty(\mathbb{R}^d),$$

which shows that $g := -i\widetilde{\partial}_1 f$. □

Remark that Proposition 1.20 allows one to give a new characterization of the Sobolev spaces $H^k(\mathbb{R}^d)$.

Proposition 1.21. *For $\xi \in \mathbb{R}^d$ denote $\langle \xi\rangle = \sqrt{1 + |\xi|^2}$. There holds*

$$H^k(\mathbb{R}^d) = \big\{ f \in L^2(\mathbb{R}^d) : \langle \xi\rangle^k \widehat{f} \in L^2(\mathbb{R}^d) \big\}.$$

Proof. Denote $A := \big\{ f \in L^2(\mathbb{R}^d) : \langle \xi \rangle^k \widehat{f} \in L^2(\mathbb{R}^d) \big\}$.

Let $f \in H^k(\mathbb{R}^d)$, then by Proposition 1.20 one has $\xi^\alpha \widehat{f} \in L^2(\mathbb{R}^d)$ for $|\alpha| \le k$. Using $\langle \xi \rangle \le 1 + |\xi_1| + \cdots + |\xi_d|$ we estimate

$$\Big| \langle \xi \rangle^k \widehat{f} \Big| \le (1 + |\xi_1| + \cdots + |\xi_d|)^k |\widehat{f}| \le \sum_{|\alpha| \le k} b_\alpha |\xi^\alpha \widehat{f}|,$$

where $b_\alpha > 0$ are some constants. By Proposition 1.20 each summand on the right-hand side is an L^2-function, which shows that $\langle \xi \rangle^k \widehat{f} \in L^2(\mathbb{R}^d)$. This gives the inclusion $H^k(\mathbb{R}^d) \subset A$.

On the other hand, let $f \in A$. For $|\alpha| \le k$ one has $|\xi^\alpha| \le \langle \xi \rangle^{|\alpha|} \le \langle \xi \rangle^k$, therefore, $|\xi^\alpha \widehat{f}| \le \langle \xi \rangle^k |\widehat{f}| \in L^2(\mathbb{R}^d)$, implying $\xi^\alpha \widehat{f} \in L^2(\mathbb{R}^d)$. By Proposition 1.20 this means that $\widetilde{\partial^\alpha} f \in L^2(\mathbb{R}^d)$. As this holds for arbitrary α with $|\alpha| \le k$, one arrives at the inclusion $A \subset H^k(\mathbb{R}^d)$. $\square$

The following important result is given without proof (it can be proved using a combination of a cut-off and a regularization as in Example 1.19, and it will certainly be done or was already done in one of the PDE courses):

Proposition 1.22. *The set $C_c^\infty(\mathbb{R}^d)$ is dense in any $H^k(\mathbb{R}^d)$.*

With the preceding notions and construction, let us now discuss a very important example of Laplacians.

Example 1.23 (Laplacians in $\mathbb{R}^d$). Take $\mathcal{H} = L^2(\mathbb{R}^d)$ and consider several operators in $\mathcal{H}$ associated with the differential expression

$$P = -\Delta = -\sum_{j=1}^d \partial_j^2$$

called *Laplacian*. Namely, define

$$\begin{gathered} T_0 = -\Delta u, \quad D(T_0) = C_c^\infty(\mathbb{R}^d), \\ T_1 = P_{\min}, \quad T_2 = P_{\max}. \end{gathered}$$

Recall that by the preceding definitions the following holds:

- T_1 is the closure of T_0,
- $T_2 = T_0^*$,
- Both T_1 and T_2 act as $u \mapsto -\widetilde{\Delta} u$,
- $D(T_2) = \big\{ u \in L^2(\mathbb{R}^d) : -\widetilde{\Delta} u \in L^2(\mathbb{R}^d) \big\}$.

We are going to show that $\overline{T}_0 = T_2$. (It means that $T_1 = T_2$, that T_0 is essentially self-adjoint and that T_2 is self-adjoint.) In addition we will relate the domain of T_2 to the Sobolev spaces.

By Proposition 1.20 we have $D(T_2) = \big\{u \in L^2(\mathbb{R}^d) : |\xi|^2\widehat{u} \in L^2(\mathbb{R}^d)\big\}$, which is equivalent to

$$D(T_2) = \big\{u \in L^2(\mathbb{R}^d) : (1+|\xi|^2)\widehat{u} \in L^2(\mathbb{R}^d)\big\}.$$

By Proposition 1.21 we have $D(T_2) = H^2(\mathbb{R}^d)$.

We further remark that for $u \in D(T_2)$ its graph norm is given by

$$\|u\|^2_{T_2} = \|u\|^2_{L^2(\mathbb{R}^d)} + \|\widetilde{\Delta} u\|^2_{L^2(\mathbb{R}^d)} \equiv \|\widehat{u}\|^2_{L^2(\mathbb{R}^d)} + \||\xi|^2\widehat{u}\|^2_{L^2(\mathbb{R}^d)},$$

while its H^k-norm is given by

$$\|u\|^2_{H^k} = \sum_{|\alpha|\le 2} \|\widetilde{\partial^\alpha} u\|^2_{L^2(\mathbb{R}^d)} \equiv \sum_{|\alpha|\le 2} \|\xi^\alpha \widehat{u}\|^2_{L^2(\mathbb{R}^d)},$$

and using the same computation as in the proof of Proposition 1.21 one easily shows that the graph norm of T_2 is equivalent to the H^k-norm. Proposition 1.22 shows then that $C_c^\infty(\mathbb{R}^d)$ is dense in $D(T_2)$ in the graph norm, i.e. that $(T_1 =)\overline{T_0} = T_2$. $\square$

Definition 1.24 (Free Laplacian in $\mathbb{R}^d$). The operator T in $L^2(\mathbb{R}^d)$ defined by

$$D(T) = H^2(\mathbb{R}^d), \quad Tu = -\Delta u,$$

is called the *free Laplacian in* $\mathbb{R}^d$ (here we drop the sign $\widetilde{\ }$: in fact, in advanced PDEs by a derivative one usually means a weak derivative). As discussed in Example 1.23, it is a self-adjoint operator.

The free Laplacian T will be of importance for the rest of the course. In fact, many operators we are going to study will be of the form $T + V$, where the operator V will be a suitable (small) perturbation.

Therefore, for $\Omega = \mathbb{R}^d$ and $P = -\Delta$ one has $P_{\min} = P_{\max}$ with $D(P_{\max})$. Anyway, these equalities do not hold for general domains.

For example, let use continue with $P = -\Delta$, ane let Ω be a bounded open set with a smooth boundary (so that one can apply the integration by parts). It is clear that $C^2(\overline{\Omega}) \subset D(P_{\max})$. On the other hand, for $u, v \in C^2(\overline{\Omega})$ one has

$$\langle u, P_{\max} v\rangle - \langle Pu, v\rangle = \int_\Omega \overline{\Delta u}\, v\, dx - \int_\Omega \overline{u}\, \Delta v\, dx = \int_{\partial\Omega} \big(\overline{\partial_n u}\, v - \overline{u}\, \partial_n v\big)\, ds,$$

and it is clear that u and v can be chosen in such a way that the result is non-zero. In follows, that $P_{\max}$ is not symmetric (so it cannot be self-adjoint). On the other hand, $P_{\min}$ is always symmetric, so $P_{\min} \neq P_{\max}$. In fact one needs to take a restriction of $P_{\max}$ in order to obtain a self-adjoint operator, and usually such a restriction is formulated in terms of a boundary conditions. We also remark that in general $D(P_{\max}) \neq H^2(\Omega)$.

Exercise 4. Let $\mathcal{H} = L^2(0, 2\pi)$. Consider the operator $T : u \mapsto -u''$ with the domain

$$D(T) = \{u \in C^\infty(0, 2\pi) : u \text{ extends to a } 2\pi\text{-periodic function on } \mathbb{R}\}.$$

Show that T is essentially self-adjoint and describe its closure.

Hint: One can use the Fourier series.

Exercise 5. Let $\Omega = (0, +\infty) \times \mathbb{R}$ and $P = -\Delta$. Choose $\chi \in C_c^\infty(\mathbb{R}^2)$ with $\chi(x) = 1$ for $|x| < 1$ and consider the function

$$u : \Omega \ni x \mapsto \chi(x) \ln |x| \in \mathbb{C}.$$

Show that $u \in D(P_{\max})$ but $u \notin H^2(\Omega)$.

Hint: All weak derivatives of u can be easily computed.

Exercise 6. Show that $H^k(\mathbb{R}^d) \subset C^0(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ for $k > \dfrac{d}{2}$.

Hint: Look at the Fourier inversion formula (1.3) for $f \in C_c^\infty(\mathbb{R}^d)$ and mutliply the subintegral function by $1 = \langle \xi \rangle^{-k} \langle \xi \rangle^k$.

2 Operators and forms

2.1 Operators defined by forms

A *sesquilinear form* t in a Hilbert space $\mathcal{H}$ with domain $D(t) \subset \mathcal{H}$ is a map $t : \mathcal{H} \times \mathcal{H} \supset D(t) \times D(t) \to \mathbb{C}$ which is linear with respect to the second argument and is antilinear with respect to the first one. By default we assume that $D(t)$ is a dense subset of $\mathcal{H}$. (In the literature, one uses also the terms *bilinear form* and *quadratic form*.) A sesquilinear form t is called

- *symmetric* (or *Hermitian*) if $t(u,v) = \overline{t(v,u)}$ for all $u, v \in D(t)$,
- *semibounded from below* if t is symmetric and for some $c \in \mathbb{R}$ one has $t(u,u) \geq -c\|u\|^2$ for all $u \in D(t)$; in this case we write $t \geq -c$,
- *closed* if $t \geq -c$ and the domain $D(t)$ equipped with the scalar product

  $$\langle u, v\rangle_t := t(u,v) + (c+1)\langle u, v\rangle_{\mathcal{H}}$$

  is a Hilbert space. (It is an easy exercise to show that this property does not depend on the particular choice of c).

Definition 2.1 (Operator generated by a closed form). Let t be a closed sesquilinear form in $\mathcal{H}$. The operator T *generated by* or *associated with* the form t is defined by

$$\big(v \in D(T) \text{ and } f = Tv\big) \text{ iff } v \in D(t) \text{ with } t(u,v) = \langle u, f\rangle_{\mathcal{H}} \text{ for all } u \in D(t).$$

The following result is of crucial importance for many subsequent examples and computations. In fact, many operators we are going to study will be defined through their sesquilinear forms.

Theorem 2.2. *In the setting of Definition 2.1, the operator T is self-adjoint in $\mathcal{H}$, and $D(T)$ is a dense subset of $D(t)$.*

Proof. We consider the case $t \geq 1$, for which $\langle u, v\rangle_t = t(u,v)$ and $t(u,u) = \|u\|_t^2 \geq \|u\|_{\mathcal{H}}^2$. (The general case is an easy exercise for the reader.)
Remark first that for $v \in D(T)$ we have $\|v\|_{\mathcal{H}}^2 \leq t(v,v) = \langle v, Tv\rangle_{\mathcal{H}} \leq \|v\|_{\mathcal{H}} \|Tv\|_{\mathcal{H}}$ and then $\|Tv\|_{\mathcal{H}} \geq \|v\|_{\mathcal{H}}$, which shows that T is injective.
Now let us show that T is surjective. Let $f \in \mathcal{H}$. For $u \in D(t)$ one has $\big|\langle u, f\rangle_{\mathcal{H}}\big| \leq \|u\|_{\mathcal{H}} \cdot \|f\|_{\mathcal{H}} \leq \|f\|_{\mathcal{H}} \|u\|_t$. Hence, $D(t) \ni u \mapsto \langle u, f\rangle_{\mathcal{H}} \in \mathbb{C}$ is a continuous antilinear map, and by the Riesz theorem there is $v \in D(t)$ with $\langle u, f\rangle_{\mathcal{H}} = \langle u, v\rangle_t \equiv t(u,v)$ for all $u \in D(t)$. By definition this means that $v \in D(T)$ with $f = Tv$. This shows the surjectivity.
We further remark that for any $u, v \in D(T)$ we have, using the symmetry of t,

$$\langle u, Tv\rangle_{\mathcal{H}} = t(u,v) = \overline{t(v,u)} = \overline{\langle v, Tu\rangle_{\mathcal{H}}} = \langle Tu, v\rangle_{\mathcal{H}}.$$

Therefore, T is symmetric, and then T^{-1} is symmetric as well (using the same argument as in Proposition 1.18). Hence, the operator T^{-1} is symmetric and defined everywhere, hence, it is self-adjoint. Then $T = (T^{-1})^{-1}$ is self-adjoint by Proposition 1.18.

To prove the remaining statement let $h \in D(t)$ with $\langle v, h\rangle_t = 0$ for all $v \in D(T)$, then we need to show that $h = 0$. Remark that by assumption we have

$$0 = \langle v, h\rangle_t = t(v, h) = \overline{t(h, v)} = \overline{\langle h, Tv\rangle_{\mathcal{H}}} = \langle Tv, h\rangle_{\mathcal{H}}.$$

As the vectors Tv cover the whole of $\mathcal{H}$ as v runs through $D(T)$, one has $h = 0$. □

For what follows we will need an additional notion:

Definition 2.3 (Closable form). We say that a symmetric sesquilinear form t is *closable*, if there exists a closed sesquilinear form extending t. The closed sesquilinear form extending t and having the smallest domain is called the *closure* of t and denoted $\overline{t}$.

The following proposition is rather obvious.

Proposition 2.4 (Domain of the closure of a form). *If t is a closable form with $t \geq -c$, then $D(\overline{t})$ is exactly the completion of $D(t)$ with respect to the scalar product $\langle u, v\rangle_t := t(u, v) + (c + 1)\langle u, v\rangle$, and $\overline{t}$ is the extension of t by continuity.*

It is time to look at examples!

Example 2.5 (There exist non-closable forms). Take $\mathcal{H} = L^2(\mathbb{R})$ and consider the form $t(u, v) = \overline{u(0)}v(0)$ defined on $D(t) = L^2(\mathbb{R}) \cap C^0(\mathbb{R})$. This form is densely defined, symmetric and positive. Let us show that it is not closable. By contradiction, assume that there exists the closure $\overline{t}$ of t, then one should then have the following property: if (u_n) is a sequence of vectors from $D(t)$ which is Cauchy with respect to $\langle \cdot, \cdot\rangle_t$, and $u := \lim u_n$ in $\mathcal{H}$, then $t(u, u) = \lim t(u_n, u_n)$. But for any $u \in \mathcal{H}$ one can construct two sequences (u_n) and (v_n) in $D(t)$ such that

- both converge to u in the L^2-norm,
- $u_n(0) = 1$ and $v_n(0) = 0$ for all n.

Then both sequences are t-Cauchy and have the same limit u in $\mathcal{H}$, but the limits of $t(u_n, u_n)$ and $t(v_n, v_n)$ are different. This shows that $\overline{t}$ cannot exist. □

Now let us give some "canonical" examples of operators defined by forms. We will see them very often.

Example 2.6 (Free Laplacian revisited). Consider $\mathcal{H} = L^2(\mathbb{R}^d)$ and the form

$$t(u, v) = \int_{\mathbb{R}^d} \overline{\nabla u} \cdot \nabla v \, dx, \quad D(t) = H^1(\mathbb{R}^d),$$

which is clearly closed: in fact, $\langle \cdot, \cdot \rangle_t$ is the H^1-scalar product, and H^1 spaces are known to be complete, as we mentioned above. Let us find the associated operator T, which is already known to be self-adjoint due to Theorem 2.2.

Let $v \in D(T)$ and $f := Tv$, then for any $u \in H^1(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} \overline{\nabla u} \nabla v \, dx = \int_{\mathbb{R}^d} \overline{u} f \, dx.$$

In particular, this equality holds for $u \in C_c^\infty(\mathbb{R}^d) \subset H^1(\mathbb{R}^d)$, which gives

$$\int_{\mathbb{R}^d} \overline{u} f \, dx = \int_{\mathbb{R}^d} \overline{\nabla u} \cdot \nabla v \, dx = \int_{\mathbb{R}^d} \overline{(-\Delta u)} f \, dx,$$

hence, $f = -\Delta v \in L^2(\mathbb{R}^d)$ (here, the derivatives are taken in the weak sense). It follows that T is a restriction of the free Laplacian in $\mathbb{R}^d$ (see Definition 1.24). The maximality property of self-adjoint operators (Exercise 2) implies that T is exactly the free Laplacian in $\mathbb{R}^d$.

Example 2.7 (Neumann boundary condition on an interval). In the Hilbert space $\mathcal{H} = L^2(0,1)$ consider the form

$$t(u,v) = \int_0^1 \overline{u'(x)} v'(x) dx, \quad D(t) = H^1(0,1).$$

The form is closed (which is again just equivalent to the completeness of $H^1(0,1)$), so let us describe the associated operator T.

Let $v \in D(T)$, then there exists $f \in \mathcal{H}$ such that

$$\int_0^1 \overline{u'(x)} v'(x) dx = \int_0^1 \overline{u(x)} f(x) dx$$

for all $u \in H^1(0,1)$. Taking here $u \in C_c^\infty$ we obtain just the definition of the weak derivatives: $f := -(v')' = -v''$. As $f \in L^2(0,1)$, the function v must be in $H^2(0,1)$, and $Tv = -v''$.

Now note that for $v \in H^2(0,1)$ and $u \in H^1(0,1)$ there holds, using the integration by parts,

$$\int_0^1 \overline{u'(x)} v'(x) dx = \overline{u(x)} v'(x) \Big|_{x=0}^{x=1} - \int_0^1 \overline{u(x)} v''(x) dx.$$

(The identity is obvious for $u, v \in C^\infty\big([0,1]\big)$, and it is then extended by density, as $C^\infty\big([0,1]\big)$ is dense in all $H^k(0,1)$ as mentioned previously.) Hence, in order to obtain the requested inequality $t(u,v) = \langle u, Tv \rangle_{\mathcal{H}}$, the boundary term must vanish for all $u \in H^1(0,1)$, which is equivalent to the additional condition $v'(0) = v'(1) = 0$.

Therefore, the associated operator $T_N := T$ acts as $T_N v = -v''$ on the domain $D(T_N) = \big\{ v \in H^2(0,1) : v'(0) = v'(1) = 0 \big\}$. It will be referred as the (positive) Laplacian with the Neumann boundary condition or simply the Neumann Laplacian on $(0,1)$. $\square$

Example 2.8 (Dirichlet boundary condition on an interval). Take again $\mathcal{H} = L^2(0,1)$ and consider the following sesquilinear form which is a restriction of the one from the previous example,

$$t_0(u,v) = \int_0^1 \overline{u'(x)}v'(x)dx, \quad D(t_0) = H_0^1(0,1) := \overline{C_c^\infty(0,1)}^{H^1(0,1)}.$$

The form is still semibounded from below and closed (as H_0^1 is complete by construction), and we denote the associated self-adjoint operator by T_0. Using the same argument as in the preceding example one shows that $D(T_0) \subset H^2(0,1) \cap H_0^1(0,1)$ and that $T_0 v = -v''$. On the other hand, one can easily show (using the density argument) that for $v \in H^2(0,1) \cap H_0^1(0,1)$ and $u \in H_0^1(0,1)$ there holds

$$\int_0^1 \overline{u'(x)}v'(x)dx = -\int_0^1 \overline{u(x)}v''(x)dx,$$

i.e. the boundary term vanishes identically (due to the fact that $u(0) = u(1) = 0$). Hence, $D(T_0) = H^2(0,1) \cap H_0^1(0,1)$. In fact, by additional efforts one can show that this domain coincides with $\{v \in H^2(0,1) : v(0) = v(1) = 0\}$. The operator $T_D := T_0$ be referred to as the (positive) Laplacian with the Dirichlet boundary condition or the Dirichlet Laplacian on $(0,1)$. □

Remark 2.9. In the two previous examples we see several important features:

- Closed sesquilinear forms do not have the maximality property, i.e. a closed sesquilinear form have can a closed extension with a strictly larger domain,
- The fact that one closed form extends another closed form does *not* imply the same relation for the associated operators. □

Example 2.10 (Neumann/Dirichlet Laplacians: general case). The two previous examples can be generalized to the multidimensional case. Let Ω be an open subset of $\mathbb{R}^d$. In $\mathcal{H} = L^2(\Omega)$ consider two sesquilinear forms:

$$\begin{aligned} t_0(u,v) &= \int_\Omega \overline{\nabla u} \cdot \nabla v \, dx, & D(t_0) &= H_0^1(\Omega) := \overline{C_c^\infty(\Omega)}^{H^1(\Omega)}, \\ t(u,v) &= \int_\Omega \overline{\nabla u} \cdot \nabla v \, dx, & D(t) &= H^1(\Omega). \end{aligned}$$

Both these forms are closed and semibounded from below (≥ 0), and one can easily show that the respective operators A and A_0 act both as $u \mapsto -\Delta u$, but the description of their domains is a difficult task. The operator A_0 is called the *Dirichlet Laplacian* in Ω and A is called the *Neumann Laplacian* on Ω. By a more careful and advanced analysis and, for example, for a bounded smooth $\partial\Omega$, one can show that

$$D(A_0) = H^2(\Omega) \cap H_0^1(\Omega) = \{u \in H^2(\Omega) : u|_{\partial\Omega} = 0\},$$

$$D(A) = \{u \in H^2(\Omega) : \frac{\partial u}{\partial n}\big|_{\partial\Omega} = 0\},$$

where n denotes the outward pointing unit normal vector on $\partial\Omega$, and the restrictions to the boundary should be understood in a suitably generalized sense. If the boundary is not regular, the domains become more complicated, in particular, the domains of A and A_0 are not necessarily contained in $H^2(\Omega)$. Indeed, $A = A_0$ if $\Omega = \mathbb{R}^d$, as $H^1(\mathbb{R}^d) = H^1_0(\mathbb{R}^d)$. It can also be shown that there are domains Ω with boundaries such that $H^1(\Omega) = H^1_0(\Omega)$ and $A = A_0$.

2.2 Semibounded operators and Friedrichs extensions

We now arrive to a rather canonical construction of self-adjoint operators, which will us to associate self-adjoint operators with some differential expressions having non-smooth coefficients.

Definition 2.11 (Semibounded operator). A symmetric operator T in $\mathcal{H}$ is called *semibounded from below* if there exists a constant $c \in \mathbb{R}$ such that

$$\langle u, Tu\rangle \geq -c\langle u, u\rangle_{\mathcal{H}} \text{ for all } u \in D(T),$$

and in this will be written as $T \geq -c$. □

Proposition 2.12. *Let T be a semibounded from below linear operator, then the induced sesquilinear form t in $\mathcal{H}$ given by*

$$t(u, v) = \langle u, Tv\rangle, \quad D(t) = D(T), \tag{2.1}$$

is semibounded from below and closable.

Proof. The semiboundedness of t is obvious due to the definition.

To show the closability we remark that without loss of generality one can assume $T \geq 1$ (the general case is reduced to this one by an easy exercise). By Proposition 2.4, the domain $\mathcal{V}$ of the closure of t must be the completion of $D(T)$ with respect to the norm $p(u) = \sqrt{t(u, u)}$. More concretely, a vector $u \in \mathcal{H}$ belongs to $\mathcal{V}$ iff there exists a sequence $u_n \in D(T)$ which is p-Cauchy and such that u_n converges to u in $\mathcal{H}$. The natural candidate for the norm of u is $p(u) = \lim p(u_n)$, and the closability of t is equivalent to the fact that this limit is independent of the choice of the sequence u_n. Using the standard arguments we are reduced to prove the following:

Assertion. If $(u_n) \subset D(t)$ is a p-Cauchy sequence converging to 0 in $\mathcal{H}$, then $\lim p(u_n) = 0$.

To prove this assertion we observe first that $p(u_n)$ is a non-negative Cauchy sequence, and is convergent to some limit $\alpha \geq 0$. We suppose that $\alpha > 0$ and try to arrive at a contradiction. Remark first that $t(u_n, u_m) = t(u_n, u_n) + t(u_n, u_m - u_n)$ and that $\big|t(u_n, u_m - u_n)\big| \leq p(u_n)p(u_m - u_n)$ by the Cauchy-Schwartz inequality for the norm p. As $p(u_m - u_n)$ goes to zero and $p(u_n)$ converges to α (hence, bounded) for large

m, n, we conclude that for any $\varepsilon > 0$ there exists $N > 0$ such that $\big|t(u_n, u_m) - \alpha^2\big| \le \varepsilon$ for all $n, m > N$. Take $\varepsilon = \alpha^2/2$ and the associated N, then for $n, m > N$ we have $\big|\langle u_n, T u_m\rangle\big| \equiv \big|t(u_n, u_m)\big| \ge \frac{1}{2}\alpha^2$. On the other hand, the term on the left-hand side goes to 0 as $n \to \infty$ (as u_n converges to 0 by assumption). So we obtain a contradiction, and the assertion is proved. □

Definition 2.13 (Friedrichs extension). Let T be a semibounded from below linear operator in $\mathcal{H}$. Define a sesquilinear form t by (2.1). The self-adjoint operator T_F generated by $\bar{t}$ is called the *Friedrichs extension* of T. □

Corollary 2.14. *A semibounded operator always has a self-adjoint extension.*

Remark 2.15 (Form domain). If T is a self-adjoint operator semibounded from below, then it is the Friedrichs extension of itself. The domain of the associated form $\bar{t}$ is usually called the *form domain* of T and is denoted $Q(T)$. The form domain plays an important role in the analysis of self-adjoint operators, in particular, in the variational characterization of eigenvalues using the min-max principle, which will be a central point later. □

Example 2.16 (Schrödinger operators). A basic example for the Friedrichs extension is delivered by Schrödinger operators with semibounded potentials. Let $V \in L^2_{\text{loc}}(\mathbb{R}^d)$ and $V \ge -C$, $C \in \mathbb{R}$ (i.e. V is real-valued and semibounded from below). In $\mathcal{H} = L^2(\mathbb{R}^d)$ consider the operator T acting as $Tu(x) = -\Delta u(x) + V(x)u(x)$ on the domain $D(T) = C_c^\infty(\mathbb{R}^d)$. One has clearly $T \ge -C$ as for $u, v \in D(T)$ there holds

$$\langle u, Tv\rangle = \int_{\mathbb{R}^d} \overline{u}(-\Delta v)\, dx + \int_{\mathbb{R}^d} V\overline{u}v\, dx = \int_{\mathbb{R}^d} \overline{\nabla u} \cdot \nabla v\, dx + \int_{\mathbb{R}^d} V\overline{u}v\, dx,$$

$$\langle u, Tu\rangle = \int_{\mathbb{R}^d} |\nabla u|^2\, dx + \int_{\mathbb{R}^d} V|u|^2 dx \ge -C\|u\|^2.$$

The Friedrichs extension T_F of T will be called the *Schrödinger operator with the potential* V. The sesqulinear form t associated with T is given by

$$t(u, v) = \int_{\mathbb{R}^d} \overline{\nabla u}\nabla v dx + \int_{\mathbb{R}^d} V\overline{u}v dx,$$

and one can easily show the inclusion

$$D(\bar{t}) \subset H^1_V(\mathbb{R}^d) := \big\{u \in H^1(\mathbb{R}^d) : \sqrt{|V|}u \in L^2(\mathbb{R}^d)\big\}.$$

Note that actually we have the equality $D(\bar{t}) = H^1_V(\mathbb{R}^d)$ (the proof needs some advanced machinery), but the inclusion will be sufficient for our purposes.

Let us extend the above example by including a class of potentials V which are not semibounded from below. This will be done using the following classical inequality.

Proposition 2.17 (Hardy inequality). *Let $d \ge 3$ and $u \in C_c^\infty(\mathbb{R}^d)$, then*

$$\int_{\mathbb{R}^d} \big|\nabla u(x)\big|^2 dx \ge \frac{(d-2)^2}{4} \int_{\mathbb{R}^d} \frac{\big|u(x)\big|^2}{|x|^2}\, dx.$$

Proof. For any $\gamma \in \mathbb{R}$ one has

$$\int_{\mathbb{R}^d} \Big|\nabla u(x) + \gamma\, \frac{x u(x)}{|x|^2}\Big|^2 dx \ge 0,$$

which may be rewritten in the form

$$\begin{aligned}\int_{\mathbb{R}^d} \big|\nabla u(x)\big|^2 dx + \gamma^2 \int_{\mathbb{R}^d} \frac{\big|u(x)\big|^2}{|x|^2}\, dx \qquad\qquad\qquad\qquad\qquad\\ \ge -\gamma \int_{\mathbb{R}^d} \Big(x\cdot \overline{\nabla u(x)}\, \frac{u(x)}{|x|^2} + x\cdot \nabla u(x)\, \frac{\overline{u(x)}}{|x|^2}\Big)\, dx. \quad (2.2)\end{aligned}$$

Using the identities

$$\nabla |u|^2 = \overline{u}\, \nabla u + u \overline{\nabla u}, \quad \operatorname{div} \frac{x}{|x|^2} = \frac{d-2}{|x|^2},$$

and the integration by parts we obtain

$$\begin{aligned}\int_{\mathbb{R}^d} \Big(x\cdot \overline{\nabla u(x)}\, \frac{u(x)}{|x|^2} + x\cdot \nabla u(x)\, \frac{\overline{u(x)}}{|x|^2}\Big)\, dx &= \int_{\mathbb{R}^d} \nabla \big|u(x)\big|^2 \cdot \frac{x}{|x|^2}\, dx\\ &= -\int_{\mathbb{R}^d} \big|u(x)\big|^2 \operatorname{div} \frac{x}{|x|^2}\, dx = -(d-2)\int_{\mathbb{R}^d} \frac{\big|u(x)\big|^2}{|x|^2}\, dx.\end{aligned}$$

Inserting this equality into (2.2) gives

$$\int_{\mathbb{R}^d} \big|\nabla u(x)\big|^2 dx \ge \gamma\big((d-2)-\gamma\big) \int_{\mathbb{R}^d} \frac{\big|u(x)\big|^2}{|x|^2}\, dx,$$

and in order to optimize the coefficient on the right-hand side we take $\gamma = (d-2)/2$. □

Note that the integral on the right-hand side of the Hardy inequality is not defined for $d \le 2$, because the function $x \mapsto |x|^{-2}$ is not integrable anymore.

By combining the Hardy inequality with the constructions of Example 2.16 one easily shows the following result:

Corollary 2.18. *Let $d \ge 3$ and $V \in L^2_{\text{loc}}(\mathbb{R}^d)$ be real-valued with $V(x) \ge -\frac{(d-2)^2}{4|x|^2}$, then the operator $T = -\Delta + V$ defined on $C_c^\infty(\mathbb{R}^d)$ is semibounded from below (in fact, $T \ge 0$) and, hence, has a self-adjoint extension (Friedrichs extension).*

Example 2.19 (Coulomb potential). We would like to show that the operator $T = -\Delta + q/|x|$ in $L^2(\mathbb{R}^3)$ is semibounded from below for any real q. The operator is of importance in quantum physics, the potential $q/|x|$ is referred to as the Coulomb potential of charge q placed at the origin. For $q \ge 0$ we are in the situation of Example 2.16 (the potential is ≥ 0), while for $q < 0$ we are going to use the Hardy inequality. For any $u \in C_c^\infty(\mathbb{R}^3)$ and any $p \in \mathbb{R} \setminus \{0\}$ we have:

$$\begin{aligned}\int_{\mathbb{R}^3} \frac{|u|^2}{|x|} dx = \int_{\mathbb{R}^3} p|u| \frac{|u|}{p|x|} dx \\ \leq \frac{p^2}{2} \int_{\mathbb{R}^3} |u|^2 dx + \frac{1}{2p^2} \int_{\mathbb{R}^3} \frac{|u|^2}{|x|^2} dx \\ \leq \frac{p^2}{2} \int_{\mathbb{R}^3} |u|^2 dx + \frac{1}{8p^2} \int_{\mathbb{R}^3} |\nabla u|^2 dx,\end{aligned}$$

and

$$\begin{aligned}\langle u, Tu \rangle = \int_{\mathbb{R}^3} |\nabla u|^2 dx + q \int_{\mathbb{R}^3} \frac{|u|^2}{|x|} dx \geq \int_{\mathbb{R}^3} |\nabla u|^2 dx - |q| \int_{\mathbb{R}^3} \frac{|u|^2}{|x|} dx \\ \geq \left(1 - \frac{|q|}{8p^2}\right) \int_{\mathbb{R}^3} |\nabla u|^2 dx - \frac{|q|p^2}{2} \int_{\mathbb{R}^3} |u|^2 dx,\end{aligned}$$

and for $p = \sqrt{|q|/8}$ one has $\langle u, Tu \rangle \geq -\frac{|q|^2}{16} \int_{\mathbb{R}^3} |u|^2 dx$. Therefore, for any $q \in \mathbb{R}$ the above operator T is semibounded from below and, as a consequence, has a self-adjoint extension (Friedrichs extension). □

Exercise 7.

1. We would like to show the following inequality:

$$\text{For all } a>0,\ f\in H^1(0,a) \text{ and } \ell\in(0,a) \text{ there holds}$$
$$\big|f(0)\big|^2 \le \ell \int_0^a |f'|^2 + \frac{2}{\ell}\int_0^a |f|^2. \tag{2.3}$$

We take first $f \in C^\infty([0,a])$.

   (a) For $x\in(0,a)$ show the inequality $\Big|\displaystyle\int_0^x f'(t)\,dt\Big|^2 \le x\|f'\|^2_{L^2(0,a)}$.

   (b) Show that $|f(0)|^2 \le 2\big|f(x)\big|^2 + 2\Big|\displaystyle\int_0^x f'(t)\,dt\Big|^2$ for $x\in(0,a)$.

   (c) Show that $\ell|f(0)|^2 \le 2\displaystyle\int_0^\ell |f|^2 + \ell^2\|f'\|^2_{L^2(0,a)}$.

Now prove the inequality (2.3).

2. In the Hilbert space $\mathcal{H}=L^2(0,1)$ consider the following sesquilinear form:

$$t(u,v)=\int_0^1 \overline{u'(t)}\,v'(t)\,dt + \alpha\overline{u(0)}v(0), \quad D(t)=H^1(0,1),$$

where $\alpha\in\mathbb{R}$ is a constant. Show that t is closed (in partcular, semibounded from below) and describe the associated self-adjoint operator acting in $\mathcal{H}$.

Exercise 8. This exercise shows a possible way of constructing the sum of two unbounded operators under the assumption that one of them is "smaller" that the other one. In a sense, we are going to extend the construction of Exercise 3.

1. Let $\mathcal{H}$ be a Hilbert space, t be a closed sesquilinear form in $\mathcal{H}$, and T be a self-adjoint operator in $\mathcal{H}$ generated by the form t. Let B be a symmetric linear operator in $\mathcal{H}$ such that $D(t)\subset D(B)$ and for which there exist constants $\alpha>0$ and $\beta>0$ with

$$\|Bu\|^2 \le \alpha\, t(u,u)+\beta\|u\|^2 \text{ for all } u\in D(t).$$

Consider the operator S' defined by $S'u=Tu+Bu$ on the domain $D(S')=D(T)$. We are going to show that S' is self-adjoint.

   (a) Consider the sesquilinear form $s(u,v)=t(u,v)+\langle u,Bv\rangle$ with domain $D(s)=D(t)$. Show that s is closed.

   (b) Let S be the operator in $\mathcal{H}$ generated by the form s. Show that $D(S)=D(T)$ and that $Su=Tu+Bu$ for all $u\in D(T)$.

   (c) Show that S' is self-adjoint.

2. Application: Schrödinger operators with L^2 potentials.

   (a) Show the inequality

   $$\|f\|^2_{L^\infty(\mathbb{R})} \leq \varepsilon \int_{\mathbb{R}} |f'|^2 + \frac{1}{\varepsilon} \int_{\mathbb{R}} |f|^2 \text{ for all } f \in H^1(\mathbb{R}) \text{ and } \varepsilon > 0. \tag{2.4}$$

   Hint: One can start with $|f(x)|^2 = \displaystyle\int_{-\infty}^{x} (|f|^2)'$ for $f \in C_c^\infty(\mathbb{R})$.

   (b) Let $V \in L^2(\mathbb{R})$ be real-valued. Show that the operator A having as domain $D(A) = H^2(\mathbb{R})$ and acting by $Af(x) = -f''(x) + V(x)f(x)$ is a self-adjoint operator in $L^2(\mathbb{R})$. Hint: Use the first part of the exercise with T :=the free Laplacian and B :=the multiplication by V.

3 Spectrum: first observations

In this section we collect first definitions concerning the spectrum. Some of this notion are supposed to be known the functional analysis course when applied to bounded operators. Nevertheless, we reinterpret these notions from the point of view of unbounded operators and see some new aspects.

3.1 Definitions and examples

Definition 3.1 (Resolvent set, spectrum, point spectrum). Let T be a linear operator in a Hilbert space $\mathcal{H}$. The *resolvent set* $\operatorname{res} T$ consists of the complex numbers z for which the operator $T - z : D(T) \ni u \mapsto Tu - zu \in \mathcal{H}$ is bijective and the inverse $(T-z)^{-1}$ is bounded. The *spectrum* $\operatorname{spec} T$ of T is defined by $\operatorname{spec} T := \mathbb{C} \setminus \operatorname{res} T$. The *point spectrum* $\operatorname{spec}_p T$ is defined as the set of the eigenvalues of T. □

Note that very often the resolvent set and the spectrum of T are often denoted by $\rho(T)$ and $\sigma(T)$, respectively.

Proposition 3.2. *If* $\operatorname{res} T \neq \emptyset$*, then* T *is a closed operator.*

Proof. Let $z \in \operatorname{res} T$, then $\operatorname{gr}(T-z)^{-1}$ is closed by the closed graph theorem, but then the graph of $T - z$ is also closed, as $\operatorname{gr}(T-z)$ and $\operatorname{gr}(T-z)^{-1}$ are isometric in $\mathcal{H} \times \mathcal{H}$. □

Proposition 3.3. *For a* closed *operator* T *one has the following equivalence:*

$$z \in \operatorname{res} T \quad \textit{iff} \quad \begin{cases} \ker(T-z) = \{0\}, \\ \operatorname{ran}(T-z) = \mathcal{H}. \end{cases}$$

Proof. The $\Rightarrow$ direction follows from the definition.

Now let T be closed and $z \in \mathbb{C}$ with $\ker(T-z) = \{0\}$ and $\operatorname{ran}(T-z) = \mathcal{H}$. The inverse $(T-z)^{-1}$ is then defined everywhere and has a closed graph (as the graph of $T - z$ is closed), and is then bounded by the closed graph theorem. □

Proposition 3.4 (Properties of the resolvent). *The set* $\operatorname{res} T$ *is open and the set* $\operatorname{spec} T$ *is closed. The operator function*

$$\operatorname{res} T \ni z \mapsto R_T(z) := (T-z)^{-1} \in \mathcal{L}(\mathcal{H})$$

called the resolvent *of* T *is holomorphic and satisfies the identities*

$$R_T(z_1) - R_T(z_2) = (z_1 - z_2) R_T(z_1) R_T(z_2), \tag{3.1}$$

$$R_T(z_1) R_T(z_2) = R_T(z_2) R_T(z_1), \tag{3.2}$$

$$\frac{d}{dz} R_T(z) = R_T(z)^2 \tag{3.3}$$

for all $z, z_1, z_2 \in \operatorname{res} T$.

Proof. Let $z_0 \in \operatorname{res} T$. We have the equality

$$T - z = (T - z_0)\big(1 - (z - z_0)R_T(z_0)\big).$$

If $|z - z_0| < 1/\|R_T(z_0)\|$, then the operator on the right had sinde has a bounded inverse, which means that $z \in \operatorname{res} T$. Moreover, one has the series representation

$$R_T(z) = \big(1 - (z - z_0)R_T(z_0)\big)^{-1} R_T(z_0) = \sum_{j=0} (z - z_0)^j R_T(z_0)^{j+1}, \tag{3.4}$$

which shows that R_T is holomorphic. The remaining properties can be proved in a similar way. □

We now consider a series of examples showing several situations where an explicit computation of the spectrum is possible.

Example 3.5. Consider the multiplication operator M_f from Example 1.6. Recall that the *essential range* of a function f is defined by

$$\operatorname{ess\,ran} f = \Big\{\lambda : \mu\big\{x : |f(x) - \lambda| < \varepsilon\big\} > 0 \text{ for all } \varepsilon > 0\Big\}.$$

Clearly, this notion makes sense in any measure space. For a continuous function f and the Lebesgue measure μ, the essential range coincides with the closure of the usual range.

Proposition 3.6 (Spectrum of the multiplication operator). *There holds*

$$\begin{aligned} \operatorname{spec} M_f &= \operatorname{ess\,ran} f, \\ \operatorname{spec}_p M_f &= \big\{\lambda : \mu\{x : f(x) = \lambda\} > 0\big\}. \end{aligned}$$

Proof. Let $\lambda \notin \operatorname{ess\,ran} f$, then the operator $M_{1/(f-\lambda)}$ is bounded, and one easily checks that this is the inverse for $M_f - \lambda$. On the other hand, let $\lambda \in \operatorname{ess\,ran} f$. For any $m \in \mathbb{N}$ denote

$$\widetilde{S}_m := \big\{x : |f(x) - \lambda| < 2^{-m}\big\}$$

and choose a subset $S_m \subset \widetilde{S}_m$ of strictly positive but finite measure. If ϕ_m is the indicator function of S_m, one has

$$\big\|(M_f - \lambda)\phi_m\big\|^2 = \int_{S_m} \big|f(x) - \lambda\big|^2 \big|\phi_m(x)\big|^2 dx \le 2^{-2m} \big\|\phi_m\big\|^2,$$

and the operator $(M_f - \lambda)^{-1}$ cannot be bounded.

To prove the second assertion we remark that the condition $\lambda \in \operatorname{spec}_p M_f$ is equivalent to the existence of $\phi \in L^2(\mathbb{R}^d)$ such that $\big(f(x) - \lambda\big)\phi(x) = 0$ for a.e. x. This means that $\phi(x) = 0$ for a.e. x with $f(x) \ne \lambda$. If $\mu\{x : f(x) = \lambda\} = 0$, then $\phi = 0$ a.e., and $\lambda \notin \operatorname{spec}_p M_f$. On the other hand, if $\mu\{x : f(x) = \lambda\} > 0$, one can choose a subset $\Sigma \subset \{x : f(x) = \lambda\}$ of a strictly positive finite measure, then the indicator function ϕ of Σ is an eigenfunction of M_f corresponding to the eigenvalue λ. □

Example 3.7. It can be shown that the spectrum is invariant under unitary transformations (see Exercise 1):

Proposition 3.8 (Spectrum and unitary equivalence). *Let two operators A an B be unitarily equivalent, then* $\operatorname{spec} A = \operatorname{spec} B$ *and* $\operatorname{spec}_\mathrm{p} A = \operatorname{spec}_\mathrm{p} B$.

Example 3.9. Let T be the free Laplacian in $\mathbb{R}^d$ (see Definition 1.24). As seen above, T is unitarily equivalent to the multiplication operator $f(p) \mapsto p^2 f(p)$ in $L^2(\mathbb{R}^d)$. By Propositions 3.6 and 3.8 there holds $\operatorname{spec} T = [0, +\infty)$ and $\operatorname{spec}_\mathrm{p} T = \emptyset$.

Example 3.10 (Discrete multiplication operator). Take $\mathcal{H} = \ell^2(\mathbb{Z})$. Consider an aribtrary function $a : \mathbb{Z} \to \mathbb{C}$, $n \mapsto a_n$, and the associated operator T:

$$D(T) = \big\{(\xi_n) \in \ell^2(\mathbb{Z}) : (a_n \xi_n) \in \ell^2(\mathbb{Z})\big\}, \quad (T\xi)_n = a_n \xi_n.$$

Similarly to examples 1.6 and 3.6 one can show that T is a closed operator and that

$$\operatorname{spec} T := \overline{\{a_n : n \in \mathbb{Z}\}}, \quad \operatorname{spec}_\mathrm{p} T := \{a_n : n \in \mathbb{Z}\}.$$

Example 3.11 (Harmonic oscillator). Let $\mathcal{H} = L^2(\mathbb{R})$. Consider the operator $T_0 = -d^2/dx^2 + x^2$ defined on $C_c^\infty(\mathbb{R})$. This operator is semibounded from below and denote by T its Friedrichs extension. The operator T is called the *harmonic oscillator*; it is one of the basic models appearing in quantum mechanics.

One can easily that the functions ϕ_n given by $\phi_n(x) = c_n(-d/dx + x)^{n-1}\phi_1(x)$, $\phi_1(x) = c_1 \exp(-x^2/2)$, are L^2-solutions to $(-d^2/dx^2 + x^2)\phi_n = (2n-1)\phi_n$, where c_n are normalizing constants and $n \in \mathbb{N}$. It is known that the functions (ϕ_n) (called *Hermite functions*) form an orthonormal basis in $L^2(\mathbb{R})$. We further remark that $\phi_n \in D(\overline{T_0})$ for all n. In order to see these inclusions, one takes $\chi \in C_c^\infty(\mathbb{R})$ with $\chi(x) = 1$ for $|x| \le 1$, and for $N \in \mathbb{N}$ one defines $f_N(x) = \chi(x/N)\phi_n(x)$. By an easy computation one shows that f_N and $T_0 f_N$ converge in L^2 to ϕ_n and $(-d^2/dx^2 + x^2)\phi_n$ respectively, which shows the claim. Then it follows that T_0 is essentially self-adjoint (see Exercise 1c), in particular, $T = \overline{T_0}$.

Furthermore, using the unitary map $U : L^2(\mathbb{R}) \to \ell^2(\mathbb{N})$, $Uf(n) = \langle \phi_n, f\rangle$, one easily checks that the operator T is unitarily equivalent to the operator of multiplication by $(2n-1)$ in $\ell^2(\mathbb{N})$, cf. Example 3.10, which gives

$$\operatorname{spec} T = \operatorname{spec}_\mathrm{p} T = \{2n - 1 : \quad n \in \mathbb{N}\}.$$

Example 3.12 (Empty spectrum). Take $\mathcal{H} = L^2(0,1)$ and $z \in \mathbb{C}$ and consider the operator

$$A_z f(x) = \int_0^x e^{z(x-t)} f(t)\, dt,$$

which is clearly continuous and $\operatorname{ran} A_z \in C^0(0,1)$. Let us show that A_z is injective. Assume that $A_z f = 0.$, then the function $g : t \mapsto e^{-zt} f(t)$ is orthogonal to the indicator functions of $(0, x)$ for all x and, as a consequence, to the indicator functions of all subintervals of $(0,1)$. Hence $g = 0$ a.e., and then $f = 0$ a.e. It follows that there exists the inverse $B_z := A_z^{-1}$. Remark that for $f \in C^0\big([0,1]\big)$ and $h := A_z f$ one has $h \in C^1\big([0,1]\big)$ with $h(0) = 0$ and $h' = zh + f$. It follows that for $h \in C^1\big([0,1]\big)$ with $h(0) = 0$ one has $B_z h = h' - zh$. Using the density argument we see the following:

let C be the linear operator in $L^2(0,1)$ given as $Ch = h'$ on the domain

$$D(C) = \big\{h \in C^1\big([0,1]\big) : h(0) = 0\big\},$$

then the closure $T = \overline{C}$ is such that $(T - z)^{-1} = A_z \in \mathcal{L}\big(L^2(0,1)\big)$ for any $z \in \mathbb{C}$. It folows that the spectrum of T is an empty set. With some additional work one can show that $Tf = f'$ (weak derivative) on the domain $D(T) = \big\{f \in H^1(0,1) : f(0) = 0\big\}$.

Example 3.13 (Empty resolvent set). Let us modify the previous example. Take $\mathcal{H} = L^2(0,1)$ and consider the operator T acting as $Tf = f'$ on the domain $D(T) = H^1(0,1)$. Now for any $z \in \mathbb{C}$ we see that the function $\phi_z(x) = e^{zx}$ belongs to $D(T)$ and satisfies $(T - z)\phi_z = 0$. Therefore, $\operatorname{spec}_p T = \operatorname{spec} T = \mathbb{C}$.

As we can see in the two last examples, for general operators one cannot say much on the location of the spectrum. In what follows we will study mostly self-adjoint operators, whose spectral theory is now understood much better than for the non-self-adjoint case.

3.2 Basic facts on the spectra of self-adjoint operators

The following proposition will be of intensive use.

Proposition 3.14. *Let T be a closable operator in a Hilbert space $\mathcal{H}$ and $z \in \mathbb{C}$, then*

$$\ker(T^* - \overline{z}) = \operatorname{ran}(T - z)^\perp, \tag{3.5}$$

$$\overline{\operatorname{ran}(T - z)} = \ker(T^* - \overline{z})^\perp. \tag{3.6}$$

Proof. Note that the second equality can be obtained from the first one by taking the orthogonal complement in the both parts. Let us prove the first equality. As $D(T)$ is dense, the condition $f \in \ker(T^* - \overline{z})$ is equivalent to $\langle (T^* - \overline{z})f, g\rangle = 0$ for all $g \in D(T)$, which can be also rewritten as

$$\langle T^* f, g\rangle = z\langle f, g\rangle \text{ for all } g \in D(T).$$

By the definition of T^*, one has $\langle T^* f, g\rangle = \langle f, Tg\rangle$ and

$$\langle f, Tg\rangle - z\langle f, g\rangle \equiv \langle f, (T - z)g\rangle = 0 \text{ for all } g \in D(T),$$

i.e. $f \perp \operatorname{ran}(T - z)$. $\square$

Proposition 3.15 (Spectrum of a self-adjoint operator is real). *Let T be a self-adjoint operator in a Hilbert space $\mathcal{H}$, then* $\operatorname{spec} T \subset \mathbb{R}$, *and for any $z \in \mathbb{C} \setminus \mathbb{R}$ there holds*

$$\big\|(T - z)^{-1}\big\| \le \frac{1}{|\Im z|}. \tag{3.7}$$

Proof. Let $z \in \mathbb{C} \setminus \mathbb{R}$ and $u \in D(T)$. We have

$$\langle u, (T-z)u\rangle = \langle u, Tu\rangle - \Re z\langle u, u\rangle - i\Im z\langle u, u\rangle.$$

As T is self-adjoint, the number $\langle u, Tu\rangle$ is real. Therefore,

$$|\Im z|\, \|u\|^2 \le \big|\langle u, (T-z)u\rangle\big| \le \big\|(T-z)u\big\| \cdot \|u\|,$$

which shows that

$$\big\|(T-z)u\big\| \ge |\Im z| \cdot \|u\|. \tag{3.8}$$

It follows from here that $\operatorname{ran}(T-z)$ is closed, that $\ker(T-z) = \{0\}$ and, by proposition 3.14, than $\operatorname{ran}(T-z) = \mathcal{H}$. Therefore, $(T-z)^{-1} \in \mathcal{L}(\mathcal{H})$, and the estimate (3.7) follows from (3.8). □

The following proposition is of importance when studying bounded operators (and it is certainly already known, but we include the proof for completeness).

Proposition 3.16 (Spectrum of a continuous operator). *Let $T \in \mathcal{L}(\mathcal{H})$, then* $\operatorname{spec} T$ *is a non-empty subset of* $\big\{z \in \mathbb{C} : |z| \le \|T\|\big\}$.

Proof. Let $z \in \mathbb{C}$ with $|z| > \|T\|$. Represent $T - z = -z(1 - T/z)$. As $\|T/z\| < 1$, the inverse to $T - z$ is defined by the series,

$$(T-z)^{-1} = -\sum_{n=0}^{\infty} T^n z^{-n-1}.$$

and $z \in \operatorname{res} T$. This implies the sought inclusion.

Let us show that the spectrum is non-empty. Assume that it is not the case. Then for any $f, g \in \mathcal{H}$ the function $\mathbb{C} \ni z \mapsto F(z) := \langle f, R_T(z)g\rangle \in \mathbb{C}$ is holomorphic in $\mathbb{C}$ by proposition 3.4. On the other hand, it follows from the above series representation for the resolvent that for large z the norm of $R_T(z)$ tends to zero. It follows that $F(z) \to 0$ as $|z| \to \infty$ and that F is bounded. By Liouville's theorem, F is constant, and, moreover, $F(z) = \lim_{|z|\to+\infty} F(z) = 0$. Therefore, $\langle f, R_T(z)g\rangle = 0$ for all $z \in \mathbb{C}$ and $f, g \in \mathcal{H}$, which means that $R_T(z) = 0$. This contradicts the definition of the resolvent and shows that the spectrum of T must be non-empty. □

Proposition 3.17 (Location of spectrum of bounded self-adjoint operators). *Let $T = T^* \in \mathcal{L}(\mathcal{H})$. Denote*

$$m = m(T) = \inf_{u \ne 0} \frac{\langle u, Tu\rangle}{\langle u, u\rangle}, \quad M = M(T) = \sup_{u \ne 0} \frac{\langle u, Tu\rangle}{\langle u, u\rangle},$$

then $\operatorname{spec} T \subset [m, M]$ *and* $\{m, M\} \subset \operatorname{spec} T$.

Proof. We proved already that $\operatorname{spec} T \subset \mathbb{R}$. For $\lambda \in (M, +\infty)$ we have

$$\|u\| \cdot \|(T-\lambda)u\| \ge \big|\langle u, (\lambda - T)u\rangle\big| \ge (\lambda - M)\|u\|^2,$$

i.e. $\|(T-\lambda)u\| \geq (\lambda - M)^{-1}\|u\|$. It follows that $\ker(T-\lambda) = \{0\}$, that $\operatorname{ran}(T-\lambda)$ is closed, and due to $\operatorname{ran}(T-\lambda)^\perp = \ker(T-\lambda)$, is dense. Hence, $(T-\lambda)^{-1} \in \mathcal{L}(\mathcal{H})$. In the same way one shows that $\operatorname{spec} T \cap (-\infty, m) = \emptyset$.

Let us show that $M \in \operatorname{spec} T$ (for m the proof is similar). Using the Cauchy-Schwarz inequality for the semi-scalar product $(u, v) \mapsto \langle u, (M-T)v\rangle$ we obtain

$$\big|\langle u, (M-T)v\rangle\big|^2 \leq \langle u, (M-T)u\rangle \cdot \langle v, (M-T)v\rangle.$$

Taking the supremum over all $u \in \mathcal{H}$ with $\|u\| \leq 1$ we arrive at

$$\big\|(M-T)v\big\|^2 \leq \|M-T\| \cdot \big\langle v, (M-T)v\big\rangle.$$

By assumption, one can construct a sequence (u_n) with $\|u_n\| = 1$ such that $\langle u_n, Tu_n\rangle \to M = M\langle u, u\rangle$ as $n \to \infty$. By the above inequality we have then $(M-T)u_n \to 0$, and the operator $M - T$ cannot have bounded inverse. Thus $M \in \operatorname{spec} T$. □

Corollary 3.18. *If $T = T^* \in \mathcal{L}(\mathcal{H})$ and $\operatorname{spec} T = \{0\}$, then $T = 0$.*

Proof. By proposition 3.17 we have $m(T) = M(T) = 0$. This means that $\langle x, Tx\rangle = 0$ for all $x \in \mathcal{H}$, and the polar identity shows that $\langle x, Ty\rangle = 0$ for all $x, y \in \mathcal{H}$. □

Let us combine all of the above to show the following fundamental fact.

Theorem 3.19 (Non-emptiness of spectrum). *The spectrum of a self-adjoint operator in a Hilbert space is a non-empty closed subset of the real line.*

Proof. In view of the preceding discussion, it remains to show the non-emptyness of the spectrum. Let T be a self-adjoint operator in a Hilbert space $\mathcal{H}$. By contradiction, assume that $\operatorname{spec} T = \emptyset$. Then, first of all, $T^{-1} \in \mathcal{L}(\mathcal{H})$. Let $\lambda \in \mathbb{C} \setminus \{0\}$, then the operator

$$L_\lambda := -\frac{T}{\lambda}\Big(T - \frac{1}{\lambda}\Big)^{-1} \equiv -\frac{1}{\lambda} - \frac{1}{\lambda^2}\Big(T - \frac{1}{\lambda}\Big)^{-1}$$

belongs to $\mathcal{L}(\mathcal{H})$ with $(T^{-1} - \lambda)L_\lambda = \mathrm{Id}_\mathcal{H}$ and $L_\lambda(T^{-1} - \lambda) = \mathrm{Id}_\mathcal{H}$. Therefore, $\lambda \in \operatorname{res}(T^{-1})$. As λ was an arbitrary non-zero complex number, we have $\operatorname{spec}(T^{-1}) \subset \{0\}$. As T^{-1} is bounded, its spectrum is non-empty, hence, $\operatorname{spec} T^{-1} = \{0\}$. On the other hand, T^{-1} is self-adjoint by Proposition 1.18, and $T^{-1} = 0$ by Corollary 3.18, which contradicts the definition of the inverse operator. □

We further remark that a partial analog of Proposition 3.17 can be proved for the semibounded self-adjoint operators.

Proposition 3.20. *Let T be a self-adjoint operator semibounded from below, $T \geq -c$, and t its sesquilinear form (i.e. T is generated by t in the sense of Definition 2.1). Then $\operatorname{spec} T \subset [-c, \infty)$, moreover,*

$$\inf \operatorname{spec} T = \inf_{u \in D(T)} \frac{\langle u, Tu\rangle}{\langle u, u\rangle} = \inf_{u \in D(t)} \frac{t(u, u)}{\langle u, u\rangle}.$$

Proof. The whole proof is almost identical to the proof of Proposition 3.17, so we leave it as an exercise. The equality

$$\inf_{u \in D(T)} \frac{\langle u, Tu \rangle}{\langle u, u \rangle} = \inf_{u \in D(t)} \frac{t(u,u)}{\langle u, u \rangle}$$

follows from the density of $D(T)$ in $D(t)$ stated in Theorem 2.2. □

The last proposition shows that some spectral information for an operator T can be deduced directly through its sesquilinear form (i.e. without computing the domain of T). This link will be even more explicit through the min-max principle, which will be introduced later.

Exercise 9. 1. Let two operators A and B be unitarily equivalent (see Exercise 1). Show that the $\operatorname{spec} A = \operatorname{spec} B$ and $\operatorname{spec}_{\mathrm{p}} A = \operatorname{spec}_{\mathrm{p}} B$.

2. Let $\mu \in \operatorname{res} A \cap \operatorname{res} B$. Show that A and B are unitarily equivalent iff their resolvents $R_A(\mu)$ and $R_B(\mu)$ are unitarily equivalent.

Exercise 10. 1. Let $\Omega \subset \mathbb{R}^n$ be a non-empty open set and let $L : \Omega \to M_2(\mathbb{C})$ be a continuous 2×2 matrix function such that $L(x)^* = L(x)$ for all $x \in \Omega$. Define an operator A in $H = L^2(\Omega, \mathbb{C}^2)$ by

$$Af(x) = L(x)f(x), \quad D(A) = \Big\{ f \in H : \int_\Omega \|L(x)f(x)\|^2_{\mathbb{C}^2} dx < +\infty \Big\}.$$

Show that A is self-adjoint and explain how to calculate its spectrum using the eigenvalues of $L(x)$.

Hint: For each $x \in \Omega$, let $\xi_1(x)$ and $\xi_2(x)$ be suitably chosen eigenvectors of $L(x)$ forming an orthonormal basis of $\mathbb{C}^2$. Consider the map

$$U : H \to H, \quad Uf(x) = \begin{pmatrix} \big\langle \xi_1(x), f(x) \big\rangle_{\mathbb{C}^2} \\ \big\langle \xi_2(x), f(x) \big\rangle_{\mathbb{C}^2} \end{pmatrix}$$

and the operator $M = UAU^*$.

2. In $H = l^2(\mathbb{Z})$ consider the operator T given by

$$Tf(n) = f(n-1) + f(n+1) + V(n)f(n), \quad V(n) = \begin{cases} 4 & \text{if } n \text{ is even,} \\ -2 & \text{if } n \text{ is odd.} \end{cases}$$

Calculate its spectrum.

Hint: Consider the operators

$$U : l^2(\mathbb{Z}) \to l^2(\mathbb{Z}, \mathbb{C}^2), \quad Uf(n) := \begin{pmatrix} f(2n) \\ f(2n+1) \end{pmatrix}, \quad n \in \mathbb{Z},$$
$$F : \ell^2(\mathbb{Z}, \mathbb{C}^2) \to L^2\big((0,1), \mathbb{C}^2\big), \quad (Ff)(\theta) = \sum_{n \in \mathbb{Z}} f(n) e^{2\pi i n \theta}.$$

Write explicit expressions for the operators $S := UTU^*$ and $\widehat{S} := FSF^*$ and use the item (1).

3.3 Compactness and spectra

The present section contains a lot of repetititons from earlier lectures, but they are important for what follows.

A linear operator T acting from a Hilbert space $\mathcal{H}_1$ to a Hilbert space $\mathcal{H}_2$ is called compact, if the image of the unit ball in $\mathcal{H}_1$ is relatively compact in $\mathcal{H}_2$. We denote by $\mathcal{K}(\mathcal{H}_1, \mathcal{H}_2)$ the set of all such operators. The definition can also be reformulated as follows: an operator $T : \mathcal{H}_1 \to \mathcal{H}_2$ is compact iff any bounded sequence $(x_n) \subset \mathcal{H}_1$ has a subsequence (x_{n_k}) such that Tx_{n_k} converges in $\mathcal{H}_2$.

Recall also that any compact operator is continuous. If A is a continuous operator and B is a compact one, then the products AB and BA are compact. It is also known that the norm limit of a sequence compact operators is compact, and that any finite-dimensional operator (i.e. an operator having a finite-dimensional range) is compact. It is also known that the adjoint of a compact operator is compact (Schauder's theorem). A classical example of a compact operator is an integral operator,

$$T : L^2(\Omega) \to L^2(\Omega), \quad Af(x) = \int_\Omega K(x,y) f(y)\, dy,$$

whose integral kernels K satisfies

$$\int_\Omega \int_\Omega |K(x,y)|^2\, dx\, dy,$$

In fact, such an operator A is a Hilbert-Schmidt one, i.e. for any orthonormal basis (e_n) there holds

$$\sum_n \|Ae_n\|^2 < \infty,$$

which is slightly stronger that the usual compactness (i.e. there are compact operators which are not Hilbert-Schmidt ones.)

Recall the following fundamental result, which is based on Fredholm's alternative and is proved in the functional analysis course:

Theorem 3.21 (Spectrum of a compact operator). *Let $\mathcal{H}$ be an infinite-dimensional Hilbert space and $T \in \mathcal{K}(\mathcal{H})$, then*

(a) $0 \in \operatorname{spec} T$,

(b) $\operatorname{spec} T \setminus \{0\} = \operatorname{spec}_p T \setminus \{0\}$,

(c) we are in one and only one of the following situations:

- $\operatorname{spec} T \setminus \{0\} = \emptyset$,
- $\operatorname{spec} T \setminus \{0\}$ *is a finite set,*
- $\operatorname{spec} T \setminus \{0\}$ *is a sequence convergent to* 0.

(d) Each $\lambda \in \operatorname{spec} T \setminus \{0\}$ is isolated (i.e. has a neighborhood containing no other values of the spectrum), and $\dim\ker(T - \lambda) < \infty$.

The result has the following important corollary:

Theorem 3.22 (Spectrum of compact self-adjoint operator). *Let $T = T^* \in \mathcal{K}(\mathcal{H})$, then can construct an orthonormal basis consisting of eigenvectors of T, and the respective eigenvalues form a real sequence convergent to* 0.

Proof. Let $(\lambda_n)_{n\geq 1}$ be the distinct non-zero eigenvalues of T. As T is self-adjoint, these eigenvalues are real. Set $\lambda_0 = 0$, and for $n \geq 0$ denote $E_n := \ker(T - \lambda_n)$. One can easily see that $E_n \perp E_m$ for $n \neq m$. Denote by F the linear hull of $\cup_{n\geq 0} E_n$. We are going to show that F is dense in $\mathcal{H}$.

Clearly, we have $T(F) \subset F$. Due to the self-adjointness of T we also have $T(F^\perp) \subset F^\perp$. Denote by $\widetilde{T}$ the restriction of T to $F^\perp$, then $\widetilde{T}$ is compact, self-adjoint, and its spectrum equals $\{0\}$, so $\widetilde{T} = 0$. But this means that $F^\perp \subset \ker T = E_0 \subset F$ and shows that $F^\perp = \{0\}$. Therefore F is dense in $\mathcal{H}$.

Now taking an orthonormal basis in each subspace E_n we obtain an orthonormal basis in the whole space $\mathcal{H}$. □

The above can be used for a discussion of a class of unbounded operators. Namely, one says that an operator T in $\mathcal{H}$ *has compact resolvent* if $\operatorname{res} T \neq \emptyset$ and for some (and then for all) $z \in \operatorname{res} T$ the resolvent $(T - z)^{-1}$ is a compact operator.

Similar to the preceding constructions one can show:

Proposition 3.23 (Spectra of semibounded operators with compact resolvents). *Let T be a semibounded from below self-adjoint operator with compact resolvent in an infinite-dimensional Hilbert space, then:*

- $\operatorname{spec} T = \operatorname{spec}_p T$,
- *for each $\lambda \in \operatorname{spec} T$ there holds $\dim\ker(T - \lambda) < \infty$.*
- *the eigenvalues of T form a sequence converging to $+\infty$.*

Proof. Let $T \geq -c$, then $-(c+1) \in \operatorname{res} T$ (Proposition 3.20), and $(T + c + 1)^{-1}$ is a bounded self-adjoint operator which is compact by assumption. Moreover, this operator is non-negative: for any $u \in \mathcal{H}$ denote $v := (T + c + 1)^{-1} \in D(T)$, then

$$\langle u, (T + c + 1)^{-1}u\rangle = \langle (T + c + 1)v, v\rangle \geq \|v\|^2 \geq 0.$$

By Theorem 3.22, there exists an orthonormal basis (e_n) of $\mathcal{H}$ such that each e_n ia an eigenfunction of $(T + c + 1)^{-1}$: $(T + c + 1)^{-1}e_n = \lambda_n e_n$, where $\lambda_n > 0$ form a sequence converging to 0. We then have $(T + c + 1)e_n = \lambda_n^{-1}$, i.e. each e_n is an eigenvector of T with eigenvalue $\mu_n := \lambda_n^{-1} - c - 1$, and the multiplicity of this eigenvalue is the same as that of λ_n as an eigenvalue of $(T + c + 1)^{-1}$, e.g. is finite. The operator T is then essentially self-adjoint on finite linear combinations of e_n (Exercise 1). Moreover, if one introduces the unitary transform $V : \mathcal{H} \to \ell^2(\mathbb{N})$ by $Vu(n) = \langle e_n, u\rangle$, then one sees that T is unitarily equivalent to the multiplication by (μ_n) in $\ell^2(\mathbb{N})$ (Example 3.10), hence, its $\operatorname{spec} T = \overline{\{\mu_n,\, n \in \mathbb{N}\}}$. As $\mu_n \to +\infty$ for $n \to +\infty$, one has $\operatorname{spec} T = \{\mu_n,\, n \in \mathbb{N}\} = \operatorname{spec}_p T$, as each μ_n is an eigenvalue. □

Now we would like to obtain a class of operators with compact resolvents.

Theorem 3.24. *Let T be a self-adjoint operator generated by a closed sesquilinear form t in $\mathcal{H}$. Assume that the Hilbert space $D(t)$ is compactly embedded in $\mathcal{H}$, then T has compact resolvent.*

Proof. Without loss of generality we will assume that $t(u,u) \geq \|u\|^2_{\mathcal{H}}$ for all $u \in D(t)$, hence, $\|u\|_t^2 = t(u,u)$, and then $T \geq 1$. Moreover, for any $u \in D(T)$ we have:

$$\|u\|_{\mathcal{H}} \|Tu\|_{\mathcal{H}} \geq \big|\langle u, Tu\rangle_{\mathcal{H}}\big| = |t(u,u)| = \|u\|_t \|u\|_t \geq \|u\|_t \|u\|_{\mathcal{H}},$$

i.e. $\|Tu\|_{\mathcal{H}} \geq \|u\|_t$, hence, $\|T^{-1}v\|_t \leq \|v\|_{\mathcal{H}}$ for all $v \in \mathcal{H}$, and $T^{-1} \in \mathcal{L}\big(\mathcal{H}, D(t)\big)$. Now let $j : D(t) \to \mathcal{H}$ be the embedding, which is compact by assumption, then $T^{-1} = jL$, where $L : \mathcal{H} \ni v \mapsto T^{-1}v \in D(t)$. Hence T^{-1} is compact as a composition of a bounded operator and a compact one. □

In order to look at concrete examples we recall the following classical criterion of compactness in $L^2(\mathbb{R}^d)$ (sometimes referred to as the Riesz-Kolmogorov-Tamarkin criterion)[1]:

Proposition 3.25. *A subset $A \subset L^2(\mathbb{R}^d)$ is relatively compact in $L^2(\mathbb{R}^d)$ if and only if the following three conditions are satisfied:*

(a) A is bounded,

(b) there holds $\displaystyle\int_{|x|\geq R} |u(x)|^2 dx \to 0$ as $R \to \infty$ uniformly for $u \in A$,

(c) $\|u_h - u\| \to 0$ as $h \to 0$ uniformly for $u \in A$. Here, for $h \in \mathbb{R}^d$ and $v \in L^2(\mathbb{R}^d)$, the symbol v_h denote the function defined by $v_h(x) = v(x+h)$.

Example 3.26 (Schrödinger operators with growing potentials). Let us discuss a particular class of operators with compact resolvents.
Now let $V \in L^2_{\text{loc}}(\mathbb{R}^d)$ and $V \geq -C$. Consider the Schrödinger operator $T = -\Delta + V$ defined as the Friedrichs extension starting from $C_c^\infty(\mathbb{R}^d)$ and discussed in Example 2.16. We know already that T is a self-adjoint and semibounded from below operator in $\mathcal{H} = L^2(\mathbb{R}^d)$. We would like to identify a reasonable large class of potentials V for which T has compact resolvent.

Theorem 3.27. *For $r \geq 0$ denote*

$$w(r) := \inf_{|x|\geq r} V(x). \tag{3.9}$$

If $\lim_{r\to+\infty} w(r) = +\infty$, then the associated Schrödinger operator $T = -\Delta + V$ has a compact resolvent.

[1]An interested reader may study H. Hanche-Olsena, H. Holden: *The Kolmogorov-Riesz compactness theorem.* Expositiones Mathematicae, Vol. 28, Issue 4 (2010), pp. 385–394 for a proof and various generalizations.

Proof. Without loss of generality we assume $V \geq 0$. Recall (Example 2.16) that the associated sesquilinear form is

$$t(u,u) = \int_{\mathbb{R}^d} \big(|\nabla u|^2 + V|u|^2\big)\, dx,$$

and the domain $\mathcal{V} = D(t)$ is the closure of $C_c^\infty(\mathbb{R}^d)$ in $H^1_V(\mathbb{R}^d)$, equipped with the norm $\|u\|^2_W = \|u\|^2_{H^1} + \|\sqrt{V}u\|_{L^2}$. Let B be the unit ball in $\mathcal{V}$. We are going to show that B is relatively compact in $L^2(\mathbb{R}^d)$ using Proposition 3.25:

- The condition (a) holds due to the obvious inequality $\|u\|_{L_2} \leq \|u\|_V$.
- The condition (b) follows from

$$\int_{|x|\geq R} |u(x)|^2 dx \leq \frac{1}{w(R)} \int_{|x|\geq R} V(x)|u(x)|^2 \leq \frac{\|\sqrt{V}u\|^2_{L^2}}{w(R)} \leq \frac{\|u\|^2_V}{w(R)}.$$

- For the condition (c) we have, for $u \in C_c^\infty(\mathbb{R}^d)$,

$$\begin{aligned}\int_{\mathbb{R}^d} \big|u(x+h) - u(x)\big|^2 dx &= \int_{\mathbb{R}^d} \Big| \int_0^1 \frac{d}{dt} u(x+th)dt\Big|^2 dx\\ = \int_{\mathbb{R}^d} \Big| \int_0^1 h\cdot \nabla u(x+th)dt\Big|^2 dx &\leq h^2 \int_{\mathbb{R}^d}\int_0^1 \big|\nabla u(x+th)\big|^2 dt\,dx\\ &\leq h^2 \int_0^1 \int_{\mathbb{R}^d} \big|\nabla u(x+th)\big|^2 dx\,dt = h^2\|\nabla u\|^2_{L^2} \leq h^2 \|u\|^2_V,\end{aligned}$$

  which then extends by density to the whole of H^1_V.

The compactness of B implies the compactness of the embedding $j : D(t) \to L^2(\mathbb{R}^d)$, and the result follows by Theorem 3.24. $\square$

The assumption of Theorem 3.27 is rather easy to check, e.g. the assumptions hold for $V(x) = |x|^\alpha +$ bounded, $\alpha > 0$, but the condition (3.9) in not an optimal one. For example, it is known that the operator $-\Delta + W$ with $W(x_1, x_2) = x_1^2 x_2^2$ has a compact resolvent, while the condition cleraly fails. A rather simple necessary and sufficient condition is known in the one-dimensional case, which we mention without proof:

Proposition 3.28 (Molchanov criterium). *The operator $T = -d^2/dx^2 + V$ has a compact resolvent iff*

$$\lim_{x\to\infty} \int_x^{x+\delta} V(s)ds = +\infty$$

for some $\delta > 0$ (and then for any $\delta > 0$).

Necessary and sufficient conditions are also available for the multi-dimensional case, but their form is much more complicated.[2]

Example 3.29 (Dirichlet and Neumann Laplacians). Let $\Omega \subset \mathbb{R}^d$ be a non-empty open set. Recall that the associated Dirichlet and Neumann Laplacians T_D and T_N are defined as the self-adjoint operators generated by the closed sesquilinear forms

$$t_D(u,u) = \int_\Omega |\nabla u|^2\,dx \qquad\qquad D(t_D) = H^1(\Omega),$$
$$t_N(u,u) = \int_\Omega |\nabla u|^2\,dx \qquad\qquad D(t_N) = H_0^1(\Omega).$$

Proposition 3.30. *Let Ω be bounded, then the embedding $H_0^1(\Omega) \to L^2(\Omega)$ is compact, and T_D has compact resolvent by Theorem 3.24.*

Proof. For a function u defined on Ω, we denote $\widetilde{u}$ its extension by zero to the whole of $\mathbb{R}^d$. For $u \in C_c^\infty(\Omega)$ one has $\widetilde{u} \in C^\infty(\mathbb{R}^d)$ with $\|u\|_{H^1(\Omega)} = \|\widetilde{u}\|_{H^1(\mathbb{R}^d)}$. As $H_0^1(\Omega)$ was defined as the closure of $C_c^\infty(\Omega)$ in $H^1(\Omega)$, it follows that the map $\iota : u \mapsto \widetilde{u}$ extends to an isometric embedding of $j : H_0^1(\Omega) \to H^1(\mathbb{R}^d)$, while $\operatorname{ran}\iota$ is clearly contained in

$$\widetilde{H}_0^1(\Omega) = \big\{u \in H^1(\mathbb{R}^d) :\ u = 0 \text{ outside } \Omega\big\}.$$

Using literally the same argument as in the proof of Theorem 3.27 one shows that the embedding $j : \widetilde{H}_0^1(\Omega) \to L^2(\mathbb{R}^d)$ is compact. Now let $k : L^2(\mathbb{R}^d) \to L^2(\Omega)$ be the operator of restriction to Ω, $(ku)(x) = u(x)$ for all $x \in \Omega$, which is clearly bounded. Then the product $kj\iota$ is exactly the embedding of $H^1(\Omega) \to L^2(\Omega)$, which is then compact (as j is compact and ι and k are bounded). $\square$

The eigenvalues μ_n^D of T_D (ordered in the non-decresing order and counted according to multiplicities) are called the *Dirichlet eigenvalues of* Ω while the respective eigenfunctions are called the *Dirichlet eigenfunctions*. It is an important domain of the modern analysis to study the relations between the geometrical and topological properties of Ω and the associated Dirichlet eigenvalues and eigenfunctions (in fact is the main topic of the spectral geometry). One should also remark that there are unbounded Ω such that that T^D still has compact resolvent. This will be addressed later in this course.

There is no literal extension of Proposition 3.30: there are bounded domains Ω such that T_N is not with compact resolvent (and the embedding of $H^1(\Omega)$ in $L^2(\Omega)$ is not compact.) Anyway, under some additional assumptions one prove an analogous result:

[2] For a discussion of such questions, an advanced reader may refer to the paper V. Kondrat'ev, M. Shubin: *Discreteness of spectrum for the Schrödinger operators on manifolds of bounded geometry.* The Maz'ya anniversary collection, Vol. 2 (Rostock, 1998), pp. 185–226 (Operator Theory: Advances and Applications, vol. 110), Birkhäuser, Basel, 1999.

Definition 3.31. A domain $\Omega \subset \mathbb{R}^d$ is called an extension domain, if there exist a bounded operator $E : H^1(\Omega) \to H^1(\mathbb{R}^d)$ such that $Ef(x) = f(x)$ for all $f \in H^1(\Omega)$ and all $x \in \Omega$. Such an operator E is usually called an *extension operator*.

One can show that if the boundary of Ω is not "too bad" (e.g. smooth, or, more generally, Lipschitz), then Ω is a extension domain.

Proposition 3.32. *If Ω is a bounded extension domain, then $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, and T_N has compact resolvent.*

Proof. One can reduce the problem to the H_0^1 case. Namely, let $E : H^1(\Omega) \to H^1(\mathbb{R}^d)$ be an extension operator and $\chi \in C_c^\infty(\mathbb{R}^d)$ with $\chi = 1$ in Ω and vanishing outside a ball Θ. Define $E_0 : H^1(\Omega) \ni u \mapsto \chi E u \in H^1(\mathbb{R}^d)$ and remark that $E_0 H^1(\Omega)$ is contained in $\widetilde{H}_0^1(\Theta)$ defined in Proposition 3.30, in particular, the embedding $j : H_0^1(\Theta) \to L^2(\mathbb{R}^d)$ is compact. Finally, let $k : L^2(\mathbb{R}^d) \to L^2(\Omega)$ be the operator of restriction to Ω, $(ku)(x) = u(x)$ for all $x \in \Omega$, which is clearly bounded. Now wee that the embedding of $H^1(\Omega)$ in $L^2(\Omega)$ is represented as the composition kjE_0, and it is a compact operator (as j is compact and k and E_0 are bounded). $\square$

The eigenvalues μ_2^N of T^N are the *Neumann eigenvalues of* Ω, and, similarly to the Dirichlet ones, they are a subject of intensive study. One of the hot topics in the modern analysis is the so-called *hot spots conjectures* discussing the properties of the eigenfunctions associated with the second Neumann eigenvalue μ_2^N (the first one is always zero).

We will use the following regularity result (will be proved somewhere else): if $I \subset \mathbb{R}$ is an interval and a function u is a weak solution of a linear ordinary differential equation with constant coefficients in I, then u is a C^∞ function (and then it is a solution in the usual sense). There is no easy multi-dimensional realization!

Exercise 11. Let $\ell > 0$. Compute explicitly Dirichlet and Neumann eigenvalues and eigenfunctions of the interval $(0, \ell)$.

Exercise 12. In $\mathcal{H} = L^2(0, 2\pi)$ consider the operator T already seen in Exercise 4: $T : u \mapsto -u''$ with the domain

$$D(T) = \big\{u \in C^\infty(0, 2\pi) : \ u \text{ extends to a } 2\pi\text{-periodic function on } \mathbb{R}\big\}.$$

Show that the closure $S := \overline{T}$ is with compact resolvent. Compute the eigenvalues and the eigenfunctions of S.

4 Spectral theorem

The main points of this lecture are as follows:

- Theorem 4.13 showing that each self-adjoint operator is unitarily equivalent to a multilplcation operator in a suitable L^2-space. Moreover, a kind of normal form for the multiplication operator is shown.
- Theorem 4.16 stating that the operators $f(T)$ are well-defined in a unique way for arbitrary bounded Borel functions f.

A good understanding of these results is of great importance for all subsequent considerations. The formulations are much more important as the proof: the complete proofs involve a number of technicalities of lower importance (especially in Subsection 4.1).[3]

To be provided with a certain motivation, let T be either a compact self-adjoint operator or a self-adjoint operator with a compact resolvent in a Hilbert space $\mathcal{H}$. As shown in the previous section, there exists an orthonormal basis (e_n) in $\mathcal{H}$ and real numbers λ_n such that, with

$$Tx = \sum_n \lambda_n \langle e_n, x\rangle e_n \quad \text{for all } x \in D(T),$$

and the domain $D(T)$ is characterized by

$$D(T) = \Big\{ x \in \mathcal{H} : \sum_n \lambda_n^2 \big|\langle e_n, x\rangle\big|^2 < \infty \Big\}.$$

For $f \in C_0(\mathbb{R})$ one can define an operator $f(T) \in \mathcal{L}(\mathcal{H})$ by

$$f(T)x = \sum_n f(\lambda_n)\langle e_n, x\rangle e_n.$$

This map $f \mapsto f(T)$ enjoys a number of properties. For example, $(fg)(T) = f(T)g(T)$, $\overline{f}(T) = f(T)^*$, $\operatorname{spec} f(T) = \overline{f(\operatorname{spec} T)}$ etc. The existence of such a construction allows one to write rather explicit expressions for solutions of some equations. For example, one can easily show that the initial value problem

$$-ix'(t) = Tx(t),\ x(0) = y \in D(T), \quad x : \mathbb{R} \to D(T),$$

has a solution that can be written as $x(t) = f_t(T)y$ with $f_t(x) = e^{itx}$. Informally speaking, for a large class of equations involving the operator T one may first assume that T is a real constant and obtain a formula for the solution, and then one can give this formula an operator-valued meaning using the above map $f \mapsto f(T)$.

Moreover, if we introduce the map $U : \mathcal{H} \to \ell^2(\mathbb{N})$ defined by $Ux =: (x_n)$, $x_n = \langle e_n, x\rangle$, then the operator UTU^* becomes a multiplication operator $(x_n) \mapsto (\lambda_n x_n)$.

[3]A complete proof can be found in Chapter 2 of the book E. B. Davies: *Spectral theory and differential operators.* Cambridge University Press, 1995.

At this point, all the preceding facts are proved for compact self-adjoint operators and for self-adjoint operators with compact resolvents only. The aim of the present section is to develop a similar theory for general self-adjoint operators.

To avoid potential misunderstanding let us recall that $C_0(\mathbb{R})$ denotes the class of the continuous functions $f : \mathbb{R} \to \mathbb{C}$ with $\lim_{|x|\to+\infty} f(x) = 0$, which becomes a Banach space if equipped with the sup-norm. This should not be confused with the set $C^0(\mathbb{R})$ of the continuous functions on $\mathbb{R}$.

4.1 Continuous functional calculus

We say that a function $f : \mathbb{C} \to \mathbb{C}$ belongs to $C^\infty(\mathbb{C})$ if the function of two real variables $\mathbb{R}^2 \ni (x, y) \mapsto f(x + iy) \in \mathbb{C}$ belongs to $C^\infty(\mathbb{R}^2)$. In the similar way one defines the classes $C_c^\infty(\mathbb{C})$, $C^k(\mathbb{C})$ etc. In what follows we always use the notation $\Re z =: x$, $\Im z =: y$ for $z \in \mathbb{C}$. Using $x = \dfrac{z + \overline{z}}{2}$ and $y = \dfrac{z - \overline{z}}{2i}$, for $f \in C^1(\mathbb{C})$ one defines the derivative

$$\frac{\partial}{\partial \overline{z}} := \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)$$

Clearly, $\partial g/\partial \overline{z} = 0$ if g is a holomorphic function. Recall the *Stokes formula* written in this notation: if $f \in C^\infty(\mathbb{C})$ and $\Omega \subset \mathbb{C}$ is a domain with a sufficiently regular boundary, then

$$\iint_\Omega \frac{\partial f}{\partial \overline{z}}\, dx\, dy = \frac{1}{2i} \oint_{\partial\Omega} f\, dz.$$

The following fact is actually known, but is presented in a slightly unusual form.

Lemma 4.1 (Cauchy integral formula). *Let $f \in C_c^\infty(\mathbb{C})$, then for any $w \in \mathbb{C}$ we have*

$$\frac{1}{\pi} \iint_{\mathbb{C}} \frac{\partial f}{\partial \overline{z}}\, \frac{1}{w - z}\, dx\, dy = f(w).$$

Proof. We note first that the singularity $1/z$ is integrable in two dimensions, and the integral is well-defined. Let Ω be a large ball containing the support of f and the point w. For small $\varepsilon > 0$ denote $B_\varepsilon := \{z \in \mathbb{C} : |z - w| \le \varepsilon\}$, and set $\Omega_\varepsilon := \Omega \setminus B_\varepsilon$. Using the Stokes formula we have:

$$\begin{aligned}
\frac{1}{\pi} \iint_{\mathbb{C}} \frac{\partial f}{\partial \overline{z}}\, \frac{1}{w - z}\, dx\, dy &= \frac{1}{\pi} \iint_{\Omega} \frac{\partial f}{\partial \overline{z}}\, \frac{1}{w - z}\, dx\, dy \\
&= \lim_{\varepsilon\to 0} \frac{1}{\pi} \iint_{\Omega_\varepsilon} \frac{\partial f}{\partial \overline{z}}\, \frac{1}{w - z}\, dx\, dy = \lim_{\varepsilon\to 0} \frac{1}{\pi} \iint_{\Omega_\varepsilon} \frac{\partial}{\partial \overline{z}}\Big(f(z)\, \frac{1}{w - z}\Big)\, dx\, dy \\
&= \lim_{\varepsilon\to 0} \frac{1}{2\pi i} \oint_{\partial\Omega_\varepsilon} f(z) \frac{1}{w - z}\, dz \\
&= \frac{1}{2\pi i} \oint_{\partial\Omega} f(z) \frac{1}{w - z}\, dz - \lim_{\varepsilon\to 0} \frac{1}{2\pi i} \oint_{|z-w|=\varepsilon} f(z) \frac{1}{w - z}\, dz.
\end{aligned}$$

The first term on the right-hand side is zero, because f vanishes at the boundary of Ω. The second term can be calculated explicitly:

$$\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\oint_{|z-w|=\varepsilon} f(z)\frac{1}{w-z}\,dz = \lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_0^{2\pi} f(w+\varepsilon e^{it})\frac{i\varepsilon e^{it}dt}{w-(w+\varepsilon e^{it})}$$
$$= -\lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_0^{2\pi} f(w+\varepsilon e^{it})dt = -f(w),$$

which gives the result. □

The main idea of the subsequent presentation is to define the operators $f(T)$, for a self-adjoint operator T, using an operator-valued generalization of the Cauchy integral formula.

Introduce first some notation. For $z\in\mathbb{C}$ we write

$$\langle z\rangle := \sqrt{1+|z|^2}.$$

For $\beta<0$ denote by $\mathcal{S}_\beta$ the set of the smooth functions $f:\mathbb{R}\to\mathbb{C}$ satisfying the estimates

$$\left|f^{(n)}(x)\right| \le c_n\langle x\rangle^{\beta-n}$$

for any $n\ge 0$ and $x\in\mathbb{R}$, where the positive constant c_n may depend on f. Set $\mathcal{A} := \bigcup_{\beta<0}\mathcal{S}_\beta$; one can show that $\mathcal{A}$ is an alebra. Moreover, if $f=P/Q$, where P and Q are polynomials with $\deg P<\deg Q$ and $Q(x)\neq 0$ for $x\in\mathbb{R}$, then $f\in\mathcal{A}$. For any $n\ge 1$ one can introduce the norms on $\mathcal{A}$:

$$\|f\|_n := \sum_{r=0}^{n}\int_{\mathbb{R}}\left|f^{(r)}(x)\right|\langle x\rangle^{r-1}dx.$$

One can easily see that the above norms on $\mathcal{A}$ induce continuous embeddings $\mathcal{A}\to C_0(\mathbb{R})$. Moreover, one can prove that $C_c^\infty(\mathbb{R})$ is dense in $\mathcal{A}$ with respect to any norm $\|\cdot\|_n$.

Now let $f\in C^\infty(\mathbb{R})$. Pick $n\in\mathbb{N}$ and a smooth function $\tau:\mathbb{R}\to\mathbb{R}$ such that $\tau(s)=1$ for $|s|<1$ and $\tau(s)=0$ for $|s|>2$. For $x,y\in\mathbb{R}$ set $\sigma(x,y):=\tau(y/\langle x\rangle)$. Define $\widetilde{f}\in C^\infty(\mathbb{C})$ by

$$\widetilde{f}(z) = \left[\sum_{r=0}^{n} f^{(r)}(x)\frac{(iy)^r}{r!}\right]\sigma(x,y).$$

Clearly, for $x\in\mathbb{R}$ we have $\widetilde{f}(x)=f(x)$, so $\widetilde{f}$ is an extension of f. One can check the following identity:

$$\frac{\partial\widetilde{f}}{\partial\bar{z}} = \frac{1}{2}\left[\sum_{r=0}^{n} f^{(r)}(x)\frac{(iy)^r}{r!}\right]\Big(\sigma_x+i\sigma_y\Big)+\frac{1}{2}f^{(n+1)}(x)\frac{(iy)^n}{n!}\sigma. \tag{4.1}$$

Now let T be a self-adjoint operator in a Hilbert space $\mathcal{H}$. For $f \in \mathcal{A}$ define an operator $f(T)$ in $\mathcal{H}$ by

$$f(T) := \frac{1}{\pi} \iint_{\mathbb{C}} \frac{\partial \widetilde{f}}{\partial \overline{z}} \, (T - z)^{-1} \, dx \, dy. \tag{4.2}$$

This integral expression is called the *Helffer-Sjöstrand formula*. We need to show several points: that the integral is well-defined, that it does not depend in the choice of σ and n etc. This will be done is a series of lemmas.

Note first that, as shown in Proposition 3.15, we have the norm estimate $\|(T - z)^{-1}\| \leq 1/|\Im z|$, and one can see from (4.1) that $\widetilde{\partial} f/\partial \overline{z}(x + iy) = O(y^n)$ for any fixed x, so the subintegral function in (4.2) is locally bounded. By additional technical efforts one can show that the integral is convergent and defines an continuous operator with $\|f(T)\| \leq c\|f\|_{n+1}$ for some $c > 0$. Using this observation and the density of $C_c^\infty(\mathbb{R})$ in $\mathcal{A}$ the most proofs will be provided for $f \in C_c^\infty$ and extended to $\mathcal{A}$ and larger spaces using the standard density arguments.

Lemma 4.2. *If $F \in C_c^\infty(\mathbb{C})$ and $F(z) = O(y^2)$ as $y \to 0$, then*

$$A := \frac{1}{\pi} \iint_{\mathbb{C}} \frac{\partial F}{\partial \overline{z}} \, (T - z)^{-1} \, dx \, dy = 0.$$

Proof. By choosing a sufficiently large $N > 0$ one may assyme that the support of F is contained in $\Omega := \{z \in \mathbb{C} : |x| < N, |y| < N\}$. For small $\varepsilon > 0$ define $\Omega_\varepsilon := \{z \in \mathbb{C} : |x| < N, \varepsilon < |y| < N\}$. Using the Stokes formula we have

$$A = \lim_{\varepsilon \to 0} \frac{1}{\pi} \iint_{\Omega_\varepsilon} \frac{\partial F}{\partial \overline{z}} \, (T - z)^{-1} \, dx \, dy = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \oint_{\partial \Omega_\varepsilon} F(z) \, (T - z)^{-1} \, dz.$$

The boundary $\partial\Omega_\varepsilon$ consists of eight segments. The integral over the vertical segments and over the horizontal segments with $y = \pm N$ are equal to 0 because the function F vanishes on these segments. It remains to estimate the integrals over the segments with $y = \pm\varepsilon$. Here we have $\|(T - z)^{-1}\| \leq \varepsilon^{-1}$ and

$$\|A\| \leq \lim_{\varepsilon \to 0} \frac{1}{2\pi} \oint_{\partial\Omega_\varepsilon} \Big(|F(x + i\varepsilon)| + |F(x - i\varepsilon)|\Big)\varepsilon^{-1} dx = 0. \qquad \square$$

Corollary 4.3. *For $f \in \mathcal{A}$ the integral in* (4.2) *is independent of the choice of $n \geq 1$ and σ.*

Proof. For $f \in C_c^\infty(\mathbb{C})$ the assertion follows from the definition of $\widetilde{f}$ and Lemma 4.2. This is extended to $\mathcal{A}$ using the density arguments. $\square$

Lemma 4.4. *Let $f \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp} f \cap \operatorname{spec} T = \emptyset$, then $f(T) = 0$.*

Proof. If $f \in C_c^\infty(\mathbb{R})$, then obviously $\widetilde{f} \in C_c^\infty(\mathbb{C})$. One can find a finite family of closed curves γ_r which do not meet the spectrum of T and enclose a domain U containing $\operatorname{supp} \widetilde{f}$. Using the Stokes formula we have

$$f(T) = \frac{1}{\pi} \iint_U \frac{\partial \widetilde{f}}{\partial \overline{z}} \, (T - z)^{-1} \, dx \, dy = \sum_r \frac{1}{2\pi i} \oint_{\gamma_r} \widetilde{f}(z) \, (T - z)^{-1} dz.$$

All the terms in the sum are zero, because $\widetilde{f}$ vanishes on γ_r. $\square$

Lemma 4.5. *For $f, g \in \mathcal{A}$ one has $(fg)(T) = f(T)g(T)$.*

Proof. By the density arguments is it sufficient to consider the case $f, g \in C_c^\infty(\mathbb{R})$. Let K and L be large balls containing the supports of $\widetilde{f}$ and $\widetilde{g}$ respectively. Using the notation $w = u + iv$, $u, v \in \mathbb{R}$, one can write:

$$f(T)g(T) = \frac{1}{\pi^2} \iiiint\limits_{K\times L} \frac{\partial \widetilde{f}}{\partial \overline{z}} \frac{\partial \widetilde{g}}{\partial \overline{w}} (T-z)^{-1}(T-w)^{-1}\, dx\, dy\, du\, dv.$$

Using the resolvent identity

$$(T-z)^{-1}(T-w)^{-1} = \frac{1}{w-z}(T-w)^{-1} - \frac{1}{w-z}(T-z)^{-1}$$

we rewrite the preceding integral in the form

$$\begin{aligned} f(T)g(T) = \frac{1}{\pi^2} \iint_L \frac{\partial \widetilde{g}}{\partial \overline{w}} (T-w)^{-1} \Big(\iint_K \frac{\partial \widetilde{f}}{\partial \overline{z}} \frac{1}{w-z} dx\, dy \Big) du\, dv \\ - \frac{1}{\pi^2} \iint_K \frac{\partial \widetilde{f}}{\partial \overline{z}} (T-z)^{-1} \Big(\iint_L \frac{\partial \widetilde{g}}{\partial \overline{w}} \frac{1}{w-z} du\, dv \Big) dx\, dy. \end{aligned}$$

By Lemma 4.1 we have

$$\iint_K \frac{\partial \widetilde{f}}{\partial \overline{z}} \frac{1}{w-z} dx\, dy = \pi f(w), \quad \iint_L \frac{\partial \widetilde{g}}{\partial \overline{w}} \frac{1}{w-z} du\, dv = -\pi g(z),$$

and we arrive at

$$\begin{aligned} f(T)g(T) &= \frac{1}{\pi} \iint_L \widetilde{f}(w) \frac{\partial \widetilde{g}}{\partial \overline{w}} (T-w)^{-1} du\, dv + \frac{1}{\pi} \iint_K \widetilde{g}(z) \frac{\partial \widetilde{f}}{\partial \overline{z}} (T-z)^{-1} dx\, dy \\ &= \frac{1}{\pi} \iint_{K\cup L} \frac{\partial (\widetilde{f}\widetilde{g})}{\partial \overline{z}} (T-z)^{-1} dx\, dy \\ &= \frac{1}{\pi} \iint_{\mathbb{C}} \frac{\partial \widetilde{fg}}{\partial \overline{z}} (T-z)^{-1} dx\, dy + \frac{1}{\pi} \iint_{\mathbb{C}} \frac{\partial (\widetilde{f}\widetilde{g} - \widetilde{fg})}{\partial \overline{z}} (T-z)^{-1} dx\, dy \\ &= (fg)(T) + \frac{1}{\pi} \iint_{\mathbb{C}} \frac{\partial (\widetilde{f}\widetilde{g} - \widetilde{fg})}{\partial \overline{z}} (T-z)^{-1} dx\, dy. \end{aligned}$$

By direct calculation one can see that $(\widetilde{fg} - \widetilde{f}\widetilde{g})(z) = O(y^2)$ for small y, and Lemma 4.2 shows that the second integral is zero. $\square$

Lemma 4.6. *Let $w \in \mathbb{C} \setminus \mathbb{R}$. Define a function r_w by $r_w(z) = (z-w)^{-1}$. Then $r_w(T) = (T-w)^{-1}$.*

Proof. We provide just the main line of the proof without technical details (they can be easily recovered). Use first the independence of n and σ. We take $n=1$ and put $\sigma(z)=\tau(\lambda y/\langle x\rangle)$ where $\lambda>0$ is sufficiently large, to have $w\notin\operatorname{supp}\sigma$. Without loss of generality we assume $\Im w>0$. For large $m>0$ consider the region

$$\Omega_m:=\{z\in\mathbb{C}:\quad |x|<m,\quad \frac{\langle x\rangle}{m}<y<2m\}.$$

Using the definition and the Stokes formula we have

$$r_w(T)=\lim_{m\to\infty}\frac{1}{\pi}\iint_{\Omega_m}\frac{\partial\widetilde{r}_w}{\partial\overline{z}}\,(T-z)^{-1}\,dx\,dy=\lim_{m\to\infty}\frac{1}{2\pi i}\oint_{\partial\Omega_m}\widetilde{r}_w(z)\,(T-z)^{-1}\,dz.$$

By rather technical explicit estimates (which are omitted here) one can show that

$$\lim_{m\to\infty}\oint_{\partial\Omega_m}\Big(\widetilde{r}_w(z)-r_w(z)\Big)\,(T-z)^{-1}\,dz=0.$$

and we arrive at

$$r_w(T)=\frac{1}{2\pi i}\lim_{m\to\infty}\oint_{\partial\Omega_m}\frac{1}{z-w}\,(T-z)^{-1}\,dz.$$

For sufficiently large m one has the inclusion $w\in\Omega_m$. For any $f,g\in\mathcal{H}$ the function $\mathbb{C}\ni z\mapsto\langle f,(T-z)^{-1}g\rangle\in\mathbb{C}$ is holomorphic in Ω_m, so applying the Cauchy formula, for large m we have

$$\frac{1}{2\pi i}\oint_{\partial\Omega_m}\frac{1}{z-w}\,\big\langle f,(T-z)^{-1}g\big\rangle\,dz=\big\langle f,(T-w)^{-1}g\big\rangle,$$

which shows that $r_w(T)=(T-w)^{-1}$. $\square$

Lemma 4.7. *For any $f\in\mathcal{A}$ we have:*

- **(a)** $\overline{f}(T)=f(T)^*$,
- **(b)** $\big\|f(T)\big\|\le\|f\|_\infty$.

Proof. The item (a) follows directly from the equalities

$$\big((T-z)^{-1}\big)^*=(T-\overline{z})^{-1},\quad \overline{\widetilde{f}(z)}=\widetilde{\overline{f}}(\overline{z}).$$

To show (b), take an arbitrary $c>\|f\|_\infty$ and define $g(s):=c-\sqrt{c^2-|f(s)|^2}$. One can show that $g\in\mathcal{A}$. There holds $\overline{f}f-2cg+g^2=0$, and using the preceding lemmas we obtain $f(T)^*f(T)-cg(T)-cg(T)^*+g(T)^*g(T)=0$, and

$$f(T)^*f(T)+\big(c-g(T)\big)^*\big(c-g(T)\big)=c^2.$$

Let $\psi\in\mathcal{H}$. Using the preceding equality we have:

$$\begin{aligned}\big\|f(T)\psi\big\|^2&\le\big\|f(T)\psi\big\|^2+\Big\|\big(c-g(T)\big)\psi\Big\|^2\\&=\Big\langle\psi,f(T)^*f(T)\psi\Big\rangle+\Big\langle\psi,\big(c-g(T)\big)^*\big(c-g(T)\big)\psi\Big\rangle\\&=c^2\|\psi\|^2.\end{aligned}$$

As $c>\|f\|_\infty$ was arbitrary, this concludes the proof. $\square$

All the preceding lemmas put together lead us to the following fundamental result.

Theorem 4.8 (Spectral theorem, continuous functional calculus). *Let T be a self-adjoint operator in a Hilbert space $\mathcal{H}$. There exists a unique linear map*

$$C_0(\mathbb{R}) \ni f \mapsto f(T) \in \mathcal{L}(\mathcal{H})$$

with the following properties:

- *$f \mapsto f(T)$ is an algebra homomorphism,*
- *$\overline{f}(T) = f(T)^*$,*
- *$\|f(T)\| \le \|f\|_\infty$,*
- *if $w \notin \mathbb{R}$ and $r_w(s) = (s-w)^{-1}$, then $r_w(T) = (T-w)^{-1}$,*
- *if $\operatorname{supp} f$ does not meet $\operatorname{spec} T$, then $f(T) = 0$.*

Proof. Existence. If one replaces C_0 by $\mathcal{A}$, everything is already proved. But $\mathcal{A}$ is dense in $C_0(\mathbb{R})$ in the sup-norm, because $C_c^\infty(\mathbb{R}) \subset \mathcal{A}$, so we can use the density argument.

Uniqueness. If we have two such maps, they coincide on the functions f which are linear combinations of r_w, $w \in \mathbb{C} \setminus \mathbb{R}$. But such functions are dense in C_0 by the Stone-Weierstrass theorem, so by the density argument both maps coincide on C_0. □

Remark 4.9. One may wonder why to introduce the class of functions $\mathcal{A}$: one could just start by C_c^∞ which is also dense in C_0. The reason in that we have no intuition on how the operator $f(T)$ should look like if $f \in C_c^\infty$. On the other hand, it is naturally expected that for $r_w(s) = (s-w)^{-1}$ we should have $r_w(T) = (T-w)^{-1}$, otherwise there are no reasons why we use the notation $r_w(T)$. So it is important to have an explicit formula for a sufficiently large class of functions containing all such r_w.

4.2 L^2 spectral representation

Now we would like to extend the functional calculus to more general functions, not necessarily continuous and not necessarily vanishing at infinity.

Definition 4.10 (Invariant and cyclic subspaces). Let $\mathcal{H}$ be a Hilbert space, L be a closed linear subspace of $\mathcal{H}$, and T be a self-adjoint linear operator in $\mathcal{H}$.

Let T be bounded. We say that L is an *invariant subspace* of T (or just T-invariant) if $T(L) \subset L$. We say that L is a *cyclic subspace* of T with cyclic vector v if L coincides with the closed linear hull of all vectors $p(T)v$, where p are polynomials.

Let T be general. We say that L is an *invariant subspace* of T (or just T-invariant) if $(T-z)^{-1}(L) \subset L$ for all $z \notin \mathbb{R}$. We say that L is a *cyclic subspace* of T with cyclic vector v if L coincides with the closed linear space of all vectors $(T-z)^{-1}v$ with $z \notin \mathbb{R}$. □

Clearly, if L is T-invariant, then $L^\perp$ is also T-invariant.

Proposition 4.11. *Both definitions of an invariant/cyclic subspace are equivalent for bounded self-adjoint operators.*

Proof. Let $T = T^* \in \mathcal{L}(\mathcal{H})$. We note first that $\operatorname{res} T$ is a connected set.

Let a closed subspace L be T-invariant in the sense of the definition for bounded operators. If $z \in \mathbb{C}$ and $|z| > \|T\|$, then $z \notin \operatorname{spec} T$ and

$$(T-z)^{-1} = -z\Big(1-\frac{T}{z}\Big)^{-1} = \sum_{n=0}^{\infty} z^{-n-1}T^n.$$

If $x \in L$, then $T^n x \in L$ for any n. As the series on the right hand side converges in the operator norm sense and as L is closed, $(T-z)^{-1}x$ belongs to L.

Let us denote $W = \big\{z \in \operatorname{res} T : (T-z)^{-1}(L) \subset L\big\}$. As just shown, W is non-empty. On the other hand, W is closed in $\operatorname{res} T$ in the relative topology: if $x \in L$, $z_n \in W$ and z_n converge to $z \in W$, then $(T-z_n)^{-1}x \in L$ and $(T-z_n)^{-1}x$ converge to $(T-z)^{-1}x$. On the other hand, W is open: if $z_0 \in W$ and $|z-z_0|$ is sufficiently small, then

$$(T-z)^{-1} = \sum_{n=0} (z-z_0)^n (T-z_0)^{-n-1},$$

see (3.4), and $(T-z)^{-1}L \subset L$. Therefore, $W = \operatorname{res} T$, which shows that L is T-invariant in the sense of the definition for general operators.

Now let $T = T^* \in \mathcal{L}(\mathcal{H})$, and assume that L is T-invariant in the sense of the definition for general operators, i.e. $(T-z)^{-1}(L) \subset L$ for any $z \notin \mathbb{R}$. Pick any $z \notin \mathbb{R}$ and any $f \in L$. We can represent $Tf = g + h$, where $g \in L$ and $h \in L^\perp$ are uniquely defined vectors. As $L^\perp$ is T-invariant, $(T-z)^{-1}h \subset L^\perp$. On the other hand

$$\begin{aligned}
(T-z)^{-1}h &= (T-z)^{-1}(Tf - g)\\
&= (T-z)^{-1}\big((T-z)f + zf - g\big)\\
&= f + (T-z)^{-1}(zf-g).
\end{aligned}$$

As $zf - g \in L$, both vectors on the right-hand side are in L. Therefore, $(T-z)^{-1}h \in L$, and finally $(T-z)^{-1}h = 0$ and $h = 0$, which shows that $Tf = g \in L$. The equivalence of the two definitions of an invariant subspace is proved.

On the other hand, for both definitions, L is T-cyclic with cyclic vector v iff L is the smallest T-invariant subspace containing v. Therefore, both definitions of a cyclic subspace also coincide for bounded self-adjoint operators. □

Theorem 4.12 (L^2 spectral representation, cyclic case). *Let T be a self-adjoint linear operator in $\mathcal{H}$ and let $S := \operatorname{spec} T$. Assume that $\mathcal{H}$ is a cyclic subspace for T with a cyclic vector v, then there exists a bounded measure μ on S with $\mu(S) \le \|v\|^2$ and a unitary map $U : \mathcal{H} \to L^2(S, d\mu)$ with the following properties:*

- *a vector $x \in \mathcal{H}$ is in $D(T)$ iff $hUx \in L^2(S, d\mu)$, where h is the function on S given by $h(s) = s$,*
- *for any $\psi \in U\big(D(T)\big)$ there holds $UTU^{-1}\psi = h\psi$.*

In other words, T is unitarily equivalent to the operator M_h of the multiplciation by h in $L^2(S, d\mu)$.

Proof. Step 1. Consider the map $\phi : C_0(\mathbb{R}) \to \mathbb{C}$ defined by $\phi(f) = \big\langle v, f(T)v \big\rangle$. Let us list the properties of this map:

- ϕ is linear,
- $\phi(\overline{f}) = \overline{\phi(f)}$,
- if $f \geq 0$, then $\phi(f) \geq 0$. This follows from

$$\phi(f) = \big\langle v, f(T)v \big\rangle = \big\langle v, \sqrt{f}(T)\sqrt{f}(T)v \big\rangle = \big\|\sqrt{f}(T)v\big\|^2.$$

- $\big|\phi(f)\big| \leq \|f\|_\infty \, \|v\|^2$.

By the Riesz representation theorem there exists a uniquely defined regular Borel measure μ such that

$$\phi(f) = \int_{\mathbb{R}} f d\mu \text{ for all } f \in C_0(\mathbb{R}).$$

Moreover, for $\operatorname{supp} f \cap S = \emptyset$ we have $f(T) = 0$ and $\phi(f) = 0$, which means that $\operatorname{supp} \mu \subset S$, and we can write the above as

$$\big\langle v, f(T)v \big\rangle = \int_S f d\mu \text{ for all } f \in C_0(\mathbb{R}). \tag{4.3}$$

Step 2. Consider the map $\Theta : C_0(\mathbb{R}) \to L^2(S, d\mu)$ defined by $\Theta f = f$. We have

$$\begin{aligned}\langle \Theta f, \Theta g \rangle &= \int_S \overline{f} g \, d\mu = \phi(\overline{f} g)\\ &= \big\langle v, f(T)^* g(T) v \big\rangle = \big\langle f(T)v, g(T)v \big\rangle.\end{aligned}$$

Denote $\mathcal{M} := \big\{ f(T)v : \; f \in C_0(\mathbb{R}) \big\} \subset \mathcal{H}$, then the preceding equality means that the map

$$U : \mathcal{H} \supset \mathcal{M} \to C_0(\mathbb{R}) \subset L^2(S, d\mu), \quad U\big(f(T)v\big) = f,$$

is one-to-one and isometric. Moreover, $\mathcal{M}$ is dense in $\mathcal{H}$, because v is a cyclic vector. Furthermore, $C_0(\mathbb{R})$ is a dense subspace of $L^2(S, d\mu)$, as μ is regular. Therefore, U is uniquely extended to a unitary map from $\mathcal{H}$ to $L^2(S, d\mu)$, and we denote this extension by the same symbol.

Step 3. Let $f, f_j \in C_0(\mathbb{R})$ and $\psi_j := f_j(T)v$, $j = 1, 2$. There holds

$$\big\langle \psi_1, f(T)\psi_2 \big\rangle = \big\langle f_1(T)v, f(T) f_2(T) v \big\rangle$$

$$
\begin{aligned}
&= \big\langle v, (\overline{f}_1 f f_2)(T) v\big\rangle \\
&= \int_S f \overline{f}_1 f_2 \, d\mu \\
&= \langle U\psi_1, M_f U\psi_2\rangle,
\end{aligned}
$$

where M_f is the operator of the multiplication by f in $L^2(S, d\mu)$. In particular, for any $w \notin \mathbb{R}$ and $r_w(s) = (s-w)^{-1}$ we obtain $Ur_w(T)U^*\xi = r_w\xi$ for all $\xi \in L^2(S, d\mu)$. The operator U maps the set $\operatorname{ran} r_w(T) \equiv D(T)$ to the range of M_{r_w}. In other words, U is a bijection from $D(T)$ to

$$
\operatorname{ran} M_{r_w} = \big\{\phi \in L^2(S, d\mu) : x \mapsto x\phi(x) \in L^2(S, d\mu)\big\} = D(M_h).
$$

Therefore, if $\xi \in L^2(S, d\mu)$, then $\psi := r_w\xi \in D(M_h)$,

$$
Tr_w(T)U^*\xi = (T-w)r_w(T)U^*\xi + wr_w(T)U^*\xi = U^*\xi + wr_w(T)U^*\xi
$$

and, finally,

$$
\begin{aligned}
UTU^*\psi = UTU^*r_w\xi = UTr_w(T)U^*\xi = U\big(U^*\xi + wr_w(T)U^*\xi\big) \\
= \xi + wr_w\xi = h\psi. \quad \square
\end{aligned}
$$

Theorem 4.13 (L^2 spectral representation). *Let T be a self-adjoint operator in a Hilbert space $\mathcal{H}$ with* $\operatorname{spec} T =: S$. *Then there exists $N \subset \mathbb{N}$, a finite measure μ on $S \times N$ and a unitary operator $U : \mathcal{H} \to L^2(S \times N, d\mu)$ with the following properties.*

- *Let $h : S \times N \to \mathbb{R}$ be given by $h(s, n) = s$. A vector $x \in \mathcal{H}$ belongs to $D(T)$ iff $hUx \in L^2(S \times N, d\mu)$,*
- *for any $\psi \in U\big(D(T)\big)$ there holds $UTU^{-1}\psi = h\psi$.*

Proof. Using the induction one can find $N \subset \mathbb{N}$ and non-empty closed mutually orthogonal subspaces $\mathcal{H}_n \subset \mathcal{H}$ with the following properties:

- $\mathcal{H} = \bigoplus_{n\in N} \mathcal{H}_n$,
- each $\mathcal{H}_n$ is a cyclic subspace of T with cyclic vector v_n satisfying $\|v_n\| \leq 2^{-n}$.

The restriction T_n of T to $\mathcal{H}_n$ is a self-adjoint operator in $\mathcal{H}_n$, and one can apply to all these operators Theorem 4.12, which gives associated measures μ_n with $\mu(S) \leq 4^{-n}$, and unitary maps $U_n : \mathcal{H}_n \to L^2(S, d\mu_n)$. Now one can define a measure μ on $S \times N$ by $\mu\big(\Omega \times \{n\}\big) = \mu_n(\Omega)$, and a unitary map

$$
U : \mathcal{H} \equiv \bigoplus_{n\in N} \mathcal{H}_n \to L^2(S \times N, d\mu) \equiv \bigoplus_{n\in N} L^2(S, d\mu_n)
$$

by $U(\psi_n) = (U_n\psi_n)$, and one can easily check that all the properties are verified. $\square$

Remark 4.14. • The previous theorem shows that any self-adjoint operator is unitarily equivalent to a multiplication operator in some L^2 space, and this multiplication operator is sometimes called a *spectral representation* of T. Clearly, such a representation is not unique, for example, the decomposition of the Hilbert space in cyclic subspaces is not unique.

- The cardinality of the set N is not invariant. The minimal cardinality among all possible N is called the *spectral multiplicity* of T, and it generalizes the notion of the multiplicity for eigenvalues. Calculating the spectral multiplicity is a non-trivial problem. □

Theorem 4.13 can be used to improve the result of Theorem 4.8. In the rest of the section we use the function h and the measure μ from Theorem 4.13 without further specifications.

Introduce the set $\mathcal{B}_\infty$ consisting of the bounded Borel functions $f : \mathbb{R} \to \mathbb{C}$. In what follows, we say that $f_n \in \mathcal{B}_\infty$ converges to $f \in \mathcal{B}_\infty$ and write $f_n \xrightarrow{\mathcal{B}_\infty} f$ if the following two conditions hold:

- there exists $c > 0$ such that $\|f_n\|_\infty \leq c$,
- $f_n(x) \to f(x)$ for all x.

Definition 4.15 (Strong convergence). Wa say that a sequence $A_n \in \mathcal{L}(\mathcal{H})$ converges strongly to $A \in \mathcal{L}(\mathcal{H})$ and write $A = \mathrm{s} - \lim A_n$ if $Ax = \lim A_n x$ for any $x \in \mathcal{H}$. □

Theorem 4.16 (Borel functional calculus). *(a) Let T be a self-adjoint operator in a Hilbert space $\mathcal{H}$. There exists a map $\mathcal{B}_\infty \ni f \mapsto f(T) \in \mathcal{L}(\mathcal{H})$ extending the map from Theorem 4.8 and satisfying the same properties except that one can improve the estimate $\|f(T)\| \leq \|f\|_\infty$ by $\|f(T)\| \leq \|f\|_{\infty,T}$.*

(b) This extension is unique if we assume that the condition $f_n \xrightarrow{\mathcal{B}_\infty} f$ implies $f(T) = \mathrm{s} - \lim f_n(T)$.

Proof. Consider the map U from Theorem 4.8. Then it is sufficient to define $f(T) := U^* M_{f\circ h} U$, then one routinely check that all the properties hold, and (a) is proved. To prove (b) we remark first that the map just defined satisfies the requested condition: If $x \in L^2(S, d\mu)$ and $f_n \xrightarrow{\mathcal{B}_\infty} f$, then $f_n(h)x$ converges to $f(h)x$ in the norm of $L^2(S \times N, d\mu)$ by the dominated convergence. But this means exactly that $f(T) = \mathrm{s} - \lim f_n(T)$. On the other hand, $C_0(\mathbb{R})$ is obviously dense in $\mathcal{B}_\infty$ with respect to the $\mathcal{B}_\infty$ convergence, which proves the uniqueness of the extension. □

4.3 Some direct applications of the spectral theorem

We have a series of important corollaries, whose proof is an elementary modification of the constructions given for the multiplication operator in Example 3.6. We still use without special notification the measure μ and the function h from Theorem 4.13.

We will use without further

Corollary 4.17.

- $\operatorname{spec} T = \operatorname{ess}_\mu \operatorname{ran} h$,
- *for any* $f \in \mathcal{B}_\infty$ *one has* $\operatorname{spec} f(T) = \operatorname{ess}_\mu \operatorname{ran} f \circ h$,
- *in particular,* $\|f(T)\| = \operatorname{ess}_\mu \sup |f \circ h|$.

Example 4.18. One can also define the operators $\varphi(T)$ with unbounded functions φ by $\varphi(T) = U^* M_{\varphi\circ h} U$. These operators are in general unbounded, but they are self-adjoint for real-valued φ; this follows from the self-adjointness of the multiplication operators $M_{\varphi\circ h}$.

Example 4.19. The usual Fourier transform is a classical example of a spectral representation. For example, Take $\mathcal{H} = L^2(\mathbb{R})$ and $T = -id/dx$ with the natural domain $D(T) = H^1(\mathbb{R})$. If $\mathcal{F}$ is the Fourier transform, then $\mathcal{F}T\mathcal{F}$ is exactly the operator of multiplication $x \mapsto xf(x)$, and $\operatorname{spec} T = \mathbb{R}$.

In particular, for bounded Borel functions $f : \mathbb{R} \to \mathbb{C}$ one can define the operators $f(T)$ by $f(T)h = \mathcal{F}^* M_f \mathcal{F}$, where M_f is the operator of multiplication by f, i.e. in general one obtains a pseudodifferential operator.

Let us look at some particular examples. Consider the shift operator A in $\mathcal{H}$ which is defined by $Af(x) = f(x+1)$. It is a bounded operator, and for any $u \in \mathcal{S}(\mathbb{R})$ we have $\mathcal{F}A\mathcal{F}^*u(p) = e^{ip}u(p)$. This means that $A = e^{iT}$, and this gives the relation $\operatorname{spec} A = \{z : |z| = 1\}$. On may also look at the operator B defined by

$$Bf(x) = \int_{x-1}^{x+1} f(t)dt.$$

Using the Fourier transform one can show that $B = \varphi(T)$, where $\varphi(x) = 2\sin x/x$ with $\operatorname{spec} B = \overline{\varphi(\mathbb{R})}$.

Example 4.20. For practical computations one does not need to have the canonical representation from Theorem 4.13 to construct the Borel functional calculus. It is sufficient to represent $T = U^* M_f U$, where $U : \mathcal{H} \to L^2(X, d\mu)$ and M_f is the multiplcation operator by some function f. Then for any Borel function φ one can put $\varphi(T) = U^* M_{\varphi\circ f} U$.

For example, for the free Laplacian T in $\mathcal{H} = L^2(\mathbb{R}^d)$ the above is realized with $X = \mathbb{R}^d$ and U being the Fourier transform, and with $f(p) = p^2$. This means that the operators $\varphi(T)$ act by

$$\varphi(T)f(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \varphi(p^2)\widehat{f}(p)e^{ipx}\,dx.$$

For example,

$$\sqrt{-\Delta+1}f(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \sqrt{1+p^2}\widehat{f}(p)e^{ipx}\,dx$$

and one can show that $D(\sqrt{-\Delta+1}) = H^1(\mathbb{R}^d)$.

Theorem 4.21 (Distance to spectrum). *Let T be a self-adjoint operator in a Hilbert space $\mathcal{H}$, and $0 \neq x \in D(T)$, then for any $\lambda \in \mathbb{C}$ one has the estimate*

$$\operatorname{dist}(\lambda, \operatorname{spec} T) \leq \frac{\big\|(T-\lambda)x\big\|}{\|x\|}.$$

Proof. If $\lambda \in \operatorname{spec} T$, then the left-hand side is zero, and the inequality is valid. Assume now that $\lambda \notin \operatorname{spec} T$. By Corollary 4.17, one has, with $\rho(x) = (x-\lambda)^{-1}$,

$$\|(T-\lambda)^{-1}\| = \operatorname{ess}_\mu \sup |\rho \circ h| = \frac{1}{\operatorname{dist}(\lambda, \operatorname{spec} T)},$$

which gives

$$\|x\| = \|(T-\lambda)^{-1}(T-\lambda)x\| \leq \frac{1}{\operatorname{dist}(\lambda, \operatorname{spec} T)} \|(T-\lambda)x\|. \qquad \square$$

Remark 4.22. The previous theorem is one of the basic tools for the constructing approximations of the spectrum of the self-adjoint operators. It is important to understand that the resolvent estimate obtained in Theorem 4.21 uses in an essential way the self-adjointness of the operator T. For non-self-adjoint operators the estimate fails even in the finite-dimensional case. For example, take $\mathcal{H} = \mathbb{C}^2$ and

$$T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

then $\operatorname{spec} T = \{0\}$, and for $z \neq 0$ we have

$$(T-z)^{-1} = -\frac{1}{z^2}\begin{pmatrix} z & 1 \\ 0 & z \end{pmatrix}.$$

For the vectors $e_1 = (1,0)$ and $e_2 = (0,1)$ one has $\langle e_1, (T-z)^{-1}e_2\rangle = -z^{-2}$, which shows that the norm of the resolvent near $z = 0$ is of order z^{-2}. In the infinite dimensional-case one can construct examples with $\|(T-z)^{-1}\| \sim \operatorname{dist}(z, \operatorname{spec} T)^{-n}$ for any power n. $\square$

4.4 Spectral projections

Definition 4.23 (Spectral projection). Let T be a self-adjoint operator in a Hilbert space $\mathcal{H}$ and $\Omega \subset \mathbb{R}$ be a Borel subset. The *spectral projection* of T on Ω is the operator $E_T(\Omega) := 1_\Omega(T)$, where 1_Ω is the characteristic function of Ω.

This exchange between the index and the argument is due to the fact that the spectral projections are usually considered as functions of subsets Ω (with a fixed operator T). $\square$

The following proposition summarizes the most important properties of the spectral projections.

Proposition 4.24. *For any self-adjoint operator T acting a in a Hilbert space there holds:*

1. *for any Borel subset $\Omega \subset \mathbb{R}$ the associated spectral projection $E_T(\Omega)$ is an orthogonal projection commuting with T. In particular, $E_T(\Omega)D(T) \subset D(T)$.*
2. *$E_T\big((a,b)\big) = 0$ if and only if $\operatorname{spec} T \cap (a,b) = \emptyset$.*
3. *for any $\lambda \in \mathbb{R}$ there holds $\operatorname{ran} E_T(\{\lambda\}) = \ker(T-\lambda)$, and $f \in \ker(T-\lambda)$ iff $f = E_T(\{\lambda\})f$.*
4. *$\operatorname{spec} T = \{\lambda \in \mathbb{R} : E_T\big((\lambda-\varepsilon, \lambda+\varepsilon)\big) \neq 0$ for all $\varepsilon > 0\}$.*

Proof. To prove (1) we remark that $1_\Omega^2 = 1_\Omega$ and $1_\Omega = \overline{1_\Omega}$, which gives $E_T(\Omega)E_T(\Omega) = E_T(\Omega)$ and $E_T(\Omega) = E_T(\Omega)^*$ and shows that $E_T(\Omega)$ is an orthogonal projection. To prove the commuting with T we restrict ourselves by considering T realized as a multiplication operator from Theorem 4.8. Let $x \in D(T)$, then $hx \in L^2(S, \times N, \mu)$ and, subsequently, $h \cdot 1_\Omega \circ h \cdot x \in L^2$, which means that $1_\Omega x \in D(T)$. The commuting follows now from $h \cdot 1_\Omega \circ h \cdot x = 1_\Omega \circ h \cdot h \cdot x$.

To prove (2) we note that the condition $E_T\big((a,b)\big) = 0$ is, by definition, equivalent to $1_{(a,b)} \circ h = 0$ μ-e.a., which in turn means that $(a,b) \cap \operatorname{ess}_\mu \operatorname{ran} h = \emptyset$, and it remains to recall that $\operatorname{ess}_\mu \operatorname{ran} h = \operatorname{spec} T$, see Corollary 4.17.

The items (3) and (4) are left as elementary exercises. $\square$

As an important corollary of the assertion (4) one has the following description of the spectra of self-adjoint operators, whose proof is another simple exercise.

Corollary 4.25. *Let T be self-adjoint, then $\lambda \in \operatorname{spec} T$ if and only if there exists a sequence $x_n \in D(T)$ with $\|x_n\| \geq 1$ such that $(T-\lambda)x_n$ converge to 0.*

One can see from Proposition 4.24 that the spectral projections contains a lot of useful information about the spectrum. Therefore, it is a good idea to understand how to calculate them at least for simple sets Ω. As a simple example we consider the computation of the spectral projector to a point (which allows one to test if the point is an eigenvalue).

Proposition 4.26 (Spectral projection to a point). *For any $\lambda \in \mathbb{R}$ there holds*

$$E_T(\{\lambda\}) = -i \operatorname{s-}\lim_{\varepsilon \to 0+} \varepsilon (T - \lambda - i\varepsilon)^{-1}.$$

Proof. For $\varepsilon > 0$ consider the function

$$f_\varepsilon(x) := -\frac{i\varepsilon}{x - \lambda - i\varepsilon}.$$

One has the following properties:

- $|f_\varepsilon| \le 1$,
- $f_\varepsilon(\lambda) = 1$,
- if $x \neq \lambda$, then $f_\varepsilon(x)$ tends to 0 as ε tends to 0.

This means that $f_\varepsilon \xrightarrow{\mathcal{B}_\infty} 1_{\{\lambda\}}$. By Theorem 4.16, $E_T(\{\lambda\}) = \mathrm{s}-\lim_{\varepsilon\to 0+} f_\varepsilon(T)$, and it remains to note that $f_\varepsilon(T) = (T-\lambda-i\varepsilon)^{-1}$ by Theorem 4.8. □

As a final remark we mention that the map $\Omega \mapsto E_T(\Omega)$ can be viewed an operator-valued measure, and one can integrate reasonable scalar function (bounded Borel ones or even unbounded) with respect to this measure using e.g. the Lebesgue integral sums. Then one obtains the integral representations,

$$T = \int_{\mathbb{R}} \lambda dE_T(\lambda), \quad f(T) = \int_{\mathbb{R}} f(\lambda) dE_T(\lambda),$$

and the associated integral sums can be viewed as certain approximations of the respective operators.

4.5 Tensor products (and operators with separated variables)

We will briefly[4] review the construction of tensor products of Hilbert spaces. Let $\mathcal{H}$ and $\mathcal{H}'$ be two Hilbert spaces. Our goal is to construct their tensor product $\mathcal{H}\otimes\mathcal{H}'$, which should be interpreted as the Hilbert spaces of "products" of the vectors from $\mathcal{H}$ and $\mathcal{H}'$. We consider first the set of all (formal) finite linear combinations of the elements of $\mathcal{H}\times\mathcal{H}'$,

$$\mathcal{A} = \Big\{ \sum_{j=1}^n \alpha_j(\phi_j, \phi'_j) : \quad (\phi, \phi'_j) \in \mathcal{H}\times\mathcal{H}', \quad \alpha_j \in \mathbb{C} \Big\}.$$

and then its subset

$$\mathcal{A}_0 = \Big\{ \sum_{j,k=1}^n \alpha_j\beta_k(\phi_j, \phi'_k) - \Big(\sum_{j=1}^n \alpha_j\psi_j, \sum_{k=1}^n \beta_k\psi'_k \Big) : \quad \phi_j \in \mathcal{H},\ \phi'_k \in \mathcal{H}', \quad \alpha_j, \beta_k \in \mathbb{C} \Big\}.$$

We now consider the quotient $\mathcal{L} := \mathcal{A}/\mathcal{A}_0$, write $\varphi\otimes\varphi'$ for the equivalence class of (φ, φ'), and define

$$\big\langle \phi\otimes\phi', \psi\otimes\psi' \big\rangle = \langle \phi, \psi\rangle_{\mathcal{H}} \langle \phi', \psi'\rangle_{\mathcal{H}'},$$

which then extends by sesquilinearity to the whole of $\mathcal{L}$. It can be shown that this defines a scalar product on $\mathcal{L}$. Be definition, the tensor product $\mathcal{H}\otimes\mathcal{H}'$ is the completion of $\mathcal{L}$ with respect to this scalar product. By direct computations one shows the following result, which simplifies many manipulations with tensor products.

[4]Most constructions are just sketched here. We refer to G. Teschl: *Mathematical methods in quantum mechanics. With applications to Schrödinger operators.* AMS, 2009, §§1.4 and 4.5

Theorem 4.27. *If* (ϕ_j) *and* (ϕ'_j) *are orthonormal bases in* $\mathcal{H}$ *and* $\mathcal{H}'$ *respectively, then* $(\phi_j \otimes \phi'_k)$ *is an orthonormal basis in* $\mathcal{H} \otimes \mathcal{H}'$.

Theorem 4.28. *If* (M, μ) *and* $(M', d\mu')$ *are measure spaces, then*

$$L^2(M, \mu) \otimes L^2(M', \mu') = L^2(M \times M', \mu \times \mu').$$

Proof. The result becomes almost obvious, if one identifies $\phi \otimes \phi'$ with the function $(x, x') \mapsto \phi(x)\phi(x')$. First, this shows immediately that $L^2(M, \mu) \otimes L^2(M', \mu') \subset L^2(M \times M', \mu \times \mu')$. To show the equality, let (ϕ_j) and (ϕ'_j) be orthonormal bases in $L^2(M, \mu)$ and $L^2(M', \mu')$ respectively and $f \in L^2(M \times M', \mu \times \mu')$ be orthogonal to all $\phi_j \otimes \phi'_k$. We just need to show that $f = 0$. The orthogonality condition means that

$$\int_M \int_{M'} \overline{\phi_j(x)}\, \overline{\phi'_k(x')} f(x, x') d\mu(x') d\mu(x) = 0 \text{ for all } j, k,$$

in particular,

$$\int_M \overline{\phi_j(x)} f_k(x) d\mu(x) \text{ for all } j, k, \quad f_k(x) := \int_{M'} \overline{\phi'_k(x')} f(x, x') d\mu(x').$$

IT follows that for each k one has $f_k(x) = 0$ for μ-a.e. x, and then $f(x, x') = 0$ for μ-a.e. x and μ'-a.e. x', hence $f = 0$. $\square$

Let A_j be self-adjoint operators in Hilbert spaces $\mathcal{H}_j$, $j = 1, \dots, n$. With any monomial $\lambda_1^{m_1} \cdot \ldots \lambda_n^{m_n}$, $m_j \in \mathbb{N}$, one can associate the operator $A_1^{m_1} \otimes \cdot A_n^{m_n}$ in $\mathcal{H} := \mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_n$ defined by

$$(A_1^{m_1} \otimes \dots \otimes A_n^{m_n})(\psi_1 \otimes \dots \otimes \psi_n) = A_1^{m_1}\psi_1 \otimes \dots \otimes A_n^{m_n}\psi_n, \quad \psi_j \in D(A_j^{m_j}),$$

and then extended by linearity; here the zero powers A_j^0 equal the identity operators in the respective spaces.

Remark 4.29. For an operator A in a Hilbert space $\mathcal{H}$ the domain $D(A^n)$ is usually defined in a recursive way:

$$D(A^0) = \mathcal{H} \text{ and } D(A^n) = \big\{x \in D(A) :\ Ax \in D(A^{n-1})\big\} \text{ for } n \in \mathbb{N}.$$

As an exercise one can show that for a self-adjoint A one has $D(A^n) = \operatorname{ran} R_A(z)^n$ with any $z \in \operatorname{res} A$ and that $D(A^n)$ is dense in $\mathcal{H}$ for any n. $\square$

Using the above construction one can associate with any real-valued polynomial P of $\lambda_1, \dots, \lambda_n$ of degree N a linear operator $P(A_1, \dots, A_n)$ in $\mathcal{H}$ defined on the set $\mathcal{H}$ consisting of the linear combinations of the vectors of the form $\psi_1 \otimes \dots \otimes \psi_n$ with $\psi_j \in D(A_j^N)$.

Theorem 4.30 (Spectrum of tensor product). *Denote by* B *the closure of the above operator* $P(A_1, \dots, A_n)$*, then* B *is self-adjoint , and*

$$\operatorname{spec} B = \overline{\big\{P(\lambda_1, \dots, \lambda_n) :\ \lambda_j \in \operatorname{spec} A_j\big\}}.$$

One denotes $B := P(A_1, \dots, A_n)$.

Sketch of the proof. The complete proof involves a number of technicalities, but the main idea is rather simple. By the spectral theorem, it is sufficient to consider the case when A_j is the multiplication by a certain function f_j in $\mathcal{H}_j := L^2(M_j, d\mu_j)$. Then

$$\mathcal{H} = L^2(M, d\mu), \quad M = M_1 \times \cdots \times M_n, \quad \mu = \mu_1 \otimes \cdots \otimes \mu_n,$$

and $P(A_1, \dots, A_n)$ acts in $\mathcal{H}$ as the multiplication by p, $p(x_1, \dots, x_n) = P\big(f_1(x_1), \dots, f_n(x_n)\big)$, and its domain includes at least all the linear combinations of the functions $\psi_1 \otimes \cdots \otimes \psi_n$ where ψ_j are L^2 with compact supports. It is a routine to show that the closure of this operator is just the usual multiplication operator by p, which gives the sought relation. □

Example 4.31 (Laplacian in rectangle). A typical example of the above construction is given by the Laplacians in rectangles. Namely, let $a, b > 0$ and $\Omega = (0, a) \times (0, b) \subset \mathbb{R}^2$, $\mathcal{H} = L^2(\Omega)$, and T be the Dirichlet Laplacian in Ω. One can show that T can be obtained using the above procedure using the representation

$$T = L_a \otimes 1 + 1 \otimes L_b = P(L_a, L_b),$$

where by L_a we denote the Dirichlet Laplacian in $\mathcal{H}_a := L^2(0, a)$, i.e.

$$L_a f = -f'', \quad D(L_a) = H^2(0, a) \cap H_0^1(0, a),$$

and $P(\lambda_1, \lambda_2) = \lambda_1 + \lambda_2$. It is known (from the exercises) that the spectrum of L_a consists of the simple eigenvalues $(\pi n/a)^2$, $n \in \mathbb{N}$, with the eigenfunctions $x \mapsto \sin(\pi n x/a)$, and this means that the spectrum of T consists of the eigenvalues

$$\lambda_{m,n}(a, b) = \Big(\frac{\pi m}{a}\Big)^2 + \Big(\frac{\pi n}{b}\Big)^2, \quad m, n \in \mathbb{N},$$

and the associated eigenfunctions are the products of the respective eigenfunctions for L_a and L_b. The multiplicity of each eigenvalue λ is exactly the number of pairs $(m, n) \in \mathbb{N}^2$ for which $\lambda = \lambda_{m,n}$. Note that the closure of the set $\{\lambda_{m,n}\}$ can be omitted as this is a discrete set without accumulation points.

The same constructions hold for the Neumann Laplacians, one obtains the same formula for the eigenvalues but now with $m, n \in \mathbb{N} \cup \{0\}$. □

Exercise 13. Let T be a self-adjoint operator in a Hilber space $\mathcal{H}$.

1. Let $\lambda \in \mathbb{R}$. Show: $\lambda \in \operatorname{spec} T$ if and only if there exists a sequence $(x_n) \subset D(T)$ with $x_n \neq 0$ such that
$$\lim_{n\to+\infty} \frac{\|(T-\lambda)x_n\|}{\|x_n\|} = 0.$$
Hint: One may construct x_n explicitly using the spectral projections on intervals $(\lambda - \varepsilon_n, \lambda + \varepsilon_n)$ for suitable ε_n.

2. For $n \in \mathbb{N}$, $n \geq 2$, define $D_n(T) = \big\{x \in D(T) : Tx \in D_{n-1}(T)\big\}$, where we set $D_1(T) := D(T)$. Let T_n be the restriction of T to $D_n(T)$. Show that $D_n(T)$ is dense in $\mathcal{H}$ for any n and that $\overline{T_n} = T$.

3. Let λ be an isolated eigenvalue of T; this means that for some $\varepsilon > 0$ there holds $\operatorname{spec} T \cap (\lambda - \varepsilon, \lambda + \varepsilon) = \{\lambda\}$. Show: there exists a constant $c > 0$ such that $\|(T-\lambda)u\| \geq c\|u\|$ for all $u \in D(T) \cap \ker(T-\lambda)^\perp$.

Exercise 14. Let T be a self-adjoint operator in a Hilbert space $\mathcal{H}$. Given $t \in \mathbb{R}$ we define e^{-iTt} to be $f_t(T)$ with $f_t(s) = e^{-ist}$ for all $s \in \mathbb{R}$.

1. Prove that each operator e^{-iTt} is unitary.

2. Prove that $e^{-iT(s+t)} = e^{-iTs}e^{-iTt}$ for all $s, t \in \mathbb{R}$.

3. Show that $\lim_{s\to t} e^{-iTs}f = e^{-iTt}f$ for any $t \in \mathbb{R}$ and $f \in \mathcal{H}$.

4. Show that $e^{-iTt}D(T) \subset D(T)$.

5. Let $f \in D(T)$. Define $F : \mathbb{R} \to \mathcal{H}$ by $F(t) = e^{-iTt}f$. Show that $F \in C^1(\mathbb{R}, \mathcal{H})$ and that it solves the differential equation $iF'(t) = TF(t)$.

Exercise 15. Let $v, w \in L^\infty(\mathbb{R})$ be real-valued. Consider the self-adjoint operators $A = -d^2/dx^2 + v$ and $B = -d^2/dx^2 + w$ in $L^2(\mathbb{R})$, both defined using Friedrichs extension.

Furthermore, consider the potential $V : \mathbb{R}^2 \ni (x, y) \mapsto v(x) + w(y) \in \mathbb{R}$ and the operator $C = -\Delta + V$ in $L^2(\mathbb{R}^2)$ defined using Friedrichs extension.

Show that $C = A \otimes 1 + 1 \otimes B$.

5 Perturbations

5.1 Kato-Rellich theorem

We have seen since the beginning of the course that one needs to pay a great attention to the domains when dealing with unbounded operators. The aim of the present subsection is to describe some classes of operators in which such problems can be avoided.

Definition 5.1 (Essentially self-adjoint operator). We say that a linear operator T is essentially self-adjoint on a subspace $\mathcal{D} \subset D(T)$ if the closure of the restriction of T to $\mathcal{D}$ is a self-adjoint operator. If the closure of T is self-adjoint, then we simply say that T is essentially self-adjoint. □

Proposition 5.2. *An essentially self-adjoint operator has a unique self-adjoint extension.*

Proof. Let T be an essentially self-adjoint operator, and let S be a self-adjoint extension of T. As S is closed, the inclusion $T \subset S$ implies $\overline{T} \subset S$. On the other hand, $S = S^* \subset (\overline{T})^* = \overline{T}$ (as $\overline{T}$ is self-adjoint). This shows that $S = \overline{T}$. □

Theorem 5.3 (Self-adjointness criterion). *Let T be a closed symmetric operator in a Hilbert space $\mathcal{H}$, then the following three assertions are equivalent:*

1. *T is self-adjoint,*
2. $\ker(T^* + i) = \ker(T^* - i) = \{0\}$,
3. $\operatorname{ran}(T + i) = \operatorname{ran}(T - i) = \mathcal{H}$.

Proof. The implication $1 \Rightarrow 2$ is obvious: a self-adjoint operator cannot have non-real eigenvalues.

To show the implication $2 \Rightarrow 3$ recall first that $\ker(T^* \pm i) = \operatorname{ran}(T \mp i)^\perp$. Therefore, it is sufficient to show that the subspaces $\operatorname{ran}(T \pm i)$ are closed. For any $f \in D(T)$ we have:

$$\begin{aligned}\big\|(T \pm i)f\big\|^2 = \big\langle (T \pm i)f, (T \pm i)f\big\rangle = \langle Tf, Tf\rangle + \langle f, f\rangle \pm i\big(\langle Tf, f\rangle - \langle f, Tf\rangle\big)\\ = \|Tf\|^2 + \|f\|^2.\end{aligned}$$

Let $f_n \in \operatorname{ran}(T \pm i)$ such that f_n converge to some $f \in \mathcal{H}$. Find $\varphi_n \in D(T)$ with $f_n = (T \pm i)\varphi_n$, then due to the preceding equality (φ_n) and $(T\varphi_n)$ are Cauchy sequences. As T is closed, φ_n converge to some $\varphi \in D(T)$ and $T\varphi_n$ converge to $T\varphi$, and then $f_n = (T \pm i)\varphi_n$ converge to $(T \pm i)\varphi = f$ and $f \in \operatorname{ran}(T \pm i)$.

It remains to the prove the implication $3 \Rightarrow 1$. Let $\varphi \in D(T^*)$. Due to the surjectivity of $T - i$ one can find $\psi \in D(T)$ with $(T - i)\psi = (T^* - i)\varphi$. As $T \subset T^*$, we have $(T^* - i)(\psi - \varphi) = 0$. On the other hand, due to $\operatorname{ran}(T + i) = \mathcal{H}$ we have $\ker(T^* - i) = 0$, which means that $\varphi = \psi \in D(T)$. □

Note that during the proof we obtained the following simple fact:

Proposition 5.4. *Let T be a symmetric operator, then* $\overline{\operatorname{ran}(T \pm i)} = \operatorname{ran}(\overline{T} \pm i)$.

This leads as to the following assertion:

Corollary 5.5 (Essential self-adjointness criterion). *Let T be a symmetric operator in a Hilbert space $\mathcal{H}$, then the following three assertions are equivalent:*

1. *T is essentially self-adjoint,*
2. $\ker(T^* + i) = \ker(T^* - i) = \{0\}$,
3. $\operatorname{ran}(T + i)$ *and* $\operatorname{ran}(T - i)$ *are dense in* $\mathcal{H}$.

Remark 5.6. The above theorem can be modified in several ways. For example, it still holds if one replaces $T \pm i$ by $T \pm i\lambda$ with any $\lambda \in \mathbb{R} \setminus \{0\}$. For semibounded operators we have an alternative version:

Theorem 5.7 (Self-adjointness criterion for semibounded operators). *Let T be a closed symmetric operator in a Hilbert space $\mathcal{H}$ and $T \geq 0$ and let $a > 0$, then the following three assertions are equivalent.*

1. *T is self-adjoint,*
2. $\ker(T^* + a) = \{0\}$,
3. $\operatorname{ran}(T + a) = \mathcal{H}$.

This is left as an exercise. The analogues of Proposition 5.4 and Corollary 5.5 hold as well. □

Now we would like to apply the above assertions to the study of some perturbations of self-adjoint operators.

Definition 5.8 (Relative boundedness). Let A be a self-adjoint operator in a Hilbert space $\mathcal{H}$ and B be a linear operator with $D(A) \subset D(B)$. Assume that there exist $a, b > 0$ such that

$$\|Bf\| \leq a\|Af\| + b\|f\| \quad \text{for all } f \in D(A),$$

then B is called *relatively bounded* with respect to A or, for short, *A-bounded*. The infimum of all possible values a is called the *relative bound* of B with respect to A. If the relative bound is equal to 0, then B is called *infinitesimally small* with respect to A. □

Theorem 5.9 (Kato-Rellich). *Let A be a self-adjoint operator in $\mathcal{H}$ and let B be a symmetric operator in $\mathcal{H}$ which is A-bounded with a relative bound < 1, then the operator $A + B$ with the domain $D(A + B) = D(A)$ is self-adjoint. Moreover, if A is essentially self-adjoint on some $\mathcal{D} \subset D(A)$, then $A + B$ is essentially self-adjoint on $\mathcal{D}$ too.*

Proof. By assumption, one can find $a \in (0,1)$ and $b > 0$ such that

$$\|Bu\| \le a\|Au\| + b\|u\|, \quad \text{for all} \quad u \in D(A). \tag{5.1}$$

Step 1. As seen many times, for any $\lambda > 0$ one has

$$\big\|(A + B \pm i\lambda)u\big\|^2 = \big\|(A+B)u\big\|^2 + \lambda^2\|u\|^2.$$

Therefore, for all $u \in D(A)$ one can estimate

$$\begin{aligned}2\big\|(A + B \pm i\lambda)u\big\| \ge \big\|(A+B)u\big\| + \lambda\|u\| \ge \|Au\| - \|Bu\| + \lambda\|u\| \\ = (1-a)\|Au\| + (\lambda - b)\|u\|.\end{aligned} \tag{5.2}$$

Let us pick some $\lambda > b$.

Step 2. Let us show that $A+B$ with the domain equal to $D(A)$ is a closed operator. Let $(u_n) \subset D(A)$ and $f_n := (A+B)u_n$ such that both u_n and f_n converge in $\mathcal{H}$. By (5.2), Au_n is a Cauchy sequence. As A is closed, u_n converge to som $u \in D(A)$ and Au_n converge to Au. By (5.1), Bu_n is a Cauchy sequence and is hence convergent to some $v \in \mathcal{H}$. Let us show that Bu_n converge exactly to Bu (actually for closed B this would be automatic, but we did not assume that B is closed!). Take any $h \in D(A)$, then $\langle v, h\rangle = \lim\langle Bu_n, h\rangle = \lim\langle u_n, Bh\rangle = \langle u, Bh\rangle = \langle Bu, h\rangle$. As $D(A)$ is dense, it follows that $v = Bu$. So finally $(A+B)u_n$ converge to $(A+B)u$. This shows that $A+B$ is closed.

Step 3. Let us show that the operators $A + B \pm i\lambda : D(A) \to \mathcal{H}$ are bijective at least for large λ. As previously, we have $\|(A \pm i\lambda)u\|^2 = \|Au\|^2 + \lambda^2\|u\|^2$. Then

$$\|Bu\| \le a\|Au\| + b\|u\| \le a\big\|(A \pm i\lambda)u\big\| + \frac{b}{|\lambda|}\big\|(A \pm i\lambda)u\big\| = \big(a + \frac{b}{|\lambda|}\big)\big\|(A \pm i\lambda)u\big\|.$$

As $a \in (0,1)$, we can choose λ sufficiently large to have $a + b/|\lambda| < 1$. This means that for such λ we have $\big\|B(A \pm i\lambda)^{-1}\big\| < 1$. Now we can represent

$$A + B \pm i\lambda = \Big(1 + B(A \pm i\lambda)^{-1}\Big)(A \pm i\lambda).$$

As A is self-adjoint, the operators $A \pm i\lambda : D(A) \to \mathcal{H}$ are bijections, and $1 + B(A \pm i\lambda)^{-1}$ is a bijection from $\mathcal{H}$ to itself. Therefore, $A+B\pm i\lambda$ are bijective, in particular, $\operatorname{ran}(A + B \pm i\lambda) = \mathcal{H}$. By Theorem 5.3 and Remark 5.6, $A + B$ is self-adjoint.

The part concerning the essential self-adjointness is reduced to the proof of the relation $\overline{A + B} = \overline{A} + B$, which is an elementary exercise. □

5.2 Essential self-adjointness of Schrödinger operators

The Kato-Rellich theorem is one of the tools used to simplify the consideration of the Schrödinger operators.

Theorem 5.10. *Let $V \in L^p(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ be real-valued with $p = 2$ if $d \le 3$ and $p > d/2$ if $d > 3$, then the operator $T = -\Delta + V$ with the domain $D(T) = H^2(\mathbb{R}^d)$ is a self-adjoint operator in $L^2(\mathbb{R}^d)$, and it is essentially self-adjoint on $C_c^\infty(\mathbb{R}^d)$.*

Proof. We give the proof only for the dimension $d \le 3$. For all $f \in C_c^\infty(\mathbb{R}^d)$ and $\lambda > 0$ we have the representation

$$\begin{aligned} f(x) &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ipx} \widehat{f}(p)\, dp \\ &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \frac{1}{p^2+\lambda} (p^2+\lambda) \widehat{f}(p) dp \\ &\le \frac{1}{(2\pi)^{d/2}} \Big\| \frac{1}{p^2+\lambda} \Big\| \cdot \Big\| (p^2+\lambda)\widehat{f}(p) \Big\| \\ &\le \frac{1}{(2\pi)^{d/2}} \Big\| \frac{1}{p^2+\lambda} \Big\| \cdot \Big(\|p^2 \widehat{f}(p)\| + \lambda \|\widehat{f}\| \Big) = a_\lambda \|\Delta f\| + b_\lambda \|f\| \end{aligned}$$

with

$$a_\lambda = \frac{1}{(2\pi)^{d/2}} \Big\| \frac{1}{p^2+\lambda} \Big\|, \quad b_\lambda = \frac{\lambda}{(2\pi)^{d/2}} \Big\| \frac{1}{p^2+\lambda} \Big\|.$$

By density, for all $f \in H^2(\mathbb{R}^d)$ and all $\lambda > 0$ we have

$$\|f\|_\infty \le a_\lambda \|\Delta f\| + b_\lambda \|f\|.$$

By assumption we can represent $V = V_1 + V_2$ with $V_1 \in L^2(\mathbb{R}^d)$ and $V_2 \in L^\infty(\mathbb{R}^d)$. Using the preceding estimate we arrive at

$$\|Vf\| \le \|V_1 f\| + \|V_2 f\| \le \|V_1\|_2 \|f\|_\infty + \|V_2\| \|f\| \le \widetilde{a}_\lambda \|\Delta f\| + \widetilde{b}_\lambda \|f\|, \quad f \in H^2(\mathbb{R}^d),$$

with $\widetilde{a}_\lambda = \|V_1\|_2 a_\lambda$ and $\widetilde{b}_\lambda = \|V_1\|_2 b_\lambda + \|V_2\|_\infty$. As a_λ can be made arbitrary small by a suitable choice of λ, we see that the multiplication operator V is infinitesimally small with respect to the free Laplacian, and the result follows from the Kato-Rellich theorem.

The above proof does not work for $d \ge 3$ as the function $p \mapsto (p^2 + \lambda)^{-1}$ does not belong to $L^2(\mathbb{R}^d)$ anymore. The respective parts of argument should be replaced by suitable Sobolev embedding theorems. □

Example 5.11 (Coulomb potential). Consider the three-dimensional case and the potential $V(x) = \alpha/|x|$, $\alpha \in \mathbb{R}$. For any bounded open set Ω containing the origin, one has $1_\Omega V \in L^2(\mathbb{R}^3)$ and $(1 - 1_\Omega)V \in L^\infty(\mathbb{R}^3)$, and finally $V \in L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$. This means that the operator $T = -\Delta + \alpha/|x|$ is self-adjoint on the domain $H^2(\mathbb{R}^d)$ and is essentially self-adjoint on $C_c^\infty(\mathbb{R}^d)$.

Let us mention some other conditions guaranteeing the essential self-adjointness of the Schrödinger operators for other types of potentials. This allows one to include potentials growing at infnity (which is impossible in the preceding theorem).

Theorem 5.12. *Let $\mathcal{H} = L^2(\mathbb{R}^d)$ and let $V \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$ be real-valued such that for some $c \in \mathbb{R}$ one has the inequality*

$$\langle u, (-\Delta + V)u\rangle \geq c\|u\|^2$$

for all $u \in C_c^\infty(\mathbb{R}^d)$. Then the operator $T = -\Delta + V$ is essentially self-adjoint on $C_c^\infty(\mathbb{R}^d)$.

Proof. By adding a constant to the potential V one can assume that $T \geq 1$. In other words, using the integration by parts,

$$\int_{\mathbb{R}^d} \big|\nabla u(x)\big|^2 dx + \int_{\mathbb{R}^d} V(x)\big|u(x)\big|^2 dx \geq \int_{\mathbb{R}^d} \big|u(x)\big|^2 dx \tag{5.3}$$

for all $u \in C_c^\infty(\mathbb{R}^d)$, and this extends by density at least to all $u \in H^1_{\mathrm{comp}}(\mathbb{R}^d)$. By Theorem 5.7 it is sufficient to show that the range of T is dense.

Let $f \in L^2(\mathbb{R}^d)$ such that $\big\langle f, (-\Delta + V)u\big\rangle = 0$ for all $u \in C_c^\infty(\mathbb{R}^d)$. Note that T preserves the real-valuedness, and we can suppose without loss of generality that f is real-valued (otherwise one considers its real and imaginary parts). We have at least $(-\Delta+V)f = 0$ in the sense of $\mathcal{D}'(\mathbb{R}^d)$, and $\Delta f = Vf$. As V is locally bounded, the function Vf is in $L^2_{\mathrm{loc}}(\mathbb{R}^d)$. Using the rather standard elliptic regularity argument one then obtains $f \in H^2_{\mathrm{loc}}(\mathbb{R}^d)$ (the elliptic regularity is shown in PDE courses).

Let us pick a real-valued function $\varphi \in C_c^\infty(\mathbb{R}^d)$ such that $\varphi(x) = 1$ for $|x| \leq 1$, that $\varphi(x) = 0$ for $|x| \geq 2$ and that $0 \leq \varphi \leq 1$, and introduce functions φ_n by $\varphi_n(x) = \varphi(x/n)$. For any $u \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ we have:

$$\begin{aligned}
&\int_{\mathbb{R}^d} \nabla(\varphi_n f)\nabla(\varphi_n u) dx + \int_{\mathbb{R}^d} V\varphi_n f \varphi_n u dx\\
&\quad = \int_{\mathbb{R}^d} \Big(|\nabla\varphi_n|^2 fu + \varphi_n\nabla\varphi_n(f\nabla u + u\nabla f) + \varphi_n^2 \nabla f \nabla u\Big) dx + \int_{\mathbb{R}^d} V\varphi_n f\varphi_n u\, dx\\
&\quad\quad = \int_{\mathbb{R}^d} \big|\nabla\varphi_n\big|^2 fu\, dx + \int_{\mathbb{R}^d} (f\nabla u - u\nabla f)\varphi_n\nabla\varphi_n\, dx + I,
\end{aligned} \tag{5.4}$$

where

$$\begin{aligned}
I &= \int_{\mathbb{R}^d} \nabla f\big(2u\varphi_n\nabla\varphi_n + \varphi_n^2\nabla u\big)\, dx + \int_{\mathbb{R}^d} V\varphi_n^2 fu\, dx\\
&= \int_{\mathbb{R}^d} \nabla f\big(u\nabla(\varphi_n^2) + \varphi_n^2\nabla u\big)\, dx + \int_{\mathbb{R}^d} V\varphi_n^2 fu\, dx\\
&= \int_{\mathbb{R}^d} \nabla f\nabla(\varphi_n^2 u)\, dx + \int_{\mathbb{R}^d} V\varphi_n^2 fu\, dx\\
&= -\int_{\mathbb{R}^d} f\Delta(\varphi_n^2 u)\, dx + \int_{\mathbb{R}^d} V\varphi_n^2 fu\, dx;
\end{aligned}$$

in the last step we have used the integration by parts in the first summand. In other words $I = \langle f, T(\varphi_n^2 u)\rangle$. As $\varphi_n^2 u \in C_c^\infty(\mathbb{R}^d)$, one has $I = 0$, and then

$$\int_{\mathbb{R}^d} \nabla(\varphi_n f)\nabla(\varphi_n u)dx + \int_{\mathbb{R}^d} V\varphi_n f\varphi_n u dx$$
$$= \int_{\mathbb{R}^d} \big|\nabla\varphi_n\big|^2 fu\,dx + \int_{\mathbb{R}^d} (f\nabla u - u\nabla f)\varphi_n\nabla\varphi_n\,dx.$$

This extends by density to $u \in H^2_{\mathrm{loc}}(\mathbb{R}^d)$, and for $u = f$ one has

$$\int_{\mathbb{R}^d} \big|\nabla(\varphi_n f)\big|^2 dx + \int_{\mathbb{R}^d} V\varphi_n^2 f^2\,dx = \int_{\mathbb{R}^d} \big|\nabla\varphi_n\big|^2 f^2\,dx.$$

Using (5.3) we arrive at

$$\int_{\mathbb{R}^d} \big|\nabla\varphi_n\big|^2 f^2\,dx \ge \int_{\mathbb{R}^d} \varphi_n^2 f^2 dx \ge \int_{\Omega} \varphi_n^2 f^2 dx,$$

where Ω is any ball. As n tends to infinity, the left-hand side goes to 0: one has $\|\nabla\varphi_n\|_\infty = \frac{1}{n}\|\varphi\|_\infty$, and one can apply the dominated convergence theorem. On the other side, the restriction of $\varphi_n f$ to Ω coincides with f for sufficiently large n, and this means that f vanishes in Ω. As Ω is arbitrary, $f = 0$ a.e. in $\mathbb{R}^d$. $\square$

We show that the both results can be combined in order to deal with more general potentials in one dimension:

Theorem 5.13. *Let $V \in L^2_{\mathrm{loc}}(\mathbb{R})$ with $V \ge -c$, then $T = -\Delta + V$ is essentially self-adjoint with domain $C_c^\infty(\mathbb{R})$ is essentially self-adjoint in $L^2(\mathbb{R})$.*

Proof. Wihout loss of generality we will assume that $V \ge 1$. On each interval $[n, n+1]$, $n \in \mathbb{Z}$, one can find a continuous function u_n satisfying $u_n \ge 1$ and $\|V - u_n\|^2_{L^2(n,n+1)} \le 2^{-n}$, then the function $U := \sum_{n\in\mathbb{Z}} u_n 1_{(n,n+1)}$ satisfies

$$U \ge 1, \quad U \in L^\infty_{\mathrm{loc}}(\mathbb{R}), \quad W := V - U \in L^2(\mathbb{R}).$$

We already know (Theorem 5.12) that $T_0 := -\Delta + U$ is essentially self-adjoint on $C_c^\infty(\mathbb{R})$. By Kato-Rellich theorem (Theorem 5.9) it is now sufficient to show that

$$W \text{ is } \overline{T_0}\text{-bounded with relative bound } < 1. \tag{5.5}$$

In Exercise 8 we have shown $\|f\|^2_{L^\infty(\mathbb{R})} \le \displaystyle\int_{\mathbb{R}} \big(|f'|^2 + |f|^2\big)\,dx$ for all $f \in H^1(\mathbb{R})$. For $f \in C_c^\infty(\mathbb{R})$ we have then

$$\begin{aligned}
\|Wf\|^2_{L^2(\mathbb{R})} &\le \|W\|^2_{L^2(\mathbb{R})}\|f\|^2_{L^\infty(\mathbb{R})} \le \|W\|^2_{L^2(\mathbb{R})} \int_{\mathbb{R}} (|f'|^2 + |f|^2)\,dx \\
&\le \|W\|^2_{L^2(\mathbb{R})} \int_{\mathbb{R}} (|f'|^2 + U|f|^2)\,dx \\
&= \|W\|^2_{L^2(\mathbb{R})}\langle T_0 f, f\rangle \le \delta\|W\|^2_{L^2(\mathbb{R})}\|T_0 f\|^2 + \frac{\|W\|^2_{L^2(\mathbb{R})}}{\delta}\|f\|^2,
\end{aligned}$$

where $\delta > 0$ is arbitrary, and this extends to

$$\|Wf\|^2_{L^\infty(\mathbb{R})} \le \delta\|W\|^2_{L^2(\mathbb{R})}\|\overline{T_0}f\|^2 + \frac{\|W\|^2_{L^2(\mathbb{R})}}{\delta}\|f\|^2 \text{ for all } f \in D(\overline{T_0}).$$

For small δ we arrive at the claim (5.5), which concludes the proof. $\square$

In fact Theorem 5.13 holds in literally the same form in all dimensions (usually referred to as Kato theorem), but the proof requires some advanced PDE tools.

Exercise 16. Let T be a symmetric operator in a Hilbert space $\mathcal{H}$, $T \geq 0$. Let $a > 0$. Show that the following three assertions are equivalent:

1. T is essentially self-adjoint,
2. $\ker(T^* + a) = \{0\}$,
3. $\operatorname{ran}(T + a)$ is dense in $\mathcal{H}$.

Exercise 17. Let A be a self-adjoint operator in $\mathcal{H}$ and B be a symmetric operator which is A-bounded with relative bound < 1.

Let $\mathcal{D} \subset D(A)$ we a subspace on which A is essentially self-adjoint. Show that $A+B$ is also essentially self-adjoint on $\mathcal{D}$. (This completes the proof of Kato-Rellich theorem.)

5.3 Discrete and essential spectra

Up to now we just distinguished between the whole spectrum and the point spectrum, i.e. the set of the eigenvalues. Let us introduce another classification of spectra, which is useful when studying various perturbations.

Definition 5.14 (Discrete spectrum, essential spectrum). Let T be a self-adjoint operator in a Hilbert space $\mathcal{H}$. We define its *discrete spectrum* $\operatorname{spec}_{\mathrm{disc}} T$ by

$$\operatorname{spec}_{\mathrm{disc}} T := \Big\{\lambda \in \operatorname{spec} T : \exists \varepsilon > 0 \text{ with } \dim \operatorname{ran} E_T\big((\lambda-\varepsilon, \lambda+\varepsilon)\big) < \infty\Big\}.$$

The set $\operatorname{spec}_{\mathrm{ess}} T := \operatorname{spec} T \setminus \operatorname{spec}_{\mathrm{disc}} T$ is called the *essential spectrum* of T. $\square$

The following proposition gives an alternative description of the discrete spectrum.

Proposition 5.15. *A real λ belongs to $\operatorname{spec}_{\mathrm{disc}} T$ iff it is an isolated eigenvalue of T of finite multiplicity.*

Proof. Let $\lambda \in \operatorname{spec}_{\mathrm{disc}} T$, then there exists $\varepsilon_0 > 0$ such that the operators $E_T\big((\lambda-\varepsilon, \lambda+\varepsilon)\big)$ do not depend on ε if $\varepsilon \in (0, \varepsilon_0)$. On the the other hand, this limit operator is non-zero, as $\lambda \in \operatorname{spec} T$. This means $E_T\big(\{\lambda\}\big) = \mathrm{s}-\lim_{\varepsilon\to 0+} E_T\big((\lambda-\varepsilon, \lambda+\varepsilon)\big) \neq 0$, and $\lambda \in \operatorname{spec}_{\mathrm{p}} T$ by Proposition 4.24(3). At the same time, $E_T\big((\lambda-\varepsilon_0, \lambda)\big) = E_T\big((\lambda, \lambda+\varepsilon_0)\big) = 0$, and Proposition 4.24(2) show that λ is an isolated point of the spectrum.

Now let λ be an isolated eigenvalue of finite multiplicity. Then there exists $\varepsilon_0 > 0$ such that $E_T\big((\lambda-\varepsilon_0, \lambda)\big) = E_T\big((\lambda, \lambda+\varepsilon_0)\big) = 0$, and $\dim \operatorname{ran} E_T(\{\lambda\}) = \dim \ker(T-\lambda) < \infty$. Therefore,

$$\begin{aligned}\dim \operatorname{ran} E_T\big((\lambda-\varepsilon_0, \lambda+\varepsilon_0)\big) &= \dim \operatorname{ran} E_T\big((\lambda-\varepsilon_0, \lambda)\big)\\ &\quad + \dim \operatorname{ran} E_T\big((\lambda, \lambda+\varepsilon_0)\big) + \dim \operatorname{ran} E_T(\{\lambda\}) < \infty. \quad \square\end{aligned}$$

Therefore, we arrive at the following direct description of the essential spectrum

Proposition 5.16. *A value $\lambda \in \operatorname{spec} T$ belongs to $\operatorname{spec}_{\mathrm{ess}} T$ iff at least one of the following three conditions holds:*

- $\lambda \notin \operatorname{spec}_{\mathrm{p}} T$,
- *λ is an accumulation point of $\operatorname{spec}_{\mathrm{p}} T$,*
- $\dim \ker(T-\lambda) = \infty$.

Furthermore, the essential spectrum is a closed set.

Proof. The first part just describes the points of the spectrum which are not isolated eigenvalues of finite multiplicity. For the second part we note that $\operatorname{spec}_{\mathrm{ess}} T$ is obtained from the closed set $\operatorname{spec} T$ by removing some isolated points. As the removing an isolated point does not change the property to be closed, $\operatorname{spec}_{\mathrm{ess}} T$ is also closed. $\square$

Let us list some examples.

Proposition 5.17 (Essential spectrum for compact operators). *Let T be a compact self-adjoint operator in an infinite-dimensional space $\mathcal{H}$, then* $\operatorname{spec}_{\mathrm{ess}} T = \{0\}$.

Proof. By Theorem 3.21, for any $\varepsilon > 0$ the set $\operatorname{spec} T \setminus (-\varepsilon, \varepsilon)$ consists of a finite number of eigenvalues of finite multiplicity, hence we have: $\operatorname{spec}_{\mathrm{ess}} T \setminus (-\varepsilon, \varepsilon) = \emptyset$ and $\dim \operatorname{ran} E_T\big(\mathbb{R} \setminus (-\varepsilon, \varepsilon)\big) < \infty$. On the other hand, $\dim \mathcal{H} = \dim \operatorname{ran} E_T\big(\mathbb{R} \setminus (-\varepsilon, \varepsilon)\big) + \dim \operatorname{ran} E_T\big((-\varepsilon, \varepsilon)\big)$, and $\dim \operatorname{ran} E_T\big((-\varepsilon, \varepsilon)\big)$ must be infinite for any $\varepsilon > 0$, which means that $0 \in \operatorname{spec}_{\mathrm{ess}} T$. $\square$

Proposition 5.18 (Essential spectrum of operators with compact resolvents). *The essential spectrum of a self-adjoint operator is empty if and only if the operator has a compact resolvent.*

Proof is left as an exercise.

Sometimes one uses the following terminology:

Definition 5.19 (Purely discrete spectrum). We say that a self-adjoint operator T has a *purely discrete spectrum* in some interval (a, b) if $\operatorname{spec}_{\mathrm{ess}} T \cap (a, b) = \emptyset$. If one has simply $\operatorname{spec}_{\mathrm{ess}} T = \emptyset$, then we say simply that the spectrum of T is *purely discrete*. As follows from the previous proposition, this exactly means that T has a compact resolvent. $\square$

Example 5.20. As seen several times, the free Laplacian in $L^2(\mathbb{R}^d)$ has the spectrum $[0, +\infty)$. This set has no isolated points, so this operator has no discrete spectrum.

The main difference between the discrete and the essential spectra comes from their behavior with respect to perturbations. This will be discussed in the following sections.

5.4 Weyl criterion and relatively compact perturbations

Let T be a self-adjoint operator in a Hilbert space $\mathcal{H}$.

The following proposition is an exercise.

Proposition 5.21. *Let λ be an isolated eigenvalue of T, then there exists $c > 0$ such that $\|(T - \lambda)u\| \geq c\|u\|$ for all $u \perp \ker(T - \lambda)$.*

The following theorem gives a description of the essential spectrum using approximating sequences.

Theorem 5.22 (Weyl criterion). *The condition $\lambda \in \operatorname{spec}_{\mathrm{ess}} T$ is equivalent to the existence of a sequence $(u_n) \subset D(T)$ satisfying the following three properties:*

1. $\|u_n\| \geq 1$,

2. u_n *converge weakly to* 0,

3. $(T-\lambda)u_n$ *converge to* 0 *in the norm of* $\mathcal{H}$.

Such a sequence will be called a singular Weyl sequence *for* λ. *Moreover, as will be shown in the proof, one can replace the conditions (1) and (2) just by:*

1'. u_n *form an orthonormal sequence.*

Proof. Denote by $W(T)$ the set of all real numbers λ for which one can find a singular Weyl sequence.

Show first the inclusion $W(T) \subset \operatorname{spec}_{\text{ess}} T$. Let $\lambda \in W(T)$ and let (u_n) be an associated singular Weyl sequence, then we have at least $\lambda \in \operatorname{spec} T$. Assume by contradiction that $\lambda \in \operatorname{spec}_{\text{disc}} T$ and denote by Π the orthogonal projection to $\ker(T-\lambda)$ in $\mathcal{H}$. As Π is a finite-rank operator, it is compact, and the sequence Πu_n converge to 0. Therefore, the norms of the vectors $w_n := (1-\Pi)u_n$ satisfy $\|w_n\| \geq 1/2$ for large n. On the other hand, the vectors $(T-\lambda)w_n = (1-\Pi)(T-\lambda)u_n$ converge to 0, which contradicts to Proposition 5.21.

Conversely, let $\lambda \in \operatorname{spec}_{\text{ess}} T$. If λ is an isolated point of $\operatorname{spec} T$, then it is an eigenvalue of infinite multiplicity, so one can taka as (u_n) any orthonormal family in $\ker(T-\lambda)$. If λ is not an isolated point of $\operatorname{spec} T$, then one can find a strictly decreasing to 0 sequence (ε_n) with $E_T(I_n \setminus I_{n+1}) \neq 0$, where $I_n := (\lambda - \varepsilon_n, \lambda + \varepsilon_n)$. Now we can choose u_n with $\|u_n\| = 1$ and $E_T(I_n \setminus I_{n+1})u_n = u_n$. These vectors form an orthonormal sequence and, in particular, converge weakly to 0. On the other hand,

$$\|(T-\lambda)u_n\| = \|(T-\lambda)E_T(I_n \setminus I_{n+1})u_n\| \leq \varepsilon_n \|u_n\| = \varepsilon_n,$$

which shows that the vectors $(T-\lambda)u_n$ converge to 0. Therefore, (u_n) is a singular Weyl sequence, and $\operatorname{spec}_{\text{ess}} T \subset W(T)$. □

The following theorem provides a starting point to the study of perturbations of self-adjoint operators.

Theorem 5.23 (Stability of essential spectrum). *Let* A *and* B *be self-adjoint operators such that for some* $z \in \operatorname{res} A \cap \operatorname{res} B$ *the difference of their resolvents* $K(z) := (A-z)^{-1} - (B-z)^{-1}$ *is a compact operator, then* $\operatorname{spec}_{\text{ess}} A = \operatorname{spec}_{\text{ess}} B$.

Proof. One can easily see, using the resolvent identities (Proposition 3.4), that $K(z)$ is compact for *all* $z \in \operatorname{res} A \cap \operatorname{res} B$.

Let $\lambda \in \operatorname{spec}_{\text{ess}} A$ and let (u_n) be an associated singular Weyl sequence. Without loss of generality assume that $\|u_n\| = 1$ for all n. We have

$$\lim \Big((A-z)^{-1} - \frac{1}{\lambda - z} \Big) u_n = \lim \frac{1}{z-\lambda} (A-z)^{-1}(A-\lambda)u_n = 0. \tag{5.6}$$

On the other hand, as $K(z)$ is compact, the sequence $K(z)u_n$ converges to 0 with respect to the norm, and

$$\lim \frac{1}{z-\lambda}(B-\lambda)(B-z)^{-1}u_n = \lim\Big((B-z)^{-1} - \frac{1}{\lambda - z}\Big)u_n$$
$$= \lim\Big((A-z)^{-1} - \frac{1}{\lambda - z}\Big)u_n - \lim K(z)u_n = 0.$$

Now denote $v_n := (B-z)^{-1}u_n$. The preceding equality shows that $(B-\lambda)v_n$ converge to 0, and one can easily show that v_n converge weakly to 0. It follows again from (5.6) and from the compactness of $K(z)$ that $\lim \|v_n\| = |\lambda - z|^{-1}$. Therefore, (v_n) is a singular Weyl sequence for B and λ, and $\lambda \in \operatorname{spec}_{\text{ess}} B$. So we have shown the inclusion $\operatorname{spec}_{\text{ess}} A \subset \operatorname{spec}_{\text{ess}} B$. As the participation of A and B is symmetric, we have also $\operatorname{spec}_{\text{ess}} A \supset \operatorname{spec}_{\text{ess}} B$. □

Let us describe a class of perturbations which can be studied using the preceding theorem.

Definition 5.24 (Relatively compact operators). Let A be a self-adjoint operator in a Hilbert space $\mathcal{H}$, and let B a closable linear operator in $\mathcal{H}$ with $D(A) \subset D(B)$. We say that B is *compact with respect to A* (or simply *A-compact*) if $B(A-z)^{-1}$ is compact for at least one $z \in \operatorname{res} A$. (It follows from the resolvent identitites that this holds then for all $z \in \operatorname{res} A$. □

Proposition 5.25. *Let B be A-compact, then B is infinitesimally small with respect to A.*

Proof. We show first that

$$\lim_{\lambda\to+\infty} \big\|B(A-i\lambda)^{-1}\big\| = 0 \tag{5.7}$$

Assume that (5.7) is false. Then one can find a constant $\alpha > 0$, non-zero vectors u_n and a positive sequence λ_n with $\lim \lambda_n = +\infty$ such that $\big\|B(A-i\lambda)^{-1}u_n\big\| > \alpha\|u_n\|$ for all n. Set $v_n := (A-i\lambda_n)^{-1}u_n$. Using $\|u_n\|^2 = \big\|(A-i\lambda_n)v_n\big\|^2 = \|Av_n\|^2 + \lambda_n^2\|v_n\|^2$ we obtain

$$\|Bv_n\|^2 > \alpha^2\|Av_n\|^2 + \alpha^2\lambda_n^2\|v_n\|^2.$$

Without loss of generality one may assume the normalization $\|Bv_n\| = 1$, then the sequence Av_n is bounded and v_n converge to 0. Let $z \in \operatorname{res} A$, then $(A-z)v_n$ is also bounded, one can extract a weakly convergent subsequence $(A-z)v_{n_k}$. Due to the compactness, the vectors $B(A-z)^{-1}\cdot(A-z)v_{n_k} = Bv_{n_k}$ converge to some $w \in \mathcal{H}$ with $\|w\| = 1$. On the other hand, as shown above, v_{n_k} converge to 0, and the closability of B shows that $w = 0$. This contradiction shows that (5.7) is true.

Now, for any $a > 0$ one can find $\lambda > 0$ such that $\|B(A-i\lambda)^{-1}u\| \le a\|u\|$ for all $u \in \mathcal{H}$. Denoting $v := (A-i\lambda)^{-1}u$ and noting that $(A-i\lambda)^{-1}$ is a bijection between $\mathcal{H}$ and $D(A)$ we see that

$$\|Bv\| \le a\|(A-i\lambda)v\| \le a\|Av\| + a\lambda\|v\|$$

for all $v \in D(A)$. As $a > 0$ is arbitrary, we get the result. □

So a combination of the preceding assertions leads us to the following observation:

Theorem 5.26 (Relatively compact perturbations). *Let A be a self-adjoint operator in a Hilbert space $\mathcal{H}$ and let B be symmetric and A-compact, then the operator $A+B$ with $D(A+B)=D(A)$ is self-adjoint, and the essential spectra of A and $A+B$ coincide.*

Proof. The self-adjointness of $A+B$ follows from the Kato-Rellich theorem, and it remains to show that the difference of the resolvents of $A+B$ and A is compact. This follows directly from the obvious identity

$$(A-z)^{-1}-(A+B-z)^{-1}=(A+B-z)^{-1}B(A-z)^{-1},$$

which holds at least for all $z\notin\mathbb{R}$. $\square$

As an exercise one can show the following assertion, which can be useful in some situations.

Proposition 5.27. *Let A be self-adjoint, B be symmetric and A-bounded with a relative bound <1, and C be A-compact, then C is also $(A+B)$-compact.*

5.5 Essential spectra for Schrödinger operators

Definition 5.28 (Kato class potential). We say that a measurable function $V:\mathbb{R}^d\to\mathbb{R}$ belongs to the Kato class if for any $\varepsilon>0$ one can find real-valued $V_\varepsilon\in L^p(\mathbb{R}^d)$ and $V_{\infty,\varepsilon}\in L^\infty(\mathbb{R}^d)$ such that $V_\varepsilon+V_{\infty,\varepsilon}=V$ and $\|V_{\infty,\varepsilon}\|_\infty<\varepsilon$. Here $p=2$ for $d\le 3$ and $p>d/2$ for $d\ge 4$. $\square$

Theorem 5.29. *If V is a Kato class potential in $\mathbb{R}^d$, then V is compact with respect to the free Laplacian $T=-\Delta$ in $L^2(\mathbb{R}^d)$, and the essential spectrum of $-\Delta+V$ is equal to $[0,\infty)$.*

Proof. We give the proof for $d\le 3$ only. Let $\mathcal{F}$ denote the Fourier transform, then for any $f\in L^2(\mathbb{R}^d)$ and $z\in\operatorname{res}T$ we have

$$\big(\mathcal{F}(T-z)^{-1}f\big)(p)=(p^2-z)^{-1}\mathcal{F}f(p).$$

This means that $(T-z)^{-1}f=g_z\star f$, where g_z is the L^2 function with $\mathcal{F}g_z(p)=(p^2-z)^{-1}$, and $\star$ stands for the convolution product. In other words,

$$(T-z)^{-1}f=\int_{\mathbb{R}^d}g_z(x-y)f(y)dy.$$

Let $\varepsilon>0$ and let V_ε and $V_{\infty,\varepsilon}$ be as in Definition 5.28. The operator $V_\varepsilon(T-z)^{-1}$ is an integral one with the integral kernel $K(x,y)=V_\varepsilon(x)g_z(x-y)$, i.e.

$$V_\varepsilon(T-z)^{-1}f(x)=\int_{\mathbb{R}^d}K(x,y)f(y)\,dy.$$

One has

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\big|K(x,y)\big|^2dxdy = \int_{\mathbb{R}^d}\big|V_\varepsilon(x)\big|^2dx\int_{\mathbb{R}^d}\big|g_z(y)\big|^2dy$$
$$= \|V_\varepsilon\|_2^2\|g_z\|^2 < \infty,$$

which means that $V_\varepsilon(T-z)^{-1}$ is a Hilbert-Schmidt operator and, therefore, is compact, see Subsection 3.3. At the same time we have the estimate

$$\|V_{\infty,\varepsilon}(T-z)^{-1}\| \leq \varepsilon\|(T-z)^{-1}\|.$$

Therefore, the operator $V(T-z)^{-1}$ is compact as it can be represented as the norm limit of the compact operators $V_\varepsilon(T-z)^{-1}$ as ε tends to 0. □

Example 5.30 (Coulomb potential). The previous theorem easily applies e.g. to the operators $-\Delta+\alpha/|x|$. It is sufficient to represent

$$\frac{1}{|x|} = \frac{1_R(x)}{|x|} + \frac{1-1_R(x)}{|x|},$$

where 1_R is the characteristic function of the ball of radius $R>0$ and centered at the origin with a sufficiently large R. So the essential spectrum of $-\Delta+\alpha/|x|$ is always the same as for the free Laplacian, i.e. $[0,+\infty)$.

6 Variational principle for eigenvalues

6.1 Min-max principle

Throughout the subsection we denote by T a self-adjoint operator in an *infinite-dimensional* Hilbert space $\mathcal{H}$, and we assume that T is semibounded from below. If $\operatorname{spec}_{\mathrm{ess}} T = \emptyset$, we denote $\Sigma := +\infty$, otherwise we put $\Sigma := \inf \operatorname{spec}_{\mathrm{ess}} T$.

Theorem 6.1 (Min-max principle). *Introduce the following numbers:*

$$\Lambda_n = \Lambda_n(T) = \inf_{\substack{V\subset D(T)\\ \dim V=n}} \sup_{\varphi\in V,\, \varphi\neq 0} \frac{\langle \varphi, T\varphi\rangle}{\langle \varphi,\varphi\rangle}, \quad n \in \mathbb{N},$$

then the sequence (Λ_n) is non-decreasing, and we are in one and only one of the following situations:

(a) For any $n \in \mathbb{N}$ there holds $\Lambda_n < \Sigma$. Then T has infinitely many discrete eigenvalues in $(-\infty, \Sigma)$, and $\Lambda_n(T)$ is exactly the n-th eigenvalue of T (when counted in the non-decreasing order by taking the multiplicities into account). There holds $\lim_{n\to\infty} \Lambda_n(T) = \Sigma$.

(b) There exists $N \in \mathbb{N}$ such that $\Lambda_N < \Sigma$ and $\Lambda_{N+1} \geq \Sigma$. Then T has exactly N discrete eigenvalues in $(-\infty, \Sigma)$, and the number Λ_n is exactly the nth eigenvalue of T for $n = 1, \dots, N$, while $\Lambda_n = \Sigma$ for all $n \geq N + 1$.

In particular, if $\Lambda_n < \Sigma$ for some $n \in \mathbb{N}$, then Λ_n is the nth eigenvalue of T

Proof. We prefer to give first a direct proof of the fact that Λ_n form a non-decreasing sequence (in fact it follows from the subsequent constructions as well), just to show how to deal with these quantities. For any $W \subset D(T)$ with $\dim V = n + 1$ one can find $V \subset D(T)$ with $\dim V = n$ and $V \subset W$, and then one clearly has

$$\sup_{\varphi\in W,\, \varphi\neq 0} \frac{\langle \varphi, T\varphi\rangle}{\langle \varphi,\varphi\rangle} \geq \sup_{\varphi\in V,\, \varphi\neq 0} \frac{\langle \varphi, T\varphi\rangle}{\langle \varphi,\varphi\rangle} \geq \inf_{\substack{V\subset W\\ \dim V=n}} \sup_{\varphi\in V,\, \varphi\neq 0} \frac{\langle \varphi, T\varphi\rangle}{\langle \varphi,\varphi\rangle}.$$

It follows that

$$\begin{aligned}\Lambda_{n+1} = \inf_{\substack{W\subset D(T)\\ \dim W=n+1}} \sup_{\varphi\in W,\, \varphi\neq 0} \frac{\langle \varphi, T\varphi\rangle}{\langle \varphi,\varphi\rangle} &\geq \inf_{\substack{W\subset D(T)\\ \dim W=n+1}} \inf_{\substack{V\subset W\\ \dim V=n}} \sup_{\varphi\in V,\, \varphi\neq 0} \frac{\langle \varphi, T\varphi\rangle}{\langle \varphi,\varphi\rangle} \\ &\geq \inf_{\substack{V\subset D(T)\\ \dim V=n}} \sup_{\varphi\in V,\, \varphi\neq 0} \frac{\langle \varphi, T\varphi\rangle}{\langle \varphi,\varphi\rangle} = \Lambda_n.\end{aligned}$$

By assumption, the spectrum of T in $(-\infty, \Sigma)$ is purely discrete, and the discrete eigenvalues in $(-\infty, \Sigma)$ may only accumulate to Σ (as any accumulation point in the spectrum automatically belongs to the essential spectrum). Hence, all these eigenvalues can be enumerated in the non-decreasing order: we denote them by E_k

and denote by v_k associated eigenvectors. We may assume without loss of generality that v_k form an orthonormal family, i.e. $\langle v_j, v_k\rangle = \delta_{j,k}$.

Consider two cases:

(Case 1) There are infinitely many eigenvalues in $(-\infty, \Sigma)$. Let $n \in \mathbb{N}$ and $V_n = \operatorname{span}(v_1, \dots, v_n)$. Clearly, $V_n \subset D(T)$ with $\dim V_n = n$. For $\varphi \in V_n$ we have

$$\langle \varphi, T\varphi\rangle = \sum_{j=1}^{n} E_j |\langle \varphi, v_j\rangle|^2 \le E_n \sum_{j=1}^{n} |\langle \varphi, v_j\rangle|^2 = E_n\langle \varphi, \varphi\rangle,$$

and it follows that $\Lambda_n \le E_n$.

On the other hand, let V be any n-dimensional subspace of $D(T)$ and P be the orthogonal projector in $\mathcal{H}$ on V_{n-1}, i.e.

$$P\varphi = \sum_{j=1}^{n-1} \langle v_j, \varphi\rangle v_j.$$

As the range of P is $(n-1)$-dimensional, there is $0 \ne \varphi \in V$ with $P\varphi = 0$ (as $\dim V = n > \dim \operatorname{ran} P$), i.e. $\varphi \perp v_j$ for all $j = 1, \dots, n-1$. By construction, the range V_{n-1} of P includes $\operatorname{ran} E_T\big((-\infty, E_n)\big)$, and $P\varphi = 0$ implies $\varphi \perp \operatorname{ran} E_T\big((-\infty, E_n)\big)$, and this inclusion is equivalent to $E_T\big([E_n, +\infty)\big)\varphi = \varphi$. This implies

$$\begin{aligned}\langle \varphi, T\varphi\rangle &= \Big\langle E_T\big([E_n, +\infty)\big)\varphi, TE_T\big([E_n, +\infty)\big)\varphi\Big\rangle\\ &\ge E_n\Big\langle E_T\big([E_n, +\infty)\big)\varphi, E_T\big([E_n, +\infty)\big)\varphi\Big\rangle = E_n\|\varphi\|^2.\end{aligned}$$

This shows that $\Lambda_n \ge E_n$. Alltogether, $\Lambda_n = E_n$ for all $n \in \mathbb{N}$.

(Case 2) There is $N \in \mathbb{N}$ such that T has exactly N eigenvalues in $(-\infty, \Sigma)$. As for the case 1 one shows that $\Lambda_n = E_n$ for all $n = 1, \dots, M$ and that $\Lambda_n \ge \Sigma$ for all $n \ge N+1$. Let us show that in fact $\Lambda_n = \Sigma$ for $n \ge N+1$. As $\Sigma \in \operatorname{spec}_{\text{ess}} T$, for any $\varepsilon > 0$ the dimension of $V = \operatorname{ran} E_T\big(\Sigma - \varepsilon, \Sigma + \varepsilon)\big)$ is infinite. Let (u_j) be an infinite orthonormal family in V, then for $U_n := \operatorname{span}(u_1, \dots, u_n)$ one has $\dim U_n = n$, and for any $\varphi \in U_n$ one has $\langle \varphi, T\varphi\rangle \le (\Sigma + \varepsilon)\langle \varphi, \varphi\rangle$, i.e. $\Lambda_n \le \Sigma + \varepsilon$. As $\varepsilon > 0$ is arbitrary, one has $\Lambda_n \le \Sigma$ for all n, in particular, for $n \ge N+1$. This concludes the proof of $\Lambda_n = \Sigma$ for $n \ge N+1$.

We now see that the case 1 corresponds to the situation (a) of the claim, while the case 2 corresponds to the situation (b) of the claim, and this covers all possible situations. □

As discussed previously (Subsection 2.1), the operator T is generated by some uniquely defined closed sesquilinear form t. Moreover, many important operators (e.f. Dirichlet/Neumann Laplacian) are defined by the associated sesquilinear forms. The min-max principle can be adapted to a direct use of forms instead of operators as shown in the new theorem:

Theorem 6.2. *Let T be generated by a closed sesquilinear form t and $\mathcal{D} \subset D(t)$ be a dense subspace with respect to $\langle\cdot,\cdot\rangle_t$. Define*

$$\Lambda_n' = \inf_{\substack{V\subset\mathcal{D}\\ \dim V=n}} \sup_{\varphi\in V,\,\varphi\neq 0} \frac{t(\varphi,\varphi)}{\langle\varphi,\varphi\rangle},$$

then for any $n\in\mathbb{N}$ one has $\Lambda_n' = \Lambda_n$.

Proof. Without loss of generality assume $T \geq 1$, then $\langle u,v\rangle_t = t(u,v)$.
Define

$$\Lambda_n'' = \inf_{\substack{V\subset D(t)\\ \dim V=n}} \sup_{\varphi\in V,\,\varphi\neq 0} \frac{t(\varphi,\varphi)}{\langle\varphi,\varphi\rangle},$$

then by construction one has $\Lambda_n'' \leq \Lambda_n'$ (due to the bigger choice of n-dimensional subspaces V).
Conversely, given $\varepsilon > 0$ let V be an n-dimensional subspace of $D(t)$ such that

$$\frac{t(\varphi,\varphi)}{\langle\varphi,\varphi\rangle} \leq \Lambda_n'' + \varepsilon$$

for all $0\neq\varphi\in V$. Denote by q the restriction of t on $V\times V$, then q is a symmetric sesqulinear form on the finite-dimensional space V (viewed as a Hilbert space with the induced scalar product). Hence, there exists a self-adjoint operator L in V such that $q(u,v) = \langle u, Lv\rangle$ for all $u,v\in V$, and one can construct an orthonormal basis $(v_1,\dots,v_n)$ of V consisting of eigenfunctions of L, $Lv_j = \lambda_j$, then

$$t(v_j,v_k) = \langle v_j, Lv_k\rangle = \lambda_j\,\delta_{j,k}, \quad \lambda_j = \langle v_j, Lv_j\rangle = t(v_j,v_j) \equiv \|v_j\|_t^2 \leq \Lambda_n'' + \varepsilon.$$

As $\mathcal{D}$ is dense in $D(t)$, one can find $u_j\in\mathcal{D}$ with $t(u_j-v_j,u_j-v_j) \equiv \|u_j-v_j\|_t^2 < \varepsilon^2$ for all $j=1,\dots,n$, then one has

$$\|u_j\|_t \leq \|v_j\|_t + \|u_j-v_j\|_t \leq \sqrt{\Lambda_n''+\varepsilon^2} + \varepsilon,$$

and (using Cauchy-Schwarz inequality)

$$\begin{aligned}\big|t(u_j,u_k) - \lambda_j\delta_{j,k}\big| &= \big|t(u_j,u_k) - t(v_j,v_k)\big| = \big|t(u_j,u_k-v_k) + t(u_j-v_j,v_k)\big|\\ &\leq \|u_j\|_t\|u_j-v_k\|_t + \|u_j-v_j\|_t\|v_k\|_t \leq C\varepsilon,\end{aligned}$$

and similarly $\big|\langle u_j,u_k\rangle - \delta_{j,k}\big| \equiv \langle u_j,u_k\rangle - \langle v_j,v_k\rangle\big| \leq C'\varepsilon$, where $C,C'>0$ are independent of $\varepsilon\in(0,\varepsilon_0)$. In particular, (u_j) are linearly independent for small ε and the subspace $V' := \operatorname{span}(u_1,\dots,u_n)$ is of dimension n. Then there is $B>0$ such that for any $\varphi = \sum_{j=1}^n \xi_j u_j \in V'$ with $\xi = (\xi_1,\dots,\xi_n)\in\mathbb{C}^n$ one has

$$\langle\varphi,\varphi\rangle = \sum_{j,k=1}^n \overline{\xi_j}\xi_k\langle u_j,u_k\rangle = |\xi|^2 + \sum_{j,k=1}^n \overline{\xi_j}\xi_k\big(\langle u_j,u_k\rangle - \delta_{j,k}\big)$$

and $\big|\langle\varphi,\varphi\rangle - |\xi|^2\big| \le B\varepsilon\|\xi\|^2$, and similarly,

$$\Big|t(\varphi,\varphi) - \sum_{j=1}^{n} \lambda_j|\xi_j|^2\Big| \le B\varepsilon|\xi|^2.$$

It follows (assuming that $\varepsilon < 1/B$) that for all $\xi \in \mathbb{C}^n \setminus \{0\}$, e.g. for $\varphi \in V' \setminus \{0\}$, there holds

$$\begin{aligned}\frac{t(\varphi,\varphi)}{\langle\varphi,\varphi\rangle} &\le \frac{\sum_{j=1}^{n} \lambda_j|\xi_j|^2 + B\varepsilon|\xi^2|}{|\xi|^2\big| - B\varepsilon|\xi|^2} \\ &\le \frac{(\Lambda''_n + \varepsilon)|\xi|^2 + B\varepsilon|\xi^2|}{|\xi|^2 - B\varepsilon|\xi|^2} \le \frac{\Lambda''_n + (B+1)\varepsilon}{1 - B\varepsilon},\end{aligned}$$

i.e. $\Lambda'_n \le \dfrac{\Lambda''_n + (B+1)\varepsilon}{1 - B\varepsilon}$ due to $\dim V' = n$. By sending ε to 0 one obtains $\Lambda'_n \le \Lambda''_n$. Hence, $\Lambda''_n = \Lambda'_n$ for all n.

In order to show that the both numbers coincide with Λ_n, we remark that $D(T)$ is dense in $D(t)$ (Theorem 2.2), hence, Λ_n is just a particular case of Λ'_n for $\mathcal{D} = D(T)$. This concludes the proof. $\square$

We now mention explicitly some elementary consequences of the min-max principle (more advanced corollaries will be discussed later).

Corollary 6.3. *For any self-adjoint lower semibounded operator T there holds*

$$\inf \operatorname{spec}_{\text{ess}} T = \lim_{n\to+\infty} \Lambda_n(T) \equiv \sup_{n\in\mathbb{N}} \Lambda_n(T).$$

(Follows directly by the min-max prnciple.)

Corollary 6.4. *Let T and T' be lower semibounded seml-adjoint operators in $\mathcal{H}$ such that $\Lambda_n(T) \le \Lambda_n(T')$ for all $n \in \mathbb{N}$. Then*

1. $\inf \operatorname{spec}_{\text{ess}} T \le \inf \operatorname{spec}_{\text{ess}} T'$,
2. *if T' has N eigenvalues in $(-\infty, \inf \operatorname{spec}_{\text{ess}} T)$, then T has at least N eigenvalues in the same interval, and these eigenvalues are not greater than the respective eigenvalues of T'.*

Proof. The first claim immediately follows by Corollary 6.3. For the second claim we remark that under the assumption made one has $\Lambda_N(T) \le \Lambda_N(T') < \inf \operatorname{spec}_{\text{ess}} T$, hence, $\Lambda_j(T) < \inf \operatorname{spec}_{\text{ess}} T$ for all $j = 1, \dots, N$, and then $\Lambda_j(T)$ is the $j$the eigenvalue of T for $j = 1, \dots, N$ by the min-max principle. $\square$

The following observation is used frequently as it allows one to show that an operator has at least one eigenvalue:

Corollary 6.5. *If there exists $\varphi \in Q(T)$ satisfying the strict inequality $t(\varphi,\varphi) < \Sigma\|\varphi\|^2$, then T has at least one eigenvalue in $(-\infty, \Sigma)$.*

Proof. One has $\Lambda_1 < \Sigma$, which means that Λ_1 is an eigenvalue of T. $\square$

The min-max principle is a powerful tool for the analysis of the behavior of the eigenvalues with respect to various parameters. An important point is that it can be used to compare the spectra/eigenvalues of operators acting in *different* Hilbert spaces. We give one of the simplest versions of this observation:

Proposition 6.6. *Let T and T' be lower semibounded self-adjoint operators in Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$, generated by closed sesqulinear forms t and t'. Assume that there exists a linear map $J : D(t') \to D(t)$ such that $\|Ju\|_{\mathcal{H}} = \|u\|_{\mathcal{H}'}$ and $t(Ju, Ju) \le t'(u, u)$ for all $u \in D(t')$. Then $\Lambda_n(T) \le \Lambda_n(T')$ for all $n \in \mathbb{N}$ (and then the assertions of Corollary 6.4 hold true as well).*

Proof. Under the assumptions made, the map J is injective, in particular $\dim J(V) = \dim V$ for any subspace $V \subset D(t')$. Then one has

$$\begin{aligned}
\Lambda_n(T) &= \inf_{\substack{V\subset D(t)\\ \dim V=n}} \sup_{\varphi\in V,\,\varphi\neq 0} \frac{t(\varphi,\varphi)}{\langle\varphi,\varphi\rangle_{\mathcal{H}}} \le \inf_{\substack{V\subset J\big(D(t')\big)\\ \dim V=n}} \sup_{\varphi\in V,\,\varphi\neq 0} \frac{t(\varphi,\varphi)}{\langle\varphi,\varphi\rangle_{\mathcal{H}}}\\
&= \inf_{\substack{V\subset D(t')\\ \dim V=n}} \sup_{\varphi\in J(V),\,\varphi\neq 0} \frac{t(\varphi,\varphi)}{\langle\varphi,\varphi\rangle_{\mathcal{H}}} = \inf_{\substack{V\subset D(t')\\ \dim V=n}} \sup_{\varphi\in V,\,\varphi\neq 0} \frac{t(J\varphi,J\varphi)}{\langle J\varphi,J\varphi\rangle_{\mathcal{H}}}\\
&\le \inf_{\substack{V\subset D(t')\\ \dim V=n}} \sup_{\varphi\in V,\,\varphi\neq 0} \frac{t'(\varphi,\varphi)}{\langle\varphi,\varphi\rangle_{\mathcal{H}'}}.
\end{aligned}$$ $\square$

One of the direct applications is the following situation, which will be applied later to some specific operators:

Definition 6.7. Let T and T' be lower semibounded self-adjoint operators in a Hilbert space $\mathcal{H}$, t and t be the associated closed sesquilinear forms. We write $T \le T'$ if $Q(T) \supset Q(T')$ and $t(u, u) \le t'(u, u)$ for all $u \in D(t')$. $\square$

As a direct corollary of the max-min principle we obtain:

Corollary 6.8. *Let T and T' be self-adjoint with $T \le T'$. Then $\Lambda_n(T) \le \Lambda_n(T')$ for all $n \in \mathbb{N}$ (and the assertions of Corollary 6.4 are valid).*

Proof. We have just a particular case of Proposition 6.6 with J =identity map. $\square$

6.2 Existence of negative eigenvalues for Schrödinger operators

In a sense, the most part of the rest of the course will be based on the min-max principle. We describe here one of the most direct applications to Schrödinger operators.

As seen above in Proposition 5.27, if V is a Kato class potential in $\mathbb{R}^d$, then the associated Schrödinger operator $T = -\Delta + V$ acting in $\mathcal{H} = L^2(\mathbb{R}^d)$ has the same essential spectrum as the free Laplacian, i.e. $\operatorname{spec}_{\text{ess}} T = [0, +\infty)$ and $\Sigma = 0$. In the present section we would like to discuss the question on the existence of negative eigenvalues.

We have rather a simple sufficient condition for the one- and two-dimensional cases.

Theorem 6.9. *Let $d \in \{1, 2\}$ and $V \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ be real-valued such that*

$$V_0 := \int_{\mathbb{R}^d} V(x)dx < 0,$$

then the associated Schrödinger operator $T = -\Delta + V$ has at least one negative eigenvalue.

Proof. We assumed the boundedness of the potential just to avoid additional technical issues concerning the domains: we have simply $Q(T) = Q(-\Delta) = H^1(\mathbb{R}^d)$. Due to $|V|^2 \le \|V\|_\infty |V|$ one has $V \in L^2(\mathbb{R}^d)$, and then $\operatorname{spec}_{\text{ess}} T = [0, +\infty)$ in virtue of Theorem 5.29. By Corollary 6.5 the existence of at least one eigenvalue follows from the existence of a non-zero $\varphi \in H^1(\mathbb{R}^d)$ with

$$\tau(\varphi) := \int_{\mathbb{R}^d} \big|\nabla\varphi(x)\big|^2 dx + \int_{\mathbb{R}^d} V(x)\big|\varphi(x)\big|^2 dx < 0.$$

The existence of such φ will be shown using some simple asymptotics.

Consider first the case $d = 1$. Take any $\varepsilon > 0$ and consider the function φ_ε given by $\varphi_\varepsilon(x) := e^{-\varepsilon|x|/2}$. Clearly, $\varphi_\varepsilon \in H^1(\mathbb{R})$ for any $\varepsilon > 0$, and the direct computation shows that

$$\int_{\mathbb{R}} \big|\varphi_\varepsilon'(x)\big|^2 dx = \frac{\varepsilon}{2} \quad \text{and} \quad \lim_{\varepsilon \to 0+} \int_{\mathbb{R}^d} V(x)\big|\varphi_\varepsilon(x)\big|^2 dx = V_0 < 0.$$

Therefore, for sufficiently small ε one obtains $\tau(\varphi_\varepsilon) < 0$.

Now let $d = 2$. Take $\varepsilon > 0$ and consider $\varphi_\varepsilon(x)$ defined by $\varphi_\varepsilon(x) = e^{-|x|^\varepsilon/2}$. We have

$$\begin{aligned}
\nabla\varphi_\varepsilon(x) &= -\frac{\varepsilon x |x|^{\varepsilon-2}}{2}\, e^{-|x|^\varepsilon/2},\\
\int_{\mathbb{R}^2} \big|\nabla\varphi_\varepsilon(x)\big|^2 dx &= \frac{\varepsilon^2}{4} \int_{\mathbb{R}^2} |x|^{2\varepsilon-2}\, e^{-|x|^\varepsilon}\, dx = \frac{\pi\varepsilon^2}{2} \int_0^\infty r^{2\varepsilon-1} e^{-r^\varepsilon}\, dr\\
&= \frac{\pi\varepsilon}{2} \int_0^\infty u e^{-u}\, du = \frac{\pi\varepsilon}{2},
\end{aligned}$$

and, as previously,

$$\lim_{\varepsilon \to 0+} \int_{\mathbb{R}^d} V(x)\big|\varphi_\varepsilon(x)\big|^2 dx = V_0 < 0,$$

and for sufficiently small ε we have again $\tau(\varphi_\varepsilon) < 0$. □

We see already in the above proof that finding suitable "test functions" φ for proving the existence of eigenvalues may become very tricky and depending on various parameters. One may easily check that the analog of φ_ε for $d = 1$ does not work for $d = 2$ and vice versa. It is a remarkable fact that the analog of Theorem 6.9 does not hold for the higher dimensions due to the Hardy inequality (Proposition 2.17):

Proposition 6.10. *Let $d \geq 3$ and let $V : \mathbb{R}^d \to \mathbb{R}$ be bounded with a compact support. For $\lambda \in \mathbb{R}$ consider the Schrödinger operators $T_\lambda := -\Delta + \lambda V$, then there exists $\lambda_0 > 0$ such that* $\operatorname{spec} T_\lambda = [0, +\infty)$ *for all $\lambda \in (-\lambda_0, +\infty)$.*

Proof. Due to the compactness of $\operatorname{supp} V$ one can find $\lambda_0 > 0$ in such a way that

$$\lambda_0 \big|V(x)\big| \leq \frac{(d-2)^2}{4|x|^2} \text{ for all } x \in \mathbb{R}^d.$$

Using the Hardy inequality, for any $u \in C_c^\infty(\mathbb{R}^d)$ and any $\lambda \in (-\lambda_0, +\infty)$ we have

$$\begin{aligned}
\langle u, T_\lambda u\rangle = \int_{\mathbb{R}^d} \big|\nabla u(x)\big|^2 dx + \lambda \int_{\mathbb{R}^d} V(x) \big|u(x)\big|^2 dx \\
\geq \int_{\mathbb{R}^d} \big|\nabla u(x)\big|^2 dx - \lambda_0 \int_{\mathbb{R}^d} \big|V(x)\big| \cdot \big|u(x)\big|^2 dx \\
\geq \int_{\mathbb{R}^d} \big|\nabla u(x)\big|^2 dx - \frac{(d-2)^2}{4} \int_{\mathbb{R}^d} \frac{\big|u(x)\big|^2}{|x|^2} dx \geq 0.
\end{aligned}$$

As T_λ is essentially self-adjoint on $C_c^\infty(\mathbb{R}^d)$, see Theorem 5.10, this inequality extends to all $u \in D(T_\lambda)$, and we obtain $T_\lambda \geq 0$, and this means that $\operatorname{spec} T_\lambda \subset [0, +\infty)$. On the other hand, $\operatorname{spec}_{\text{ess}} T_\lambda = [0, +\infty)$ as λV is of Kato class (see Theorem 5.29). $\square$

Exercise 18. We will work in $\mathbb{R}^N$ with $N \geq 3$. Let $V(x) := \big(|x|^2+1\big)^{-\gamma}$ with some $\gamma \in (0,1)$. The aim of the present exercise is to show that the operator $T(\lambda) = -\Delta - \lambda V$ in $L^2(\mathbb{R}^N)$ has negative eigenvalues for all $\lambda > 0$.

1. Show that the operator $T(\lambda)$ is essentially self-adjoint on $C_c^\infty(\mathbb{R}^N)$ and semibounded from below, describe its domain and form domain.
2. Describe the essential spectrum of T.
3. Let $s \geq 1$. Show that the function f_s, $f_s(x) = (|x|^2+s^2)^{-N}$, belongs to the form domain of $T(\lambda)$.
4. Denote $W_s := (-\Delta f_s)/f_s$. Show that the operator $T_s := -\Delta - W_s$ is essentially self-adjoint on $C_c^\infty(\mathbb{R}^N)$ and that f_s is its eigenfunction.
5. Show that for any $\lambda > 0$ there exists $s \geq 1$ such that $W_s(x) \leq \lambda V(x)$ for all x.
6. Use the preceding inequality to show that $T(\lambda)$ has at least one negative eigenvalue.
7. What can be said about the negative eigenvalues of $T(\lambda)$ if $\gamma \geq 1$?

Exercise 19.

1. Let T be a lower semibounded self-adjoint operator in a Hilbert space $\mathcal{H}$. Assume that the essential spectrum of T is non-empty and denote

$$\Sigma := \inf \operatorname{spec}_{\mathrm{ess}} T.$$

Furthermore, assume that there exist N linearly independent vectors $f_1, \dots, f_N$ in $D(T)$ such that all the eigenvalues of the $N \times N$ matrix

$$\Big(\big\langle f_j, (T-\Sigma) f_k\big\rangle\Big)_{j,k=1}^N$$

are strictly negative. Show that T has at least N eigenvalues in $(-\infty, \Sigma)$.

2. Consider the following self-adjoint operator T in $\mathcal{H} = L^2(\mathbb{R})$:

$$T = \frac{d^4}{dx^4} + 2\frac{d^2}{dx^2} + 1, \quad D(T) = H^4(\mathbb{R}).$$

   (a) Compute the spectrum of T. Hint: Use the Fourier transform.
   (b) Let $V \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ be real-valued. Show that the operator

$$S := T + V, \quad D(S) = H^4(\mathbb{R}),$$

   is self-adjoint and compute its essential spectrum.

(c) Let $\mathcal{F}$ be the Fourier transform in $L^2(\mathbb{R})$ and $\widehat{V} := \mathcal{F}V$. Give an explicit expression for the operator $\widehat{S} := \mathcal{F}S\mathcal{F}^*$ and describe its domain.

(d) Let $\varphi \in C_c^\infty(\mathbb{R})$ with $\varphi \geq 0$ and $\|\varphi\|_{L^1(\mathbb{R})} = 1$. For $\varepsilon > 0$ and $q \in \mathbb{R}$ consider the following functions $\varphi_{q,\varepsilon}$,

$$\varphi_{q,\varepsilon}(p) = \frac{1}{\varepsilon}\,\varphi\Big(\frac{p-q}{\varepsilon}\Big).$$

Show that these functions belong to $D(\widehat{S})$ and that

$$\lim_{\varepsilon \to 0+} \big\langle \varphi_{q,\varepsilon}, \widehat{S}\varphi_{r,\varepsilon} \big\rangle = \widehat{V}(q-r) \quad \text{ for } q, r = \pm 1.$$

(e) Assume that $\widehat{V}(0) < 0$ and $\big|\widehat{V}(2)\big| < \big|\widehat{V}(0)\big|$. Show that the operator S has at least two negative eigenvalues.

7 Laplacian eigenvalues for bounded domains

7.1 Dirichlet and Neumann eigenvalues

In this section we discuss some application of the general spectral theory to the eigenvalues of the Dirichlet and Neumann Laplacians in bounded domains. A central role will be played by the min-max principle.

Let us recall the setting. Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with a regular boundary (for example, lipschitzian); all the domains appearing in this section will be supposed to have a regular boundary without further specifications. Then the embedding of $H^1(\Omega)$ into $\mathcal{H} := L^2(\Omega)$ is a compact operator. By definition, the Dirichlet Laplacian $T_D = -\Delta_D$ and the Neumann Laplacian $T_N = -\Delta_N$ are the self-adjoint operators in $\mathcal{H}$ associated with the sesqulinear forms t_D and t_N respectively,

$$\begin{aligned} t_D(u,v) = t_D^\Omega(u,u) &= \int_\Omega \overline{\nabla u(x)} \cdot \nabla v(x)\, dx, \quad D(t_D) = Q(T_D) = H_0^1(\Omega),\\ t_N(u,v) = t_N^\Omega(u,u) &= \int_\Omega \overline{\nabla u(x)} \cdot \nabla v(x)\, dx, \quad D(t_N) = Q(T_N) = H^1(\Omega). \end{aligned}$$

In some cases, if the domain Ω is important, we write $T^\Omega_{D/N}$ instead of simply $T_{D/N}$. We know that both T_D and T_N have compact resolvents and that their spectra are purely discrete (see Section 3.3). Denote by $\lambda_j^{D/N} = \lambda_j^{D/N}(\Omega)$, $j \in \mathbb{N}$, the eigenvalues of $T_{D/N}$ repeated according to their multiplicities and enumerated in the non-decreasing order. The eigenvalues are clearly non-negative, and they are usually referred to as the Dirichlet/Neumann eigenvalues of the domain Ω (the presence of the Laplacian is assumed implicitly). Let us discuss some basic properties of these eigenvalues.

We first remark that by the min-max principle one has

$$\lambda_j^{D/N}(\Omega) = \Lambda_n(T^\Omega_{D/N}) \text{ for any } n \in \mathbb{N} \tag{7.1}$$

(as no essential spectrum is present), which gives the principal method for the study of the eigenvalues.

At this point we need the so-called trace theorem for the Sobolev spaces, which is proved in suitable PDE courses:

Theorem 7.1 (Trace theorem). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a lipschitzian boundary $\partial\Omega$, then there exists a unique bounded map (called trace map)*

$$\gamma_0 : H^1(\Omega) \to L^2(\partial\Omega)$$

satisfying $(\gamma_0 f)(s) = f(s)$ for all $f \in C^\infty(\overline{\Omega})$ and $s \in \partial\Omega$. Moreover,

$$H_0^1(\Omega) = \{ f \in H^1(\Omega) : \gamma_0 f = 0 \};$$

in particular, it follows that $H_0^1(\Omega)$ is strictly smaller that $H^1(\Omega)$ in this case.

Proposition 7.2. *(a) $\lambda_1^N = 0$. If Ω is connected, then $\ker T_N$ is spanned by the constant function $u(x) = 1$.*

(b) $\lambda_1^D > 0$.

Proof. (a) Note that $u = 1$ is clearly an eigenfunction of T_N with the eigenvalue 0. As all the eigenvalues are non-negative, $\lambda_1^N = 0$. Now let $u \in \ker T_N$, then

$$0 = \langle u, T_N u\rangle = t_N(u,u) = \int_\Omega \big|\nabla u(x)\big|^2 dx,$$

which shows that $\nabla u = 0$. Therefore, v is constant on each maximal connected component of Ω.
(b) We have at least $\lambda_1^D \geq 0$. Assume that $\lambda_1^D = 0$ and let v be an associated eigenfunction. We have as above $\nabla v = 0$, so v must be constant on each maximal connected component of Ω. But the restriction of v to the boundary of Ω must vanish, which gives $v = 0$. $\square$

A direct application of Corollary 6.8 based on the comparison $T_N \leq T_D$ gives

Proposition 7.3. *For any $j \in \mathbb{N}$ one has $\lambda_j^N(\Omega) \leq \lambda_j^D(\Omega)$.*

Another important aspect is the dependence of the eigenvalues on the domain.

Proposition 7.4 (Monotonicity with respect to domain, Dirichlet case). *If $\Omega \subset \widetilde{\Omega}$, then $\lambda_n^D(\Omega) \geq \lambda_n^D(\widetilde{\Omega})$ for all $n \in \mathbb{N}$.*

Proof. Let $J : H_0^1(\Omega) \to H_0^1(\widetilde{\Omega})$ be the extension by zero. In fact, if $u \in C_c^\infty(\Omega)$, then clearly $Ju \in H_0^1(\widetilde{\Omega})$ with $\|Ju\|_{H_0^1(\widetilde{\Omega})} = \|u\|_{H^1(\Omega)}$, which then extends by density to the whole of $H_0^1(\Omega)$. We further have $J\partial_j u = \partial_j Ju$, which shows that

$$t_D^{\widetilde{\Omega}}(Ju, Ju) = t_D^\Omega(u,u), \quad \|Ju\|_{L^2(\widetilde{\Omega})} = \|u\|_{L^2(\Omega)} \quad \text{for all } u \in H_0^1(\Omega) = D(t_D^\Omega).$$

Now we are in the situation of Proposition 6.6, and one has $\Lambda_n(T_D^\Omega) \geq \Lambda_n(T_D^{\widetilde{\Omega}})$ for all $n \in \mathbb{N}$. We conclude the proof by using (7.1). $\square$

Note that there is no easy generalization of this result to the Neumann case. The reason can be understood at a certain abstract level. As can be seen from the proof, for $\Omega \subset \widetilde{\Omega}$ there exists an obvious embedding $\tau : H_0^1(\Omega) \to H_0^1(\widetilde{\Omega})$ (extension by zero) such that $\|\tau u\| = \|u\|$ for all $u \in H_0^1(\Omega)$. If one replaces the spaces H_0^1 by H^1, then the existence of a bounded embedding and the estimates for its norm in terms of the two domains become non-trivial. Nevertheless, we mention at least one important case where a kind of the monotonicity can be proved.

Proposition 7.5 (Neumann eigenvalues of composed domains). *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a regular boundary, and let Ω_1 and Ω_2 be non-intersecting open subsets of Ω with regular boundaries such that $\overline{\Omega} = \overline{\Omega_1 \cup \Omega_2}$, then $\lambda_n^N(\Omega_1 \cup \Omega_2) \leq \lambda_j^N(\Omega)$ for any $j \in \mathbb{N}$.*

Proof. Under the assumptions made, the spaces $L^2(\Omega)$ and $L^2(\Omega_1\cup\Omega_2)$ coincide, and any function $f\in H^1(\Omega)$ belongs to $H^1(\Omega_1\cup\Omega_2)$ with

$$\|f\|_{H^1(\Omega)}=\|f\|_{H^1(\Omega_1\cup\Omega_2)}.$$

Hence one can consider the identity map $J: D(t_N^\Omega)=H^1(\Omega)\to H^1(\Omega_1\cup\Omega)=D(t_N^{\Omega_1\cup\Omega_2})$ and use the proposition 6.6. $\square$

Remark 7.6. Under the assumptions of proposition 7.5 for any $n\in\mathbb{N}$ we have $\lambda_n^D(\Omega)\le\lambda_n^D(\Omega_1\cup\Omega_2)$, which follows from the inclusion $\Omega_1\cup\Omega_2\subset\Omega$. Therefore, for any $n\in\mathbb{N}$ one has the chain

$$\lambda^N(\Omega_1\cup\Omega_2)\le\lambda^N(\Omega)\le\lambda^D(\Omega)\le\lambda_n^D(\Omega_1\cup\Omega_2),$$

and this is the key argument of the so-called Dirichlet-Neumann bracketing which is used e.g. for estimating the asymptotic behavior of the eigenvalues (see below). $\square$

We complete this first discussion by proving the continuity of the Dirichlet eigenvalues with respect to domain.

Proposition 7.7 (Continuity with respect to domain, Dirichlet). *If $\Omega_j\subset\Omega_{j+1}$ for all $j\in\mathbb{N}$, and $\Omega=\bigcup_{j=1}^\infty\Omega_j$, then $\lambda_n^D(\Omega)=\lim_{j\to\infty}\lambda_n^D(\Omega_j)$ for any $n\in\mathbb{N}$.*

Proof. Let us pick $n\in\mathbb{N}$, and let $f_1,\dots f_n$ be the mutually orthogonal normalized eigenfunctions associated with the eigenvalues $\lambda_1^D(\Omega),\dots,\lambda_n^D(\Omega)$. If U denotes the subspace spanned by $f_1,\dots,f_n$, then for any $f\in U$ one has the estimate $\|\nabla f\|^2\le\lambda_n^D(\Omega)\|f\|^2$.

Now take an arbitrary $\varepsilon>0$. Using the density of $C_c^\infty(\Omega)$ in $H_0^1(\Omega)$ one can approximate every f_j by $u_j\in C_c^\infty(\Omega)$ in such a way that $u_1,\dots,u_n$ will be linearly independent and that $\|\nabla u\|^2_{L^2(\Omega)}\le\big(\lambda_n^D(\Omega)+\varepsilon\big)\|u\|^2_{L^2(\Omega)}$ for all u from the n-dimensional subspace V spanned by $u_1,\dots,u_n$. Let $K\subset\Omega$ be a compact subset containing the supports of all u_j and, as a consequence, the supports of all functions from V. One can find $M\in\mathbb{N}$ such that $K\subset\Omega_m$ for all $m\ge M$, and then for all $m\ge M$ we have $V\subset H_0^1(\Omega_m)=:D(t_D^{\Omega_m})$. As V is n-dimensional, there holds

$$\Lambda_n(t_D^{\Omega_m})\le\sup_{u\in V\setminus\{0\}}\frac{\|\nabla u\|^2_{L^2(\Omega_m)}}{\|u\|^2_{L^2(\Omega_m)}}=\sup_{u\in V\setminus\{0\}}\frac{\|\nabla u\|^2_{L^2(\Omega)}}{\|u\|^2_{L^2(\Omega)}}\le\lambda_n^D(\Omega)+\varepsilon,$$

hence, $\lambda_n^D(\Omega_m)\le\lambda_n^D(\Omega)+\varepsilon$ for all $m\ge M$. On the other hand, $\lambda_n^D(\Omega)\le\lambda_n^D(\Omega_m)$ by monotonicity with respect to the domain (as $\Omega_n\subset\Omega$ for all n). $\square$

7.2 Weyl asymptotics

In this subsection we will discuss some aspects of the asymptotic behavior of the Laplacian eigenvalues. We introduce the Dirichlet/Neumann counting functions $N_{D/N}(\lambda,\Omega)$ by

$$N_{D/N}(\lambda,\Omega)=\text{the number of } j\in\mathbb{N} \text{ for which } \lambda_j^{D/N}(\Omega)\in(-\infty,\lambda].$$

Clearly, $N_{D/N}(\lambda,\Omega)$ is finite for any λ, and it has a jump at each eigenvalue; the jump is equal to the multiplicity. We emphasize the following obvious properties:

$$N_D(\lambda,\Omega) \le N_N(\lambda,\Omega) \tag{7.2}$$

$$N_{D/N}(\lambda,\Omega_1\cup\Omega_2) = N_{D/N}(\lambda,\Omega_1) + N_{D/N}(\lambda,\Omega_2) \text{ for } \Omega_1\cap\Omega_2=\emptyset. \tag{7.3}$$

$$N_D(\lambda,\Omega) \le N_D(\widetilde{\Omega}) \text{ for } \Omega\subset\widetilde{\Omega}. \tag{7.4}$$

We are going to discuss the following rather general result on the behavior of the counting functions N_D as $\lambda\to+\infty$:

Theorem 7.8 (Weyl asymptotics). *We have*

$$\lim_{\lambda\to+\infty}\frac{N_D(\lambda,\Omega)}{\lambda^{d/2}} = \frac{\omega_d}{(2\pi)^d}\,\mathrm{vol}(\Omega),$$

where ω_d denotes the volume of the unit ball in $\mathbb{R}^d$.

To keep simple notation we proceed with the proof for the case $d=2$ only. Due to $\omega_2=\pi$ we are reduced to prove

$$\lim_{\lambda\to+\infty}\frac{N_D(\lambda,\Omega)}{\lambda} = \frac{\mathrm{area}(\Omega)}{4\pi}. \tag{7.5}$$

The proof consists of several steps.

Lemma 7.9. *The Weyl asymptotics is valid for rectangles, for both N_N and N_D.*

Proof. Let $\Omega=(0,a)\times(0,b)$, $a,b>0$. As shown in Example 4.31, the Neumann eigenvalues of Ω are the numbers

$$\lambda(m,n) := \Big(\frac{\pi m}{a}\Big)^2 + \Big(\frac{\pi n}{b}\Big)^2$$

with $m,n\in\mathbb{N}_0:=\{0\}\cup\mathbb{N}$, and the Dirichlet spectrum consists of the eigenvalues $\lambda(m,n)$ with $m,n\in\mathbb{N}$. Denote

$$D(\lambda) := \Big\{(x,y)\in\mathbb{R}^2 : \frac{x^2}{a^2}+\frac{y^2}{b^2}\le\frac{\lambda}{\pi^2},\, x\ge 0, y\ge 0\Big\},$$

then $N_D(\lambda,\Omega)=\#D(\lambda)\cap\big(\mathbb{N}\times\mathbb{N}\big)$ and $N_N(\lambda,\Omega)=\#D(\lambda)\cap\big(\mathbb{N}_0\times\mathbb{N}_0\big)$, where $\#$ denotes the cardinality.

First, counting the points $(n,0)$ and $(0,n)$ with $n\in\mathbb{N}_0$ inside $D(\lambda)$ we obtain the majoration

$$N_N(\lambda)-N_D(\lambda)\le\frac{a+b}{\pi}\sqrt{\lambda}+2,\quad \lambda>0.$$

At the same time, $D(\lambda)$ contains the union of the unit squares $[m-1,m]\times[n-1,n]$ with $(m,n)\in D(\lambda)\cap\big(\mathbb{N}\times\mathbb{N}\big)$. As there are exactly $N_D(\lambda,\Omega)$ such squares, we have

$$N_D(\lambda,\Omega)\le\mathrm{area}\,D(\lambda)=\frac{\lambda ab}{4\pi}.$$

We also observe that $D(\lambda)$ is contained in the union of the unit squares $[m, m+1] \times [n, n+1]$ with $(m,n) \in D(\lambda) \cap \big(\mathbb{N}_0 \times \mathbb{N}_0\big)$. As the number of such cubes is exactly $N_N(\lambda, \Omega)$, this gives

$$N_N(\lambda, \Omega) \geq \operatorname{area} D(\lambda) = \frac{\lambda ab}{4\pi}.$$

Putting all together we arrive at

$$\frac{\lambda ab}{4\pi} \leq N_N(\lambda, \Omega) \leq N_D(\lambda, \Omega) + \frac{a+b}{\pi}\sqrt{\lambda} + 2 \leq \frac{\lambda ab}{4\pi} + \frac{a+b}{\pi}\sqrt{\lambda} + 2,$$

and it remains to recall that area$(\Omega) = ab$. □

Definition 7.10 (Domains composed from rectangles). We say that a domain Ω with a regular boundary is composed from rectangles if there exists a finite family of non-intersecting open rectangles Ω_j, $j = 1, \dots, k$, with $\overline{\Omega} = \overline{\bigcup_{j=1}^k \Omega_j}$. □

Lemma 7.11. *The Weyl asymptotics holds for domains composed from rectangles.*

Proof. Let Ω be a domain composed from rectangles, ant let Ω_j, $j = 1, \dots, k$, be the rectangles as in Definition 7.10. Using Remark 7.6 and the equality (7.3) we obtain the chain

$$\begin{aligned}\frac{N_D(\lambda, \Omega_1) + \dots + N_N(\lambda, \Omega_k)}{\lambda} &= \frac{N_D(\lambda, \Omega_1 \cup \dots \cup \Omega_k)}{\lambda} \leq \frac{N_D(\lambda, \Omega)}{\lambda} \\ &\leq \frac{N_N(\lambda, \Omega)}{\lambda} \leq \frac{N_N(\lambda, \Omega_1 \cup \dots \cup \Omega_k)}{\lambda} = \frac{N_N(\lambda, \Omega_1) + \dots + N_D(\lambda, \Omega_k)}{\lambda},\end{aligned}$$

and the result is obtained by applying Lemma 7.9 to the quotients $N_{D/N}(\lambda, \Omega_j)/\lambda$ and by noting that area$(\Omega) =$ area$(\Omega_1) + \dots +$ area(Ω_k). □

Proof of Theorem 7.8. Let Ω be a domain with a regular boundary. It is a standard result of the analysis that for any $\varepsilon > 0$ one can find two domains Ω_ε and $\widetilde{\Omega}_\varepsilon$ such that:

- both Ω_ε and $\widetilde{\Omega}_\varepsilon$ are composed from rectangles,
- $\Omega_\varepsilon \subset \Omega \subset \widetilde{\Omega}_\varepsilon$,
- area$(\widetilde{\Omega}_\varepsilon \setminus \Omega_\varepsilon) < \varepsilon$.

Using (7.2) and the monotonicity of the Dirichlet eigenvalues with respect to domain we have:

$$\frac{N_D(\lambda, \Omega_\varepsilon)}{\lambda} \leq \frac{N_D(\lambda, \Omega)}{\lambda} \leq \frac{N_D(\lambda, \widetilde{\Omega}_\varepsilon)}{\lambda} \leq \frac{N_N(\lambda, \widetilde{\Omega}_\varepsilon)}{\lambda}.$$

By Lemma 7.11, we can find $\lambda_\varepsilon > 0$ such that

$$\frac{\operatorname{area}(\Omega_\varepsilon) - \varepsilon}{4\pi} \leq \frac{N_D(\lambda, \Omega)}{\lambda} \leq \frac{\operatorname{area}(\widetilde{\Omega}_\varepsilon) + \varepsilon}{4\pi} \quad \text{for } \lambda > \lambda_\varepsilon.$$

At the same time, $\operatorname{area}(\Omega_\varepsilon) \geq \operatorname{area}(\Omega) - \varepsilon$ and $\operatorname{area}(\widetilde{\Omega}_\varepsilon) \leq \operatorname{area}(\Omega) + \varepsilon$, so for $\lambda > \lambda_\varepsilon$ we have

$$\frac{\operatorname{area}(\Omega) - 2\varepsilon}{4\pi} \leq \frac{N_D(\lambda, \Omega)}{\lambda} \leq \frac{\operatorname{area}(\Omega) + 2\varepsilon}{4\pi},$$

which gives the sought result. □

We note that the Weyl asymptotics also holds for the Neumann Laplacian if the domain is sufficiently smooth, which can be proved using suitable extension theorem for Sobolev spaces. The Weyl asymptotics is one of the basic results on the relations between the Dirichlet/Neumann eigenvalues and the geometric properties of the domain. It states, in particular, that the spectrum of a bounded domain contains the information on the dimension and the volume of the domain. There are various refinements involving lower order terms with respect to λ, and the respective coefficients contain some information on the topology of the domain, on its boundary etc.

Exercise 20.

1. Let Λ_d denote the smallest Dirichlet eigenvalue of the unit ball in $\mathbb{R}^d$. Give the expression for the smallest Dirichlet eigenvalue $\lambda(r,d)$ of the ball of radius $r>0$ in $\mathbb{R}^d$.

2. Let $\Omega \subset \mathbb{R}^d$ be a non-empty open set. Denote

$$R = R(\Omega) = \sup\big\{r > 0 : \Omega \text{ contains a ball of radius } r\big\}.$$

   Let T be the Dirichlet Laplacian in Ω. Show the inequality $\inf \operatorname{spec} T \leq \lambda(R,d)$.

   Now we assume that $R(\Omega) = \infty$. (Such domains Ω are sometimes called *quasiconical*.)

3. Construct a sequence $(\varphi_n) \subset C_c^\infty(\mathbb{R}^d)$ with the following properties: $\varphi_n(x) = 1$ for $|x| \leq n-1$, $\varphi_n(x) = 0$ for $|x| \geq n$, and there exists $c > 0$ such that $\|\varphi_n\|_\infty + \sum_j \|\partial_j \varphi_n\|_\infty + \sum_{j,k} \|\partial_j \partial_k \varphi_n\|_\infty \leq c$ for all n.

4. Pick any $k \in \mathbb{R}^d$. For $n \in \mathbb{N}$ choose $a_n \in \Omega$ such that the ball of radius n centered at a_n is contained in Ω. Consider the functions $u_n(x) = e^{ikx}\varphi_n(x - a_n)$. Show that $\varphi_n \in D(T)$ and that $\lim_{n\to\infty} \big\|(T - k^2)u_n\big\| / \|u_n\| = 0$.

5. Show that $\operatorname{spec} T = [0, +\infty)$.

8 Schrödinger operators: more on eigenvalues and eigenfunctions

In the present section we are going to discuss in greater details some asymptotics related to the spectral analysis of Schrödinger operators $H = -\Delta + V$ in $L^2(\mathbb{R}^d)$ with real-valued potentials $V : \mathbb{R}^d \to \mathbb{R}$. Such operators are of importance in quantum mechanics: H can be viewed as a Hamiltonian of a particle whose interaction with the environment is described by the potential V. The spectrum of H corresponds then to possible values of the energy of the particle.

The analysis will be mostly carried out using the min-max principle.

8.1 Finiteness/infiniteness of the discrete spectrum

In Subsection 5.5 we have shown that if V is in the Kato class, then the essential spectrum of H is $[0, +\infty)$. Hence, for such potentials, the negative spectrum is purely discrete, and we have shown some sufficient conditions for the discrete spectrum to be non-empty or empty (Subsection 6.2). On the other hand, we do not know yet under which conditions the discrete spectrum is finite or infinite. This will be addressed in the present subsection.

For a semibounded from below self-adjoint operator A and $\lambda \le \inf \operatorname{spec}_{\text{ess}} A$ we will denote

$$N(A, \lambda) = \#\{ \text{ eigenvalues of } A \text{ in } (-\infty, \lambda)\} \equiv \dim \operatorname{ran} E_A\big((-\infty, \lambda)\big).$$

(We use the convention $\inf \emptyset = +\infty$.) The most basic condition establishing the finiteness of the discrete spectrum in any dimension is as follows:

Theorem 8.1. *Let $V \in L^\infty_{\text{loc}}(\mathbb{R}^d)$ be semibounded from below and let $v_0 \in \mathbb{R}$ be such that the set $S = \{x : V(x) < v_0\}$ is bounded. Then the spectrum of $H = -\Delta + V$ in $(-\infty, v_0)$ is purely discrete and finite (i.e. H has at most finitely many negative eigenvalues).*

Proof. Under the assumptions made the operator H is essentially self-adjoint on $C_c^\infty(\mathbb{R}^d)$, see Theorem 5.12. It follows that H can be viewed as the self-adjoint operator generated by the following closed sesquilinear form h:

$$h(u, v) = \int_{\mathbb{R}^d} (\overline{\nabla u} \cdot \nabla v + V \overline{u} v)\, dx, \quad D(h) = \Big\{u \in H^1(\mathbb{R}^d) : \int_{\mathbb{R}^d} V |u|^2\, dx < \infty\Big\}.$$

In fact, if one denotes T the self-adjoint operator generated by h, then one clearly has $T_0 \subset T$, where T_0 acts on the domain $D(T_0) = C_c^\infty(\mathbb{R}^d)$ with $T_0 u = -\Delta u + V u$. But this operator T_0 is essentially self-adjoint and its closure is H (Theorem 5.12), so the maximality of self-adjoint operators implies $T = H$.

Let B be an open ball containing the set S. The idea is to decouple two sides of ∂B and to compare H with two operators acting in B and $\mathbb{R}^d \setminus \overline{B}$. We consider the following closed sesquilinear form $\widetilde{h}$ extending h:

$$\widetilde{h}(u,v) = \int_{\mathbb{R}^d\setminus\partial B} (\overline{\nabla u}\cdot\nabla v + V\overline{u}v)\,dx,$$
$$D(\widetilde{h}) = \big\{u\in H^1(\mathbb{R}^d\setminus\partial B) : \int_{\mathbb{R}^d} V|u|^2\,dx<\infty\big\},$$

and let $\widetilde{H}$ be the self-adjoint operator generated by $\widetilde{h}$, then $\widetilde{H}\le H$ (see Definition 6.7) and $\Lambda_n(\widetilde{H})\le\Lambda_n(H)$ for any $n\in\mathbb{N}$. It follows that $\inf\operatorname{spec}_{\text{ess}}\widetilde{H} = \lim_{n\to\infty}\Lambda_n(\widetilde{H})\le\lim_{n\to\infty}\Lambda_n(H) = \inf\operatorname{spec}_{\text{ess}} H$, and that

$$N(H,\lambda)\le N(\widetilde{H},\lambda)\text{ for }\lambda\le\inf\operatorname{spec}_{\text{ess}}\widetilde{H}. \tag{8.1}$$

We now remark that the operator $\widetilde{H}$ is the direct sum of two operators, $H_1\oplus H_2$, where H_1 acts in $L^2(B)$ and is generated by the closed sesqulinear form h_1,

$$h_1(u,v) = \int_B (\overline{\nabla u}\cdot\nabla v + V\overline{u}v)\,dx,\quad D(h_1) = \big\{u\in H^1(B) : \int_B V|u|^2\,dx<\infty\big\},$$

while H_2 acts in $L^2(\mathbb{R}^2\setminus\overline{B})$ and is generated by the closed sesquilinear form h_2,

$$h_2(u,v) = \int_{\mathbb{R}^2\setminus\overline{B}} (\overline{\nabla u}\cdot\nabla v + V\overline{u}v)\,dx,$$
$$D(h_2) = \Big\{u\in H^1(\mathbb{R}^2\setminus\overline{B}) : \int_{\mathbb{R}^2\setminus\overline{B}} V|u|^2\,dx<\infty\Big\}.$$

Due to the compact embedding of $H^1(B)$ into $L^2(B)$, the operator H_1 is with compact resolvent and $\operatorname{spec}_{\text{ess}} H_1=\emptyset$ On the other hand, in $\mathbb{R}^2\setminus\overline{B}$ one has $V\ge v_0$, therefore, $H_2\ge v_0$ and $\operatorname{spec}_{\text{ess}} H_2\subset\operatorname{spec} H_2\subset[v_0,\infty)$. Hence, $\operatorname{spec}_{\text{ess}}\widetilde{H} = \operatorname{spec}_{\text{ess}} H_1\cup\operatorname{spec}_{\text{ess}} H_2\subset[v_0,+\infty)$, and $\inf\operatorname{spec}_{\text{ess}} H\ge\inf\operatorname{spec}_{\text{ess}}\widetilde{H}\ge v_0$. In additionn, $N(\widetilde{H},v_0) = N(H_1,v_0)+N(H_2,v_0)$. The operator H_1 is semibounded from below with compact resolvent, hence, it has finitely many eigenvalues in $(-\infty,v_0)$, i.e. $N(H_1,v_0)<\infty$. On the other hand, $N(H_2,v_0)=0$ as $H_2\ge v_0$. Therefore, $N(\widetilde{H},v_0)<\infty$, and then $N(H,v_0)<\infty$ by (8.1). □

Corollary 8.2. *Let $V\in L^\infty(\mathbb{R}^d)$ be real valued with compact support and $H=-\Delta+V$ in $L^2(\mathbb{R}^d)$, then* $\operatorname{spec}_{\text{ess}} H=[0,\infty)$, *and H has at most finitely many negative eigenvalues.*

Proof. The potential V is in Kato class, hence, $\operatorname{spec}_{\text{ess}} H=[0,+\infty)$ (see Subsection 5.5). The remaining assertion follows from Theorem 8.1 for $v_0=0$. □

We remark that the condition $V\in L^\infty_{\text{loc}}$ can be relaxed in several ways (we use it as a condition guaranteeing the essential self-adjointness), but the proof becomes rather technical at several points. Some weaker assumptions on the behavior of the potential at infinity is also possible. For example, in dimensions ≥ 3 one can use the Hardy inequality (Proposition 2.17) in order to show the finiteness of the negative spectrum without the compact support condition.

Theorem 8.3. *Let $d \geq 3$ and $V \in L^\infty_{\text{loc}}(\mathbb{R}^d)$ be real-valued and semibounded from below. Assume that there exist $R > 0$ and $0 < b < 1$ such that*

$$V(x) \geq -\frac{b(d-2)^2}{4|x|^2} \text{ for all } x \text{ with } |x| \geq R,$$

then the Schrödinger operator $H = -\Delta + V$ in $L^2(\mathbb{R}^d)$ has at most finitely many negative eigenvalues.

Proof. We first remark that for any $v > 0$ one has the inclusion

$$\{x : V(x) < -v\} \subset \{x : |x| < R\} \cup \Big\{x : |x| \geq R \text{ and } |x|^2 < \frac{b(d-2)^2}{4v}\Big\},$$

hence, $\{x : V(x) < -v\}$ is bounded, and $\operatorname{spec}_{\text{ess}} H \subset [-v, +\infty)$ by Theorem 8.1. As $v > 0$ was arbitrary, there holds $\operatorname{spec}_{\text{ess}} H \subset [0, +\infty)$.
Now let $\varphi \in C^\infty_c(\mathbb{R}^d)$. Due to the Hardy inequality there holds

$$\int_{\mathbb{R}^d} \Big(|\nabla\varphi|^2 - \frac{(d-2)^2}{4|x|^2}\,|\varphi|^2\Big)\,dx \geq 0.$$

Now consider

$$W : w \mapsto V(x) + \frac{b(d-2)^2}{4|x|^2},$$

then one can find $c > 0$ such that $W \geq -c1_{|x|\leq R}$, and

$$\begin{aligned}
\langle\varphi, H\varphi\rangle &= \int_{\mathbb{R}^d} \big(|\nabla\varphi|^2 + V|\varphi|^2\big)\,dx \\
&= \int_{\mathbb{R}^d} \big((1-b)|\nabla\varphi|^2 + W|\varphi|^2\big)\,dx + b\int_{\mathbb{R}^d} \Big(|\nabla\varphi|^2 - \frac{(d-2)^2}{4|x|^2}\,|\varphi|^2\Big)\,dx \\
&\geq \int_{\mathbb{R}^d} \big((1-b)|\nabla\varphi|^2 + W|\varphi|^2\big)\,dx \\
&\geq \int_{\mathbb{R}^d} \big((1-b)|\nabla\varphi|^2 - c1_{|x|\leq R}|\varphi|^2\big)\,dx.
\end{aligned}$$

If h is the closed sesquilinear form for H and h_b is the closed sesquilinear form for $H_b := -(1-b)\Delta - c1_{|x|\leq R}$, then by density we obtain $h(\varphi,\varphi) \geq h_b(\varphi,\varphi)$ for all $\varphi \in D(h)$. Therefore, one has the inequality $H \geq H_b$. It follows that $N(H,0) \leq N(H_b,0)$, while $N(H_b,0) < \infty$ by Corollary 8.2. □

The above results can be very informally summarized as follows: if the negative part of V is "sufficiently localized", then the negative spectrum of H is finite. Let us now turn to the typical case in which the discrete spectrum is infinite:

Theorem 8.4. *Let V be a bounded real-valued Kato class potential in $\mathbb{R}^d$ and $H = -\Delta + V$ in $L^2(\mathbb{R}^d)$. Assume that there exist $R > 0$, $c > 0$ and $p \in (0,2)$ such that*

$$V(x) \leq -\frac{c}{|x|^p} \text{ for all } x \text{ with } |x| \geq R.$$

Then the essential spectrum of H is $[0, +\infty)$, and H has infinitely many *negative eigenvalues.*

Proof. The equality for the essental spectrum follows from the discussion of Subsection 5.5. To show the infiniteness of the discrete spectrum one needs to show that $\Lambda_n(H) < 0$ for any $n \in \mathbb{N}$.

Let $\varphi \in C_c^\infty(\mathbb{R}^d)$ with $\operatorname{supp}\varphi \subset \{x : R < |x| < 2R\}$ and $\|\varphi\|_{L^2(\mathbb{R}^d)} = 1$. For $t > 1$ consider $\varphi_t(x) = t^{-d/2}\varphi(x/t)$, then $\|\varphi_t\|_{L^2(\mathbb{R}^d)} = 1$ with $\operatorname{supp}\varphi_t \subset \{x : tR < |x| < 2tR\}$,

$$
\begin{aligned}
\int_{\mathbb{R}^d} |\nabla\varphi_t|^2 dx &= \int_{\mathbb{R}^d} \frac{1}{t^{2+d}} \Big|(\nabla\varphi)\Big(\frac{x}{t}\Big)\Big|^2 dx = at^{-2}, \quad a := \int_{\mathbb{R}^d} |\nabla\varphi|^2 dx,\\
\int_{\mathbb{R}^d} V|\varphi_t|^2 dx &= \int\limits_{tR<|x|<2tR} t^{-d} V(x) \Big|\varphi\Big(\frac{x}{t}\Big)\Big|^2 dx\\
&\le -c \int\limits_{tR<|x|<2tR} t^{-d} \frac{1}{|x|^p} \Big|\varphi\Big(\frac{x}{t}\Big)\Big|^2 dx\\
&= -bct^{-p}, \qquad b := \int\limits_{R<|x|<2R} \frac{|\varphi(x)|^2}{|x|^p}\, dx > 0.
\end{aligned}
$$

As $p < 2$, one can choose $s > 1$ sufficiently large to have

$$
\langle \varphi_t, H\varphi_t\rangle = \int_{\mathbb{R}^d} \big(|\nabla\varphi_t|^2 + V|\varphi_t|^2\big) dx = at^{-2} - bct^{-p} = t^{-2}(a - bct^{2-p}) < 0 \text{ for all } t \ge s.
$$

Now for $n \in \mathbb{N}$ put $\psi_n := \varphi_{2^n s}$, then ψ_n form an orthonormal family and have mutually disjoint supports, in particular,

$$
\langle \psi_m, H\psi_n\rangle = 0 \text{ for } m \ne n, \quad \lambda_n := \langle \psi_n, H\psi_n\rangle < 0.
$$

Now take any $N \in \mathbb{N}$ and consider $F := \operatorname{span}\{\psi_1, \dots, \psi_N\}$, then $\dim F = N$. For

$$
\psi \in F, \quad \psi = \sum_{n=1}^N \xi_n\psi_n, \quad \xi = (\xi_1, \dots, \xi_N) \in \mathbb{C}^N \setminus \{0\},
$$

one has

$$
\frac{\langle \psi, H\psi\rangle}{\langle \psi, \psi\rangle} = \frac{\sum_{n=1}^N \lambda_n|\xi_n|^2}{\sum_{n=1}^N |\xi_n|^2} \le \max\{\lambda_1, \dots, \lambda_N\} < 0.
$$

Therefore,

$$
\Lambda_N(H) \le \sup_{\psi\in F,\, \psi\ne 0} \frac{\langle \psi, H\psi\rangle}{\langle \psi, \psi\rangle} \le \max\{\lambda_1, \dots, \lambda_N\} < 0.
$$

As $N \in \mathbb{N}$ was arbitrary, the assertion follows by the min-max principle. □

8.2 Weyl-type asymptotics

Under suitable assumption on the potential V one can establish some information on the behavior of $N(H, \lambda)$ as λ tends to $\inf \operatorname{spec}_{\mathrm{ess}} H$, which is quite similat to the

Weyl asymptotics for the Laplacians in bounded domains (and some components of that proof will be used).

Consider first the case $\inf\operatorname{spec} H = +\infty$.

Theorem 8.5. *Let $d \geq 2$ and $V \in C^1(\mathbb{R}^d)$ such that for some $c_1, c_2, c_3 > 0$ and $\beta > 1$ there holds, for all $x \in \mathbb{R}^d$,*

$$c_1\big(|x|^\beta - 1\big) \leq V(x) \leq c_2\big(|x|^\beta + 1\big), \tag{8.2}$$

$$\big|\nabla V(x)\big| \leq c_3(|x|^{\beta-1} + 1), \tag{8.3}$$

For $\lambda \in \mathbb{R}$ denote

$$g(\lambda, V) = \frac{\omega_d}{(2\pi)^d} \int_{V(x)\leq\lambda} \big(\lambda - V(x)\big)^{d/2} \, dx,$$

where ω_d is the volume of the unit ball in $\mathbb{R}^d$. Then for $H = -\Delta + V$ in $L^2(\mathbb{R}^d)$ there holds

$$N(H, \lambda) = g(\lambda, V) + o\big(g(\lambda, V)\big) \text{ as } \lambda \to +\infty.$$

Proof. Recall that H is generated by the closed sesquilinear form

$$h(u, v) = \int_\Omega \big(\overline{\nabla u} \cdot \nabla v + V_\pm uv\big) \, dx, \quad D(h) = \Big\{u \in H^1(\mathbb{R}^d) : \int_{\mathbb{R}^d} V|u|^2 \, dx < \infty\Big\},$$

and due to $V(x) \to +\infty$ for $|x| \to +\infty$ the spectrum is purely discrete, and $N(H, \lambda)$ is well-defined for all $\lambda \in \mathbb{R}$. For $n = (n_1, \dots, n_d) \in \mathbb{Z}^d$ denote

$$\Omega_n := (n_1, n_1 + 1) \times \cdots \times (n_d, n_d + 1), \quad V_n^- := \min_{x\in\overline{\Omega_n}} V(x), \quad V_n^+ := \max_{x\in\overline{\Omega_n}} V(x)$$

and introduce $V_\pm : \mathbb{R}^d :\to \mathbb{R}$ by

$$V_\pm(x) = V_n^\pm \text{ for } x \in \Omega_n, \text{ in particular, } V_- \leq V \leq V_+.$$

Set

$$\Omega := \bigcup_{n\in\mathbb{Z}^d} \Omega_n \qquad (\text{then } \overline{\Omega} = \mathbb{R}^d)$$

and consider the self-adjoint operators $H_\pm$ in $L^2(\mathbb{R}^d)$ given by their sesquilinear forms $h_\pm$,

$$h_\pm(u, v) = \int_\Omega \big(\overline{\nabla u} \cdot \nabla v + V_\pm uv\big) \, dx,$$

$$D(h_-) = \Big\{u \in H^1(\Omega) : \int_{\mathbb{R}^d} V_-|u|^2 \, dx < \infty\Big\},$$

$$D(h_+) = \Big\{u \in H_0^1(\Omega) : \int_{\mathbb{R}^d} V_+|u|^2 \, dx < \infty\Big\}.$$

We have then $H_- \leq H \leq H_+$ (see Definition 6.7), hence,

$$N(H_+, \lambda) \leq N(H, \lambda) \leq N(H_-, \lambda) \text{ for any } \lambda \in \mathbb{R}. \tag{8.4}$$

We then remark that $H_\pm$ are direct sums of operators $H_n^\pm$ acting in $L^2(\Omega_n)$,

$$H_n^\pm = T_n^\pm + V_n^\pm,$$

where $T_n^\pm$ is the Neumann (-) / Dirichlet (+) Laplacian in Ω_n, hence,

$$N(H_\pm, \lambda) = \sum_{n\in\mathbb{Z}^d} N(H_n^\pm, \lambda) = \sum_{n\in\mathbb{Z}^d} N(T_n^\pm, \lambda - V_n^\pm) \equiv \sum_{n: V_n^\pm<\lambda} N(T_n^\pm, \lambda - V_n^\pm).$$

We give the rest of the proof for $d = 2$ only (other dimensions need a simple readjustment of the remainders). As shown during the proof of Proposition 7.9 (the last equation in the proof), for some constant $c > 0$ there holds

$$\Big| N(T_n^\pm, E) - \frac{1}{4\pi}\, E_+ \Big| \le c\big(\sqrt{E_+} + 1\big),$$

where we denote

$$E_+ = E \text{ for } E \ge 0 \text{ and } E_+ = 0 \text{ for } E < 0.$$

Hence,

$$\Big| N(T_n^\pm, \lambda - V_n^\pm) - \frac{1}{4\pi}\, (\lambda - V_n^\pm)_+ \Big| \le c\big(\sqrt{(\lambda - V_n^\pm)_+} + 1\big). \tag{8.5}$$

so summing over n we arrive at

$$\Big| N(H_\pm, \lambda) - \frac{1}{4\pi} \int_{V_\pm(x)<\lambda} \big(\lambda - V_\pm(x)\big)\, dx \Big| \le c \int_{V_\pm(x)<\lambda} \big(\sqrt{\lambda - V_\pm(x)} + 1\big)\, dx.$$

Hence, using (8.4) and denoting

$$g(\lambda, V_\pm) := \frac{1}{4\pi} \int_{V_\pm(x)<\lambda} \big(\lambda - V_\pm(x)\big)\, dx,$$
$$\rho(\lambda, V_\pm) := \int_{V_\pm(x)<\lambda} \big(\sqrt{\lambda - V_\pm(x)} + 1\big)\, dx,$$

we arrive at

$$g(\lambda, V_+) - c\rho(\lambda, V_+) \le N(H, \lambda) \le g(\lambda, V_-) + c\rho(\lambda, V_-) \text{ for any } \lambda \in \mathbb{R}, \tag{8.6}$$

and we recall that

$$g(\lambda, V_+) \le g(\lambda, V) \le g(\lambda, V_-). \tag{8.7}$$

Now let us obtain some asymptotic estimates for all terms in (8.6).
For $x \in \Omega_n$ one has $|x - n| \le \sqrt{2}$, therefore,

$$|x| \le |n| + \sqrt{2}, \quad |n| \le |x| + \sqrt{2}.$$

For $x, y, z \in \Omega_n$ there holds, with the help of (8.3),

$$\big|V(y) - V(z)\big| \le |y - z| \sup_{y\in\Omega_n} |\nabla V(y)| \le \sqrt{2} \sup_{y\in\Omega_n} |\nabla V(y)|,$$

$$
\begin{aligned}
\sup_{y\in\Omega_n} |\nabla V(y)| &\le \sup_{y\in\Omega_n} c_3\big(|y|^{\beta-1}+1\big)\\
&\le c_3\big((|n|+\sqrt{2})^{\beta-1}+1\big)\\
&\le c_3\big((|x|+2\sqrt{2})^{\beta-1}+1\big),
\end{aligned}
$$

and it follows that $V_+(x)-V_-(x)\le \sqrt{2}c_3\big((|x|+2\sqrt{2})^{\beta-1}+1\big)$ for all $x\in\mathbb{R}^d$. To have a simpler writing, we choose $c_4>0$ sufficiently large to obtain

$$
V_+(x)-V_-(x)\le c_4(|x|^{\beta-1}+1) \text{ for all } x\in\mathbb{R}^d,
$$

and then, with some $c_5, c_6\ge 0$,

$$
c_5\big(|x|^\beta-1\big)\le V_-(x)\le V_+(x)\le c_6\big(|x|^\beta+1\big) \text{ for all } x\in\mathbb{R}^d.
$$

Using $\{x: V_+(x)<\lambda\}\subset\{x:V_-(x)<\lambda\}$ and $|x_+-y_+|\le|x-y|$ we estimate

$$
\begin{aligned}
g(\lambda,V_-)-g(\lambda,V_+) &= \frac{1}{4\pi}\int_{V_-(x)<\lambda}\Big(\big(\lambda-V_-(x)\big)_+-\big(\lambda-V_+(x)\big)_+\Big)\,dx\\
&\le \frac{1}{4\pi}\int_{V_-(x)<\lambda}\big(V_+(x)-V_-(x)\big)\,dx\\
&\le \frac{c_4}{4\pi}\int_{V_-(x)<\lambda}\big(|x|^{\beta-1}+1\big)\,dx\\
&\le \frac{c_4}{4\pi}\int_{c_5\left(|x|^\beta-1\right)<\lambda}\big(|x|^{\beta-1}+1\big)\,dx\\
&= \frac{c_4}{4\pi}\,2\pi\int_0^{\left(\frac{\lambda+c_5}{c_5}\right)^{1/\beta}} r\big(r^{\beta-1}+1\big)\,dr\\
&= \frac{c_4}{4\pi}\,2\pi\Big[\frac{1}{\beta+1}\Big(\frac{\lambda+c_5}{c_5}\Big)^{(\beta+1)/\beta}+\frac12\Big(\frac{\lambda+c_5}{c_5}\Big)^{2/\beta}\Big],
\end{aligned}
$$

and for $\lambda\to+\infty$ one has

$$
g(\lambda,V_-)-g(\lambda,V_+)=O(\lambda^{1+1/\beta}). \tag{8.8}
$$

Using

$$
\frac{1}{4\pi}\int_{c_2(|x|^\beta+1)<\lambda}\big(\lambda-c_2(|x|^\beta+1)\big)\,dx\le g(\lambda,V)\le\frac{1}{4\pi}\int_{c_1(|x|^\beta-1)<\lambda}\big(\lambda-c_1(|x|^\beta-1)\big)\,dx
$$

and similar computations in polar coordinates one concludes that there exist $b_\pm>0$ such that

$$
b_-\lambda^{1+2/\beta}\le g(\lambda,V)\le b_+\lambda^{1+2/\beta} \text{ for } \lambda\to+\infty.
$$

By combining the last inequality with (8.7) and (8.8) we conclude that

$$
g(\lambda,V_\pm)=g(\lambda,V)+o\big(g(\lambda,V)\big) \text{ for large } \lambda.
$$

In view of (8.6) it remains to show that $\rho(\lambda, V_\pm) = o\big(g(\lambda, V)\big)$ for large λ. One easily estimates, using the polar coordinates,

$$\rho(\lambda, V_\pm) \le \int_{c_5(|x|^\beta - 1) < \lambda} \sqrt{\lambda - c_5(|x|^\beta - 1)}\, dx$$

$$= 2\pi \int_0^{\left(\frac{\lambda + c_5}{c_5}\right)^{1/\beta}} r\Big(\sqrt{\lambda - c_5(r^\beta - 1)} + 1\Big)\, dr,$$

and using the substitution $r = (\frac{\lambda+c_5}{c_5})^{1/\beta}\, dt$ one easily shows that

$$\rho(\lambda, V_\pm) = O(\lambda^{1/2+2/\beta}) = o\big(g(\lambda, V)\big),$$

which completes the proof. □

By some small modifications one can extend the proof to the case $d = 1$ as well (the term on the right-hand side of (8.5) will be different: check it!), and the result holds in exactly the same formulation. The regularity assumptions on V that we used are not canonical, and can be weakened in many directions.

An almost identical analysis is possible for $\inf \operatorname{spec}_{\mathrm{ess}} H < \infty$ as well (which is a very good exercise for motivated readers):

Theorem 8.6. *Let $V \in C^1(\mathbb{R}^d)$ such that for some $c_1, c_2, c_3 > 0$ and $p \in (0, 2)$ there holds, for all $x \in \mathbb{R}^d$,*

$$-\frac{c_1}{(|x| + 1)^p} \le V(x) \le -\frac{c_2}{(|x| + 1)^p}, \quad \big|\nabla V(x)\big| \le \frac{c_3}{(|x| + 1)^{p+1}}, \tag{8.9}$$

For $\lambda > 0$ denote

$$g(\lambda, V) = \frac{\omega_d}{(2\pi)^d} \int_{V(x) \le -\lambda} \big(-\lambda - V(x)\big)^{d/2} dx,$$

then for $H = -\Delta + V$ there holds

$$N(H, -\lambda) = g(\lambda, V) + o\big(g(\lambda)\big) \text{ as } \lambda \to 0^+.$$

Exercise 21. Let $V : \mathbb{R}^2 \to \mathbb{R}$ be real-valued, continuous, with compact support. Consider the family of operators $T_\lambda = -\Delta + \lambda V$ in $L^2(\mathbb{R}^2)$, $\lambda > 0$, and denote by $N(\lambda)$ the number of strictly negative eigenvalues of T_λ, which is finite due to the previous consideration. We are going to study the behavior of this counting function as $\lambda \to +\infty$ and obtain a certain analog of the Weyl asymptotics.

Without loss of generality we assume that the support of V is included into the open square $B := (0, a) \times (0, a)$ with some $a > 0$ (this can be achieved by a suitable shift of the coordinates). Futhermore, for any function $f : \mathbb{R}^2 \to \mathbb{R}$ we define a function f_- by $f_-(x) := \max\big(0, -f(x)\big)$.

Let $n \in \mathbb{N}$. For $m = (m_1, m_2) \in (1, \dots, n) \times (1, \dots, n)$ consider the open squares

$$B_n(m) = \Big(\frac{m_1 - 1}{n}\, a, \frac{m_1}{n}\, a\Big) \times \Big(\frac{m_2 - 1}{n}\, a, \frac{m_2}{n}\, a\Big)$$

and set $B_n := \bigcup_m B_n(m)$. Introduce new potentials U_n^- and U_n^+ as follows:

$$U_n^-(x) = \begin{cases} U_{n,m}^- := \inf_{x \in B_n(m)} V(x), & x \in B_n(m), \\ 0, & \text{otherwise}, \end{cases}$$

$$U_n^+(x) = \begin{cases} U_{n,m}^+ := \sup_{x \in B_n(m)} V(x), & x \in B_n(m), \\ 0, & \text{otherwise}, \end{cases}$$

and denote by $H_n^\pm$ the operators in $L^2(\mathbb{R}^2)$ given respectively by the sesquilinear forms

$$t_n^\pm(u, u) = \int_{B_n \cup (\mathbb{R}^2 \setminus \overline{B})} \big|\nabla u(x)|^2 dx + \lambda \int_{\mathbb{R}^2} U_n^\pm \big|u(x)|^2 dx,$$
$$D(t_n^+) = H_0^1\big(B_n \cup (\mathbb{R}^2 \setminus \overline{B})\big), \quad D(t_n^-) = H^1\big(B_n \cup (\mathbb{R}^2 \setminus \overline{B})\big).$$

1. Let $\varepsilon > 0$. Show that one can find $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that for all $n \ge n_0$ one has
$$\Big| \int_{\mathbb{R}^2} \big(U_n^\pm\big)_-(x) dx - \int_{\mathbb{R}^2} V_-(x) dx \Big| < \varepsilon.$$

2. Show that $H_n^\pm$ can be represented as direct sums of some operators $L_{n,m}^\pm$ acting in $L^2\big(B_n(m)\big)$ and $L^\pm$ in $L^2(\mathbb{R}^2 \setminus B)$ whose spectra can be computed explicitly.

3. Denote by $N_n^\pm(\lambda)$ the number of strictly negative eigenvalues of $H_n^\pm$. Show that these numbers are finite and that $N_n^+(\lambda) \le N(\lambda) \le N_n^-(\lambda)$.

4. Use the Weyl asymptotics for $L_{n,m}^\pm$ to show that
$$\lim_{\lambda \to +\infty} \frac{N_n^\pm(\lambda)}{\lambda} = \frac{1}{4\pi} \int_{\mathbb{R}^2} \big(U_n^\pm\big)_-(x) dx.$$

5. Show the estimate
$$\lim_{\lambda \to +\infty} \frac{N(\lambda)}{\lambda} = \frac{1}{4\pi} \int_{\mathbb{R}^2} V_-(x) dx.$$

8.3 Decay of eigenfunctions

In the present subsection we would like to discuss the decay of the eigenfunctions of Schrödinger operators corresponding to discrete eigenvalues. By definition, the eigenfunctions belong to L^2, but in many cases a better decay can be seen. For example, the eigenfunction $f(x) = e^{-x^2/2}$ of $-d^2/dx^2 + x^2$ in $L^2(\mathbb{R})$ shows a very rapid decay at infinity. In the present course we only prove (under some additional assumptions on V) that the eigenfunctions of $-\Delta + V$ decay exponentially: a better estimate in terms of V is possible, but the argument becomes more involved. Results of this type are often referred to as Agmon estimates.

During the whole subsection we assume that the potential V is $L^\infty_{\text{loc}}(\mathbb{R}^d)$ and semi-bounded from below (hence, real-valued) and consider $H = -\Delta + V$ in $L^2(\mathbb{R}^d)$.

We will start with an integral identity:

Lemma 8.7. *Let $\Phi \in C^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ be real-valued with $\nabla\Phi \in L^\infty(\mathbb{R}^d)$. Let u be an eigenfunction of H with eigenvalue E, then*

$$\int_{\mathbb{R}^d} \Big(\big|\nabla(e^\Phi u)\big|^2 + V|e^\Phi u|^2 \Big)\, dx = \int_{\mathbb{R}^d} e^{2\Phi} \big(E + |\nabla\Phi|^2\big)|u|^2 dx.$$

Proof. For $v \in C_c^\infty(\mathbb{R}^d)$ one has

$$\begin{aligned}
&\int_{\mathbb{R}^d} \big|\nabla(e^\Phi v)\big|^2\, dx = \int_{\mathbb{R}^d} \big|v e^\Phi \nabla\Phi + e^\Phi \nabla v\big|^2\, dx \\
&= \int_{\mathbb{R}^d} e^{2\Phi}|v|^2\, |\nabla\Phi|^2\, dx + \int_{\mathbb{R}^d} e^{2\Phi}|\nabla v|^2\, dx + \int_{\mathbb{R}^d} e^{2\Phi}\nabla\Phi \cdot \big(\overline{v}\nabla v + v\overline{\nabla v}\big)\, dx. \qquad (8.10)
\end{aligned}$$

The last summand can be transformed using the integration by parts as follows:

$$\begin{aligned}
\int_{\mathbb{R}^d} e^{2\Phi}\nabla\Phi \cdot \big(\overline{v}\nabla v + v\overline{\nabla v}\big)\, dx &= \frac{1}{2}\int_{\mathbb{R}^d} \nabla(e^{2\Phi})\big(\overline{v}\nabla v + v\overline{\nabla v}\big)\, dx \\
&= -\frac{1}{2}\int_{\mathbb{R}^d} e^{2\Phi}\nabla \cdot \big(\overline{v}\nabla v + v\overline{\nabla v}\big)\, dx,
\end{aligned}$$

while $\nabla \cdot \big(\overline{v}\nabla v + v\overline{\nabla v}\big) = 2|\nabla v|^2 + \Big(\overline{v}\,\Delta v + \overline{\Delta v}\, v\Big) = 2|\nabla v|^2 + 2\Re(\overline{v}\,\Delta v)$. Therefore,

$$\int_{\mathbb{R}^d} e^{2\Phi}\nabla\Phi \cdot \big(\overline{v}\nabla v + v\overline{\nabla v}\big)\, dx = -\int_{\mathbb{R}^d} e^{2\Phi}\Big(|\nabla v|^2 + \Re(\overline{v}\,\Delta u)\Big)\, dx,$$

and the substitution into (8.10) gives

$$\int_{\mathbb{R}^d} \big|\nabla(e^\Phi v)\big|^2\, dx = \int_{\mathbb{R}^d} e^{2\Phi}\big(-\Re(\overline{v}\,\Delta v) + |\nabla\Phi|^2|v|^2\big)dx,$$

and then

$$\int_{\mathbb{R}^d} \Big(\big|\nabla(e^\Phi v)\big|^2 + V|e^\Phi v|^2 \Big)^2\, dx = \int_{\mathbb{R}^d} e^{2\Phi}\big(-\Re(\overline{v}\,\Delta v) + |\nabla\Phi|^2|v|^2 + \overline{v}\,V v\big)dx$$

$$= \int_{\mathbb{R}^d} e^{2\Phi}\big(\Re(\overline{v}\, Hv) + |\nabla\Phi|^2|v|^2\big)\,dx.$$

This extends to all $v \in D(H)$ by density (as Φ and $\nabla\Phi$ are assumed bounded). In particular, for $v := u$ one has, due to $Hu = Eu$ and $E \in \mathbb{R}$,

$$\int_{\mathbb{R}^d} \Big(\big|\nabla(e^{\Phi}v)\big|^2 + V|e^{\Phi}v|^2\Big)^2\,dx = \int_{\mathbb{R}^d} e^{2\Phi}\big(E + |\nabla\Phi|^2\big)|u|^2dx. \qquad \square$$

Theorem 8.8 (Agmon estimate). *Let u be an eigenfunction of H with eigenvalue E satisfying the strict inequality*

$$E < \liminf_{|x|\to+\infty} V(x),$$

then there exists $A > 0$ such that

$$\int_{\mathbb{R}^d} e^{A|x|}\big(|\nabla u|^2 + |u|^2\big)\,dx < \infty.$$

We remark that the constant A and the value of the integral can be controlled: the interested reader will be able to recover all necessary estimates from the proof.

Proof. Pick $R > 0$ and $v_0 > E$ such that $V(x) \ge v_0$ for all $|x| \ge R$, then the set

$$\{x : V(x) < v_0\} \subset \{x : |x| < R\}$$

is bounded. Recall that this implies $\operatorname{spec}_{\text{ess}} H \subset [v_0, +\infty)$ by Theorem 8.1. Denote

$$B := \big\{x : |x| < R\big\}, \quad B^c := \big\{x : |x| > R\big\}.$$

Consider $\phi : x \mapsto \sqrt{|x|^2 + 1}$: this is a C^∞ function with $|\nabla\phi| \le 1$. Let us pick a non-decreasing C^1 function $\theta : \mathbb{R} \to \mathbb{R}$ with the following properties:

$$\theta(t) = t \text{ for } t \le 0, \quad \theta(t) = 1 \text{ for } t \ge 2, \quad \theta'(t) \le 1 \text{ for all } t,$$

and for $L > 0$ consider $\psi(x) = L + \theta\big(\phi(x) - L\big)$, then $\psi \in C^1$ with

$$\begin{gathered} 0 \le \psi \le L + 1, \quad |\nabla\psi| = |(\theta' \circ \phi)\nabla\phi| \le |\nabla\phi| \le 1, \\ \psi(x) = \phi(x) \text{ for } \phi(x) \le L. \end{gathered}$$

For $a > 0$, to be chosen later, consider the function $\Phi : x \mapsto a\psi(x)$. Then Lemma 8.7 is applicable, and

$$\int_{\mathbb{R}^d} e^{2\Phi}\big(E + |\nabla\Phi|^2\big)|u|^2dx = \int_{\mathbb{R}^d} \Big(\big|\nabla(e^{\Phi}u)\big|^2 + V|e^{\Phi}u|^2\Big)\,dx.$$

One estimates $V \ge \inf V$ in B, $V \ge v_0$ in B^c, then

$$\int_B e^{2\Phi}\big(E + |\nabla\Phi|^2\big)|u|^2dx + \int_{B^c} e^{2\Phi}\big(E + |\nabla\Phi|^2\big)|u|^2dx$$

$$\geq \int_{\mathbb{R}^d} \big|\nabla(e^{\Phi}u)\big|^2\,dx + \inf V \int_B |e^{\Phi}u|^2\,dx + v_0 \int_{B^c} |e^{\Phi}u|^2\,dx,$$

and

$$\int_{\mathbb{R}^d} \big|\nabla(e^{\Phi}u)\big|^2\,dx + \int_{B^c} e^{2\Phi}\big(v_0 - E - |\nabla\Phi|^2\big)|u|^2\,dx \leq \int_B e^{2\Phi}\big(E - \inf V + |\nabla\Phi|^2\big)|u|^2 dx.$$

One has $v_0 - E > 0$ and $|\nabla\Phi| \leq a$, and now we assume that a is choosen sufficiently small to have

$$v_0 - E - |\nabla\Phi|^2 \geq \delta := v_0 - E - a^2 > 0,$$

then

$$\int_{\mathbb{R}^d} \big|\nabla(e^{\Phi}u)\big|^2\,dx + \delta \int_{B^c} e^{2\Phi}|u|^2\,dx \leq (E - \inf V + a^2) \int_B e^{2\Phi}|u|^2 dx$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^d} \Big(\big|\nabla(e^{\Phi}u)\big|^2 + e^{2\Phi}|u|^2\Big)\,dx \\
&\qquad = \int_{\mathbb{R}^d} \big|\nabla(e^{\Phi}u)\big|^2\,dx + \int_{B^c} e^{2\Phi}|u|^2\,dx + \int_B e^{2\Phi}|u|^2\,dx \\
&\qquad\qquad \leq \bigg[\Big(1 + \frac{1}{\delta}\Big)(E - \inf V + a^2) + 1\bigg] \int_B e^{2\Phi}|u|^2 dx. \qquad (8.11)
\end{aligned}$$

For $x \in B$ one has $\Phi(x) \leq a\psi(\sqrt{R^2+1})$. One can assume that $L > \sqrt{R^2+1}$, then $\psi(\sqrt{R^2+1}) = \sqrt{R^2+1}$, and $\Phi(x) \leq a\sqrt{R^2+1}$ for all $x \in B$. Then it follows from (8.11) that

$$\int_{\mathbb{R}^d} \Big(\big|\nabla(e^{\Phi}u)\big|^2 + e^{2\Phi}|u|^2\Big)\,dx \leq C, \qquad (8.12)$$

$$C := \bigg[\Big(1 + \frac{1}{\delta}\Big)(E - \inf V + a^2) + 1\bigg] e^{2a\sqrt{R^2+1}} \int_B |u|^2.$$

We have

$$\begin{aligned}
\big|\nabla(e^{\Phi}u)\big|^2 = |\nabla u + u\nabla\Phi|^2 e^{2\Phi} &= \big(|\nabla u|^2 + 2\Re(\overline{u}\nabla u \cdot \nabla\Phi) + |u\nabla\Phi|^2\big)e^{2\Phi} \\
&\geq \big(|\nabla u|^2 - 2|\overline{u}\nabla u \cdot \nabla\Phi|\big)e^{2\Phi}
\end{aligned}$$

and (using $|2xy| \leq \frac{1}{2}\,|x|^2 + 2|y|^2$)

$$2|\overline{u}\nabla u \cdot \nabla\Phi| \leq \frac{1}{2}|\nabla u|^2 + 2|u\nabla\Phi|^2 \leq \frac{1}{2}|\nabla u|^2 + 2a^2|u|^2.$$

Therefore,

$$\big|\nabla(e^{\Phi}u)\big|^2 \geq \Big(\frac{1}{2}|\nabla u|^2 - 2a^2|u|^2\Big)e^{2\Phi}, \text{ i.e. } |\nabla u|^2 e^{2\Phi} \leq 2\big|\nabla(e^{\Phi}u)\big|^2 + 4a^2 e^{2\Phi}|u|^2.$$

It follows from (8.12) that

$$\begin{aligned}\int_{\mathbb{R}^d} e^{2\Phi}\big(|\nabla u|^2+|u|^2\big)\,dx &\le 2\int_{\mathbb{R}^d}\big|\nabla(e^{\Phi}u)\big|^2\,dx+(4a^2+1)\int_{\mathbb{R}^d}e^{2\Phi}|u|^2\,dx\\ &\le (4a^2+3)C=:B,\end{aligned}$$

i.e. (using the explicit expression of Φ),

$$\int_{\mathbb{R}^d}\exp\Big[2a\Big(L+\theta\big(\phi(x)-L\big)\Big)\Big]\big(|\nabla u|^2+|u|^2\big)\,dx\le B.$$

The constant B is independent of L, and $L+\theta\big(\phi(x)-L\big)$ converges monotonically (as $\theta'\le 1$) to $\phi(x)=\sqrt{x^2+1}$ for any x as $L\to+\infty$. Hence, by the monotone convergence theorem,

$$\int_{\mathbb{R}^d}e^{2a\sqrt{x^2+1}}\big(|\nabla u|^2+|u|^2\big)\,dx\le B.$$

Using $e^{2a|x|}\le e^{2a\sqrt{x^2+1}}$ one obtains the claim with $A=2a$. □

One of the classical applications of the Agmon estimates is the comparison between eigenvalue problems in the whole space (or in unbounded domains) and in bounded domains. Such results are of importance when computing eigenvalues numerically: all numerical computations can only be performed for bounded domains.

From now on we will use the following notation: for $j\in\mathbb{N}$,

$$\begin{aligned}E_j(A)=\,&\text{the } j\text{th eigenvalue of a lower semibounded operator } A\\ &\text{(counted in the non-decreasing order and according to the multiplicities).}\end{aligned}$$

Theorem 8.9. *Let $v_0<\liminf_{|x|\to+\infty}V(x)$: then the spectrum of H in $(-\infty,v_0]$ consists of $N<\infty$ eigenvalues (Theorem 8.1). Assume that the potential V has at most polynomial growth at infinity: there exist $c>0$ and $m>0$ such that*

$$V(x)\le c(|x|+1)^m \text{ for all } x\in\mathbb{R}^d.$$

For $R>0$, consider $H_R=-\Delta+V$ in $L^2(B_R)$ with the Dirichlet boundary conditions, where $B_R=\{x:|x|<R\}$. Then there exists $a>0$ such that for any $n\in\{1,\dots,N\}$ there holds

$$E_n(H)=E_n(H_R)+O(e^{-aR}) \text{ as } R\to+\infty.$$

Remark that similar results can be obtained without the polynomial growth assumption, but a more advanced version of Agmon estimate is needed then (which takes into account the growth of V at infinity).

Proof. Let h and h_R be the sesquilinear forms for H and H_R. Consider the map

$$J: D(h_R)=H^1_0(B_R)\to D(h),\quad Ju:=\text{the extension of } u \text{ by zero},$$

then we are in the situation of Proposition 6.6, and $\Lambda_n(H) \le \Lambda_n(H_R)$ for any $n \in \mathbb{N}$ and any $R > 0$. We have $\Lambda_n(H) = E_n(H)$ for $n = 1, \dots, N$ and $\Lambda_n(H_R) = E_n(H_R)$ for all n (as H_R has compact resolvent), hence,

$$E_n(H) \le E_n(H_R) \text{ for all } n = 1, \dots, N \text{ and } R > 0. \tag{8.13}$$

Now we prove the lower bound for $E_n(H)$. Let u_j be eigenfunctions of H for the eigenvalues $E_j(H)$, $j = 1, \dots, n$, building an orthonormal family, i.e.

$$\langle u_j, u_k \rangle_{L^2(\mathbb{R}^d)} = \delta_{j,k}, \quad h(u_j, u_k) = E_j(H)\delta_{j,k}.$$

Due to Agmon estimate (Theorem 8.1) one can find $A > 0$ and $B > 0$ such that

$$\int_{\mathbb{R}^d} e^{A|x|}\big(|u_j|^2 + |\nabla u_j|^2\big)\,dx \le B \text{ for all } j = 1, \dots, N. \tag{8.14}$$

Let $\chi : \mathbb{R} \to \mathbb{R}$ be a C^∞ function with

$$0 \le \chi \le 1, \quad \chi(t) = 1 \text{ for } t < \frac{1}{2}, \quad \chi(t) = 0 \text{ for } t \ge \frac{3}{4}.$$

Consider the functions

$$\chi_R(x) := \chi\big(\tfrac{|x|}{R}\big) \quad \text{and} \quad v_j := \chi_R u_j, \quad j = 1, \dots, N,$$

and the subspace

$$U_n := \operatorname{span}(v_1, \dots, v_n).$$

As $\chi_R(x) = 0$ for $|x| > \frac{3}{4}R$, one has the inclusion $U_n \subset H^1_0(B_R) \equiv D(h_R)$. We are going to show that $\dim U_n = n$ and that for any non-zero $v \in U_n$ there holds $h_R(v,v)/\langle v, v\rangle_{L^2(B_R)} \le E_n(H) + O(e^{-aR})$, then the result will follow but the min-max principle.

For the quantities

$$\alpha_{j,k} := \int_{\mathbb{R}^d} (1 - \chi_R^2)\overline{u_j}\, u_k\, dx$$

one has, with the help of (8.14),

$$\begin{aligned} |\alpha_{j,j}| &= \int_{\mathbb{R}^d} (1 - \chi_R^2)|u_j|^2\,dx \le \int_{|x| > \frac{R}{2}} |u_j|^2\,dx \\ &\le e^{-AR/2} \int_{|x| > \frac{R}{2}} e^{A|x|}|u_j|^2\,dx \le B e^{-AR/2}, \end{aligned}$$

and $|\alpha_{j,k}| \le \dfrac{1}{2}\big(|\alpha_{j,j}| + |\alpha_{k,k}|\big) \le B e^{-AR/2}$. Therefore,

$$\langle v_j, v_k\rangle_{L^2(B_R)} = \langle u_j, u_k\rangle_{L^2(\mathbb{R}^d)} - \alpha_{j,k} = \delta_{j,k} + O(e^{-aR/2}). \tag{8.15}$$

In particular, the functions $v_1, \dots, v_n$ are linearly independent for large R (as their Gramian matrix is non-degenerate), and $\dim U_n = n$.

Now consider

$$\int_{B_R} \overline{\nabla v_j} \cdot \nabla v_k \, dx = \int_{\mathbb{R}^d} (\overline{\chi_R \nabla u_j + u_j \nabla \chi_R}) \cdot (\chi_R \nabla u_k + u_k \nabla \chi_R) \, dx$$
$$= \int_{\mathbb{R}^d} \chi_R^2 \overline{\nabla u_j} \cdot \nabla u_k \, dx + \int_{\mathbb{R}^d} 2\Re(\overline{u_j} \chi_R \nabla u_k \cdot \nabla \chi_R) \, dx \qquad (8.16)$$
$$+ \int_{\mathbb{R}^d} |\nabla \chi_R|^2 \overline{u_j} u_k \, dx$$

Introducing

$$\beta_{j,k} = \int_{\mathbb{R}^d} (1 - \chi_R^2) \overline{\nabla u_j} \cdot \nabla u_k \, dx$$

we obtain, again with the help of (8.14),

$$|\beta_{j,j}| = \int_{\mathbb{R}^d} (1 - \chi_R^2) |\nabla u_j|^2 \, dx \leq \int_{|x| > \frac{R}{2}} |\nabla u_j|^2 \, dx$$
$$\leq e^{-AR/2} \int_{|x| > \frac{R}{2}} e^{A|x|} |\nabla u_j|^2 \, dx \leq B e^{-AR/2},$$

and $|\beta_{j,k}| \leq \frac{1}{2}\big(|\beta_{j,j}| + |\beta_{k,k}|\big) \leq B e^{-AR/2}$, and then

$$\int_{\mathbb{R}^d} \chi_R^2 \overline{\nabla u_j} \cdot \nabla u_k \, dx = \int_{\mathbb{R}^d} \overline{\nabla u_j} \cdot \nabla u_k \, dx - \beta_{j,k}$$
$$= \int_{\mathbb{R}^d} \overline{\nabla u_j} \cdot \nabla u_k \, dx + O(e^{-AR/2}). \qquad (8.17)$$

We then estimate

$$\Big| \int_{\mathbb{R}^d} |\nabla \chi_R|^2 \overline{u_j} u_k \, dx \Big| \leq \frac{\|\chi'\|_\infty^2}{R^2} \int_{|x| \geq R/2} \frac{|u_j|^2 + |u_k|^2}{2} \, dx$$
$$\leq \frac{\|\chi'\|_\infty^2}{R^2} e^{-AR/2} \int_{|x| \geq R/2} e^{A|x|} \frac{|u_j|^2 + |u_k|^2}{2} \, dx \qquad (8.18)$$
$$\leq \frac{B\|\chi'\|_\infty^2}{R^2} e^{-AR/2}.$$

Finally,

$$\Big| \int_{\mathbb{R}^d} 2\Re(\overline{u_j} \chi_R \nabla u_k \cdot \nabla \chi_R) \, dx \Big| \leq \frac{\|\chi'\|_\infty}{R} \int_{|x| > R/2} |\overline{u_j} \nabla u_k| \, dx$$
$$\leq \frac{\|\chi'\|_\infty}{R} \int_{|x| > R/2} \frac{|u_j|^2 + |\nabla u_k|^2}{2} \, dx$$
$$\leq \frac{\|\chi'\|_\infty}{R} e^{-AR/2} \int_{|x| > R/2} e^{A|x|} \frac{|u_j|^2 + |\nabla u_k|^2}{2} \, dx \qquad (8.19)$$
$$\leq \frac{B\|\chi'\|_\infty}{R} e^{-AR/2}.$$

Using the three estimates (8.17), (8.18), (8.19) in the equality (8.16) one arrives at

$$\int_{B_R} \overline{\nabla v_j} \cdot \nabla v_k \, dx = \int_{\mathbb{R}^d} \overline{\nabla u_j} \cdot \nabla u_k \, dx + O(e^{-AR/2}). \tag{8.20}$$

Furthermore, using the polynomial estimate for V at infinity, for

$$\gamma_{j,k} := \int_{\mathbb{R}^d} (1-\chi_R^2) V \overline{u_j} u_k \, dx$$

one has

$$|\gamma_{j,k}| \le c \int_{|x|>R/2} \big(|x|+1\big)^m \overline{u_j} u_k \, dx \le c \int_{|x|>R/2} \big(|x|+1\big)^m \frac{|u_j|^2+|u_k|^2}{2}\, dx.$$

If R is sufficiently large, for all x with $|x|>R/2$ there holds

$$\big(|x|+1\big)^m \le e^{A|x|/2} \le e^{-AR/4} e^{A|x|},$$

therefore,

$$|\gamma_{j,k}| \le c e^{-AR/4} \int_{|x|>R/2} e^{A|x|} \frac{|u_j|^2+|u_k|^2}{2}\, dx \le B c e^{-AR/4},$$

$$\int_{B_R} V \overline{v_j} v_k \, dx = \int_{\mathbb{R}^d} V \overline{u_j} u_k \, dx - \gamma_{j,k} = \int_{\mathbb{R}^d} V \overline{u_j} u_k \, dx + O(e^{-AR/4}).$$

By combining with (8.20) one obtains

$$\begin{aligned} h_R(v_j,v_k) &= \int_{B_R} \big(\overline{\nabla v_j}\cdot \nabla v_k + V\overline{v_j} v_k\big) dx \\ &= \int_{\mathbb{R}^d} \big(\overline{\nabla u_j}\cdot \nabla u_k + V\overline{u_j} u_k\big) dx + O(e^{-AR/4}) \\ &= h(u_j,u_k) + O(e^{-AR/4}) \\ &= E_j(H)\delta_{j,k} + O(e^{-AR/4}). \end{aligned} \tag{8.21}$$

Now let $\xi = (\xi_1,\dots,\xi_n) \in \mathbb{C}^n$, then for $v := \xi_1 v_1 + \dots + \xi_n v_n \in U_n$ one has in view of (8.15) and (8.21):

$$\begin{aligned} \langle v,v\rangle_{L^2(B_R)} &= \big(1+O(e^{-AR/2})\big)|\xi|^2_{\mathbb{C}^n}, \\ h_R(v,v) &= \sum_{j=1}^n E_j(H)|\xi_j|^2 + O(e^{-AR/4})|\xi|^2_{\mathbb{C}^n} \le \big(E_n(H) + O(e^{-AR/4})\big)|\xi|^2_{\mathbb{C}^n}, \end{aligned}$$

where the O-terms are independent of ξ, and

$$\begin{aligned} E_n(H_R) = \Lambda_n(H_R) &\le \sup_{v\notin U_n,\, v\neq 0} \frac{h_R(v,v)}{\langle v,v\rangle_{L^2(B_R)}} \\ &= \sup_{\xi\in\mathbb{C}^n,\,\xi\neq 0} \frac{\big(E_n(H)+O(e^{-AR/4})\big)|\xi|^2_{\mathbb{C}^n}}{\big(1+O(e^{-AR/2})\big)|\xi|^2_{\mathbb{C}^n}} = E_n(H) + O(e^{-AR/4}). \end{aligned}$$

Hence, $E_n(H) \le E_n(H) + O(e^{-AR/4})$, and together with (8.13) this gives the claim with $a := A/4$. $\square$

Exercise 22 (Agmon estimate on the half-line).
Let $V \in C^0([0,\infty))$ with $V \geq 0$ and $H = -d^2/dx^2 + V$ in $L^2(0,\infty)$ with Neumann boundary condition at 0, e.g. H is generated by the sesquilinear form h:

$$h(u,v) = \int_0^\infty \big(\overline{u'}v' + V\overline{u}v\big)\,dx, \quad D(h) = \Big\{u \in H^1(0,\infty) : \int_0^\infty V|u|^2 < \infty\Big\}.$$

Let u be an eigenfunction of H with eigenvalue E satisfying

$$E < \liminf_{x\to+\infty} V(x).$$

Show the following result: there exist $R > 0$ and $a > 0$ such that

$$\int_0^\infty e^{a\phi}\big(|u'|^2 + |V-E|\,|u|^2\big)\,dx < \infty,$$

where ϕ is given by

$$\phi(x) = \begin{cases} 0, & x < R, \\ \displaystyle\int_R^x \sqrt{V(y)-E}\,dy, & x \geq R. \end{cases}$$

Hint: Mimic the proof of Theorem 8.8 with $\Phi = a\phi$, $\psi := L + \theta(\phi - L)$, with the above function ϕ.

8.4 Strong coupling asymptotics

In this subsection we consider the parameter-dependent Schrödinger operator

$$H_\lambda = -\Delta + \lambda V, \quad \lambda > 0,$$

and we will be interested in the behavior of its eigenvalues $E_j(H_\lambda)$ as $\lambda \to +\infty$. The parameter λ is usually referred to as *coupling constant*, and large values of λ corresponds to the strong coupling.

Theorem 8.10 (Strong coupling asymptotics at first order). *Let $V \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$ be real-valued with*

$$V_0 := \operatorname*{ess\,inf}_{x\in\mathbb{R}^d} V(x) \equiv \sup\big\{a \in \mathbb{R} : \big|\{x : V(x) < a\}\big| = 0\big\} > -\infty.$$

Then for any fixed $n \in \mathbb{N}$ there holds

$$\Lambda_n(H_\lambda) = V_0\lambda + o(\lambda) \text{ as } \lambda \to +\infty.$$

Proof. One has $V \geq V_0$ a.e., and for any $u \in D(H_\lambda)$ and any $\lambda > 0$ one has

$$\langle u, H_\lambda u\rangle = \int_{\mathbb{R}^d} \big(|\nabla u|^2 + \lambda V|u|^2\big)\, dx \geq \lambda V_0 \|u\|^2,$$

therefore, $\operatorname{spec} H \subset [V_0, +\infty)$, and $\Lambda_n(H_\lambda) \geq \lambda V_0$ (which does not use the fact that λ is large). It remains to show that $\limsup_{\lambda\to+\infty} \Lambda_n(H_\lambda)/\lambda \leq V_0$.
Pick any $n \in \mathbb{N}$. Let M_V be the operator of multiplication by V in $L^2(\mathbb{R}^d)$ and m_V be its sesqulinear form,

$$m_V(u,u) = \int_{\mathbb{R}^d} V\overline{u}v\, dx, \quad D(m_V) = \{u \in L^2(\mathbb{R}^d) : m_V(u,u) < \infty\},$$

then $C_c^\infty(\mathbb{R}^d)$ is dense in $D(m_V)$. Furthermore, one has $V_0 = \inf \operatorname{spec}_{\mathrm{ess}} M_V$ (in fact, it is an easy exercise to show that the spectrum of M_V is purely essential), and the min-max principle states that $\Lambda_n(M_V) = V_0$. Let $\varepsilon > 0$, then due to the definition of $\Lambda_n(M_V)$ one can find a n-dimensional subspace $U \subset C_c^\infty(\mathbb{R}^d)$ with

$$\frac{m_V(u,u)}{\langle u,u\rangle} \equiv \frac{\displaystyle\int_{\mathbb{R}^d} V|u|^2\, dx}{\langle u,u\rangle} \leq V_0 + \varepsilon \text{ for all } u \in U,\ u \neq 0.$$

As U is finite-dimensional, there exists $C > 0$ such that

$$\frac{\displaystyle\int_{\mathbb{R}^d} \overline{u}(-u'')\, dx}{\langle u,u\rangle} \leq C \text{ for all } u \in U,\ u \neq 0.$$

It follows that

$$\Lambda_n(H_\lambda) \leq \sup_{u\in U,\, u\neq 0} \frac{\langle u, H_\lambda u\rangle}{\langle u,u\rangle}$$

$$\equiv \sup_{u\in U,\, u\neq 0} \frac{\displaystyle\int_{\mathbb{R}^d} \overline{u}(-u'')\,dx + \lambda \int_{\mathbb{R}^d} V|u|^2\,dx}{\langle u, u\rangle} \leq C + \lambda(V_0+\varepsilon).$$

Sending $\lambda \to +\infty$ we obtain $\limsup_{\lambda\to+\infty} \Lambda_n(H_\lambda)/\lambda \leq V_0+\varepsilon$. As $\varepsilon>0$ is arbitrary, $\limsup_{\lambda\to+\infty} \Lambda_n(H_\lambda)/\lambda \leq V_0$. This finishes the proof. □

Corollary 8.11. *In the situation of Theorem 8.10 assume additionally that*

$$V_0 < \liminf_{|x|\to+\infty} V(x), \tag{8.22}$$

then for any $N\in\mathbb{N}$ there is $\lambda_N>0$ such that H_λ has at least N eigenvalues below the essential spectrum for all $\lambda>\lambda_N$, and for any $n=1,\dots,N$ there holds

$$\lim_{\lambda\to+\infty} E_n(H_\lambda) = V_0\,\lambda + o(\lambda) \text{ for } \lambda\to+\infty.$$

Proof. The assumption (8.22) implies that one can find $v>V_0$ and $R>0$ such that $V(x)>v$ for all $|x|>R$. By Theorem 8.1 it follows that $\operatorname{spec}_{\mathrm{ess}} H_\lambda \subset [\lambda v, +\infty)$ for any $\lambda>0$. Let $N\in\mathbb{N}$, then the asymptotics $\Lambda_N(H_\lambda)=V_0\lambda+o(\lambda)$ for large λ implies that there exists $\lambda_N>0$ such that $\Lambda_N(H_\lambda)<\lambda v\leq \inf\operatorname{spec}_{\mathrm{ess}} H_\lambda$ for all $\lambda>\lambda_N$, then automatically $\Lambda_n(H_\lambda)\leq \inf\operatorname{spec}_{\mathrm{ess}} H_\lambda$ for all $n=1,\dots,n$ and $\lambda>\lambda_N$. Then it follows by the min-max principle that $\Lambda_n(H_\lambda)$ is the nth eigenvalue of H_λ, i.e. $E_n(H_\lambda)=\Lambda_n(H_\lambda)$ for all $n=1,\dots,n$ and $\lambda>\lambda_N$. □

We are now interested in more precise asymptotic expansions for the eigenvalues $E_n(H_\lambda)$ for large λ. This problem has no general solution: in fact, the asymptotics depends on the way how V attains its minimum: it can be reached e.g. at a single point, or on a submanifold, or on an open set, and the respective eigenvalue asymptotics are different. We only consider the case when the minimum is attained at a single point, which can be viewed as the generic case (all other cases are much more involved: the respective results could be very good topics for a thesis).

We first introduce a class of potentials in which the eigenvalues of H_λ are just power functions of λ. These potentials will be then used to study more general potentials.

Definition 8.12. Let $k>0$. A function $V:\mathbb{R}^d\to\mathbb{R}$ is called **k-homogeneous** if for every $t>0$ there holds $V(tx)=t^kV(x)$ for a.e. $x\in\mathbb{R}^d$. In addition, if there exists $c>0$ such that $V(x)\geq c|x|^k$ for all $x\in\mathbb{R}^d$, then we say that V is **positive definite**.

Typical examples of positive definite k-homogeneous potentials are $|x|^k$ or, more generally, $\big(x\cdot(Ax)\big)^{k/2}$, where A is a positive definite matrix, or suitable linear combinations of such terms. The potential $x_1^4+x_2^4$ is clearly 4-homogeneous in $\mathbb{R}^2$, and it positive definite due to $x_1^4+x_2^4\geq \frac12(x_1^2+x_2^2)^2=\frac14|x|^4$. The potential $x_1^2-x_2^2$ is an example of a 2-homogeneous potential which is not positive definite.

Proposition 8.13. *Let $k>0$ and $V\in L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$ be a positive definite k-homogeneous potential. Then $H_\lambda=-\Delta+\lambda V$, $\lambda>0$, is essentially self-adjoint on $C_c^\infty(\mathbb{R}^d)$, with compact resolvent, and for any $n\in\mathbb{N}$ and any $\lambda>0$ there holds*

$$E_n(H_\lambda)=\lambda^{\frac{2}{k+2}}E_n(H_1).$$

Proof. The positive definiteness implies that $V \geq 0$: then H_λ is essentially self-adjoint on $C_c^\infty(\mathbb{R}^d)$ (Theorem 5.12), and that $\lim_{|x|\to+\infty} V(x) = +\infty$: then H_λ has compact resolvent.

Consider the unitary transform U in $L^2(\mathbb{R}^d)$ given by $(Uu)(x) = \lambda^{ds/2} u(\lambda^s x)$, where $s \in \mathbb{R}$ is to be determined later. For any $u \in C_c^\infty(\mathbb{R}^d)$ one has

$$\begin{aligned}(H_\lambda U u)(x) &= \lambda^{ds/2}\Big[\lambda^{2s}(-\Delta u)(\lambda^s x) + \lambda V(x) u(\lambda^s x)\Big]\\ &= \lambda^{ds/2}\Big[\lambda^{2s}(-\Delta u)(\lambda^s x) + \lambda^{1-ks} V(\lambda^s x) u(\lambda^s x)\Big].\end{aligned}$$

Now choose s to have $2s = 1 - ks$, i.e. $s = 1/(k+2)$, then

$$(H_\lambda U u)(x) = \lambda^{2s}\lambda^{ds/2}\big(\big(-\Delta u)(\lambda^s x) + V(\lambda^s x)u(\lambda^s x)\big) = \lambda^{\frac{2}{k+2}}(U H_1 u)(x).$$

As all H_λ are essentially self-adjoint on $C_c^\infty(\mathbb{R}^d)$, it follows that H_λ and $\lambda^{\frac{2}{k+2}} H_1$ are unitarily equivalent and, as a result, have the same eigenvalues. $\square$

The following theorem is one of the central results of the asymptotic spectral theory (and it is one of the central results of the present lecture course):

Theorem 8.14 (Detailed strong coupling asymptotics). *Let $V \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$ real-valued and such that:*

- *There exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ there holds*

$$\inf_{\varepsilon<|x|} V(x) = \inf_{\varepsilon<|x|<\varepsilon_0} V(x),$$

- *There exist $0 < k < m$ and a positive definite k-homogeneous potential U with*

$$V(x) = V(0) + U(x) + O(|x|^m) \text{ as } |x| \to 0.$$

Then for any fixed $n \in \mathbb{N}$ there holds

$$E_n(-\Delta + \lambda V) = \lambda V(0) + \lambda^{\frac{2}{2+k}} E_n(-\Delta + U) + O(\lambda^{\frac{2}{2+m}}) \text{ as } \lambda \to +\infty.$$

Informally, the assumptions of the theorem mean that 0 is the only global minimum of V, and it is non-degenerate in a suitable sense. The results say (again, very informally) that in the strong coupling regime the potential V can be replaced by its homogeneous part. The proof of Theorem 8.14 will be split into several steps.

The following definition was already used implicitly many times:

Definition 8.15. Let $\Omega \subset \mathbb{R}^d$ be an open set and $V \in L^\infty_{\mathrm{loc}}(\Omega)$ semibounded from below. The operator $T = -\Delta + V$ in $L^2(\Omega)$ **with Dirichlet boundary conditions** is defined as the unique self-adjoint operator in $L^2(\Omega)$ generated by the closed sesquilinear form

$$t(u,v) = \int_\Omega \big(\overline{\nabla u}\cdot \nabla v + V\overline{u}v\big)\,dx, \quad D(t) = \Big\{u \in H_0^1(\Omega) : \int_\Omega V|u|^2\,dx < \infty\Big\}.$$

Lemma 8.16 (Decoupling with a parameter). *Let $V \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$ be semibounded from below. For $R > 0$ denote*

$$B_R := \big\{x \in \mathbb{R}^d : |x| < R\big\}, \quad B_R^c := \big\{x \in \mathbb{R}^d : |x| > R\big\},$$

and consider the following three self-adjoint operators:

- $H = -\Delta + V$ *in* $L^2(\mathbb{R}^d)$,
- $H_R = \Delta + V$ *in* $L^2(B_{2R})$ *with Dirichlet boundary conditions,*
- $H_R^c = -\Delta + V$ *in* $L^2(B_R^c)$ *with Dirichlet boundary conditions.*

Then for all $R > 0$ and all $n \in \mathbb{N}$ there holds

$$\Lambda_n(H_{2R} \oplus H_R^c) - \frac{C}{R^2} \le \Lambda_n(H) \le \Lambda_n(H_{2R}), \tag{8.23}$$

with some constant $C > 0$ independent of V, R and n.

Proof. The inequality on the right-hand side of (8.23) is clear (and it already appeared at several places): one considers the map $J : Q(H_{2R}) \to Q(H)$ defined as the extension by zero and uses Proposition 6.6. Now let us concentrate on the lower bound for $\Lambda_n(H)$. Let $\psi_1, \psi_2 \in C^\infty(\mathbb{R})$ with

$$0 \le \psi_j \le 1, \quad \psi_1^2 + \psi_2^2 = 1, \quad \psi_1(t) = 1 \text{ for } t \le 1, \quad \psi_1(t) = 0 \text{ for } t \ge 2.$$

Such functions can be constructed as follows: let $\psi : \mathbb{R} \to \mathbb{R}$ be a C^∞ function with

$$0 \le \psi \le 1, \quad \psi(t) = 1 \text{ for } t \le 1, \quad \psi(t) = 0 \text{ for } t \ge 2.$$

then one can take

$$\psi_1(t) := \frac{\psi(t)}{\sqrt{\psi(t)^2 + \big(1 - \psi(t)\big)^2}}, \quad \psi_2(t) := \frac{1 - \psi(t)}{\sqrt{\psi(t)^2 + \big(1 - \psi(t)\big)^2}}.$$

For $R > 0$ consider the functions

$$\chi_j : \mathbb{R}^d \to \mathbb{R}, \quad \chi_j(x) = \psi_j\big(\tfrac{|x|}{R}\big), \quad j = 1, 2.$$

For any $u \in C_c^\infty(\mathbb{R}^d)$ one has the following equalities:

$$\int_{\mathbb{R}^d} \Big(\big|\nabla(\chi_1 u)\big|^2 + \big|\nabla(\chi_2 u)\big|^2\Big)\,dx = \int_{\mathbb{R}^d} \Big(|u\nabla\chi_1 + \chi_1\nabla u|^2 + |u\nabla\chi_2 + \chi_2\nabla u|^2\Big)\,dx$$
$$= \int_{\mathbb{R}^d} \Big[|u|^2\big(|\nabla\chi_1|^2 + |\nabla\chi_2|^2\big) + (\chi_1^2 + \chi_2^2)|\nabla u|^2 + (\overline{u}\nabla u + u\overline{\nabla u})(\chi_1\nabla\chi_1 + \chi_2\nabla\chi_2)\Big]\,dx.$$

One has $\chi_1^2 + \chi_2^2 = 1$ and $\chi_1\nabla\chi_1 + \chi_2\nabla\chi_2 = \frac{1}{2}\nabla(\chi_1^2 + \chi_2^2) = \frac{1}{2}\nabla 1 = 0$, therefore,

$$\int_{\mathbb{R}^d} \Big(\big|\nabla(\chi_1 u)\big|^2 + \big|\nabla(\chi_2 u)\big|^2\Big)\,dx = \int_{\mathbb{R}^d} \big(|\nabla u|^2 + W|u|^2\big)\,dx,$$

where $W := |\nabla\chi_1|^2 + |\nabla\chi_2|^2$. Due to

$$\|W\|_\infty \le \frac{C}{R^2}, \quad C := \big|(\psi_1')^2 + (\psi_2')^2\big|_\infty.$$

one has

$$\int_{\mathbb{R}^d} \Big(\big|\nabla(\chi_1 u)\big|^2 + \big|\nabla(\chi_2 u)\big|^2\Big)\,dx \le \int_{\mathbb{R}^d} |\nabla u|^2\,dx + \frac{C}{R^2}\,\|u\|^2_{L^2(\mathbb{R}^d)},$$

and using again $\chi_1^2 + \chi_2^2 = 1$,

$$\begin{aligned}\int_{\mathbb{R}^d} \Big(\big|\nabla(\chi_1 u)\big|^2 + V|\chi_1 u|^2\Big)\,dx + \int_{\mathbb{R}^d} \Big(\big|\nabla(\chi_2 u)\big|^2 + V|\chi_2 u|^2\Big)\,dx \\ \le \int_{\mathbb{R}^d} \big(|\nabla u|^2 + V|u|^2\big)\,dx + \frac{C}{R^2}\,\|u\|^2_{L^2(\mathbb{R}^d)}, \quad (8.24)\end{aligned}$$

which extends by density to all $u \in Q(H)$. By construction, $\chi_1 u \subset H^1_0(B_{2R})$ and $\chi_2 u \in H^1_0(B^c_R)$, therefore, $\chi_1 u \in Q(H_{2R})$ and $\chi_2 u \in Q(H^c_R)$. If h_{2R}, h^c_R, h are the closed sesqulinear forms for H_{2R}, H^c_R, H, then the inequality (8.24) rewrite as

$$h_{2R}(\chi_1 u, \chi_1 u) + h^c_R(\chi_2 u, \chi_2 u) \le h(u,u) + \frac{C}{R^2}\|u\|^2.$$

Now consider the closed sesquilinear form t' in $L^2(B_{2R}) \oplus L^2(B^c_R)$ given by

$$t'\big((u_1,u_2),(v_1,v_2)\big) = h_{2R}(u_1,v_1) + h^c_R(u_2,v_2), \quad D(t') = D(h_{2R}) \oplus D(h^c_R),$$

the closed sesquilinear form t in $L^2(\mathbb{R}^d)$ given by

$$t(u,v) = h(u,v) + \frac{C}{R^2}\langle u,v\rangle_{L^2(\mathbb{R}^d)}, \quad D(t) = D(h),$$

and the linear map $J : D(t) \ni u \mapsto (\chi_1 u, \chi_2 u) \in D(t')$. Then the inequality (8.24) can be rewritten as $t'(Ju,Ju) \le t(u,u)$, and due to $\chi_1^2 + \chi_2^2 = 1$ one also has

$$\begin{aligned}\|Ju\|^2_{L^2(B_{2R})\oplus L^2(B^c_R)} &= \int_{B_{2R}} |\chi_1 u|^2\,dx + \int_{B^c_R} |\chi_2 u|^2\,dx \\ &= \int_{\mathbb{R}^d} \big(|\chi_1 u|^2 + |\chi_2 u|^2\big)\,dx = \int_{\mathbb{R}^d} |u|^2\,dx = \|u\|^2_{L^2(\mathbb{R}^d)}.\end{aligned}$$

Therefore, if T' and T are the self-adjoint operators generated by t' and t, one has $\Lambda_n(T') \le \Lambda_n(T)$ for all n by Proposition 6.6. Now it is sufficient to note that

$$T' = H_{2R} \oplus H^c_R, \quad T = H + \frac{C}{R^2}, \quad \Lambda_n(T) = \Lambda_n(H) + \frac{C}{R^2}. \qquad \square$$

Now let us turn to the proof of the main result:

Proof of Theorem 8.14. First make some general remarks. Without loss of generality we may assume $V(0)=0$. As $V\in L^\infty_{\text{loc}}(\mathbb{R}^d)$, it is bounded on a small ball around the origin, and then also U is bounded in a small ball around the origin: there exist $a>0$ and $b>0$ with $|U(x)|\le b$ for $|x|\le a$. Then, using the k-homogenity, for all $x\ne 0$ one has $|U(x)|\le \Big|\big(\frac{|x|}{a}\big)^k U\big(a\frac{x}{|x|}\big)\Big|\le \frac{b}{a^k}|x|^k$, i.e. U is polynomially bounded (which is important, as below we are going to use Theorem 8.9).
Use Lemma 8.16 with V replaced by λV and $R:=\lambda^{-s}$ ($s>0$ will be chosen later):

$$\Lambda_n(H_{2\lambda^{-s}}\oplus H^c_{\lambda^{-s}})-C\lambda^{2s}\le \Lambda_n(-\Delta+\lambda V)\le \Lambda_n(H_{2\lambda^{-s}}),\quad n\in\mathbb{N},\tag{8.25}$$
$$H_{2\lambda^{-s}}:=-\Delta+\lambda V \text{ in } L^2(B_{2\lambda^{-s}}) \text{ with Dirichlet boundary condition,}$$
$$H^c_{\lambda^{-s}}:=-\Delta+\lambda V \text{ in } L^2(B^c_{\lambda^{-s}}) \text{ with Dirichlet boundary condition.}$$

Pick any $n\in\mathbb{N}$ and study $\Lambda_n(H_{2\lambda^{-s}})$. Due to the assumptions on V there exist $\lambda_s>0$ and $A>0$ such that

$$\lambda\big|V(x)-U(x)\big|\le \lambda A|x|^m\le A\lambda^{1-ms} \text{ for all } |x|<2\lambda^{-s} \text{ and } \lambda>\lambda_s.$$

Consider the operator $L_\lambda:=-\Delta+\lambda U$ in $B_{2\lambda^{-s}}$ with Dirichlet boundary conditions, then $\|H_{2\lambda^{-s}}-L_\lambda\|\le A\lambda^{1-ms}$ and by the min-max principle one has

$$\big|\Lambda_n(H_{2\lambda^{-s}})-\Lambda_n(L_\lambda)\big|\le A\lambda^{1-ms} \text{ for } \lambda>\lambda_s.$$

The operator L_λ is generated by the sesqulinear form

$$\ell_\lambda(u,u)=\int_{|x|\le 2\lambda^{-s}}\big(|\nabla u|^2+\lambda U|u|^2\big)\,dx,\quad D(\ell_\lambda)=H^1_0(B_{2\lambda^{-s}}).$$

We apply a scaling argument similar to the one used in Proposition 8.13. Namely, for $t\in\mathbb{R}$ consider the unitary transform

$$\Theta: L^2(B_{2\lambda^{t-s}})\to L^2(B_{2\lambda^{-s}}),\quad (\Theta u)(x)=\lambda^{dt/2}u(\lambda^t x),\quad x\in B_{2\lambda^{-s}},$$

then $\Theta: H^1_0(B_{2\lambda^{t-s}})\to H^1_0(B_{2\lambda^{-s}})$ is bijective, and for any $u\in H^1_0(B_{2\lambda^{t-s}})$ one has

$$\ell_\lambda(\Theta u,\Theta u)=\lambda^{dt}\int_{|x|<2\lambda^{-s}}\Big(\lambda^{2t}\big|(\nabla u)(\lambda^t x)\big|^2+\lambda^{1-kt}U(\lambda^t x)\big|u(\lambda^t x)\big|^2\Big)\,dx.$$

Choosing $t=\frac{1}{2+k}$ one obtains $2t=1-kt$ and

$$\begin{aligned}\ell_\lambda(\Theta u,\Theta u)&=\lambda^{\frac{2}{2+k}}\lambda^{dt}\int_{|x|<2\lambda^{-s}}\Big(\big|(\nabla u)(\lambda^t x)\big|^2+U(\lambda^t x)\big|u(\lambda^t x)\big|^2\,dx\\&=\lambda^{\frac{2}{2+k}}\int_{|x|<2\lambda^{\frac{1}{2+k}-s}}\big(|\nabla u|^2+U|u|^2\big)\,dx.\end{aligned}$$

The integral on the right-hand side is the sesquilinear form for the self-adjoint operator $K_\lambda:=-\Delta+U$ in $B_{2\lambda^{\frac{1}{2+k}-s}}$ with Dirichlet boundary conditions, and it follows that for any $n\in\mathbb{N}$ there holds

$$\Lambda_n(L_\lambda)=\lambda^{\frac{2}{2+k}}\Lambda_n(K_\lambda).$$

Now assume that $\frac{1}{2+k} - s > 0$, then by Theorem 8.9 one has, with some $a > 0$,

$$\Lambda_n(K_\lambda) = \Lambda_n(-\Delta + U) + O\Big(\exp(-a\lambda^{\frac{1}{2+k}-s})\Big)$$

By summarizing, if $s < \frac{1}{2+k}$, then for $\lambda \to +\infty$ one has

$$\begin{aligned}\Lambda_n(H_{2\lambda^{-s}}) &= \Lambda_n(L_\lambda) + O(\lambda^{1-ms}) = \lambda^{\frac{2}{2+k}}\lambda_n(K_\lambda) + O(\lambda^{1-ms}) \\ &= \lambda^{\frac{2}{2+k}}\Lambda_n(-\Delta + U) + O(\lambda^{1-ms}) + O\Big(\lambda^{\frac{2}{2+k}}\exp(-a\lambda^{\frac{1}{2+k}-s})\Big).\end{aligned} \tag{8.26}$$

In order to use the two-side estimate (8.25) we need some information on $\Lambda_n(H^c_{\lambda^{-s}})$ as well. Due to the assumptions on the potential V, one can find $b > 0$ such that

$$\inf_{|x|>\lambda^{-s}} V(x) \geq b(\lambda^{-s})^k = b\lambda^{-ks} \text{ as } \lambda \to +\infty.$$

It follows that for large λ one has

$$\Lambda_1(H^c_{\lambda^{-s}}) \geq \lambda \inf_{|x|>\lambda^{-s}} V(x) = b\lambda^{1-ks}.$$

Due to $s < \frac{1}{2+k}$ one has $1-ks > 1-\frac{k}{2+k} = \frac{2}{k+2}$, and then, in view of (8.26), for large λ one has $\Lambda_1(H^c_{\lambda^{-s}}) > \Lambda_n(H_{2\lambda^{-s}})$, therefore, $\Lambda_n(H_{2\lambda^{-s}} \oplus H^c_{\lambda^{-s}}) = \Lambda_n(H_{2\lambda^{-s}})$. Substituting this equality into the two-side estimate (8.25) one arrives at the asymptotics $\Lambda_n(-\Delta + \lambda V) = \Lambda_n(H_{2\lambda^{-s}}) + O(\lambda^{2s})$. Now use (8.26) again:

$$\Lambda_n(-\Delta + \lambda V) = \lambda^{\frac{2}{2+k}}\Lambda_n(-\Delta + U) + O(\lambda^{1-ms}) + O(\lambda^{2s}) + O\Big(\lambda^{\frac{2}{2+k}}\exp(-a\lambda^{\frac{1}{2+k}-s})\Big),$$

for any $s < \frac{1}{2+k}$. We now remark that the last summand is small with respect to the first two O-terms. In order to optimize the remainder we solve $1 - ms = 2s$, i.e. take $s = \frac{2}{2+m}$, which gives the sought asymptotics

$$\Lambda_n(-\Delta + \lambda V) = \lambda^{\frac{2}{2+k}}\Lambda_n(-\Delta + U) + O(\lambda^{\frac{2}{2+m}}). \qquad \square$$

It remains to recall that for large λ one has $E_n(-\Delta + \lambda V) = \Lambda_n(-\Delta + \lambda V)$ (see Corollary 8.11) and $E_n(-\Delta + U) = \Lambda_n(-\Delta + U)$ (see Proposition 8.13).

Corollary 8.17 (Approximation by harmonic oscillators). *Let $V \in C^3(\mathbb{R}^d)$ be real-valued with $V(0) < \liminf_{|x|\to+\infty} V(x)$. Assume that 0 is the unique global minimum of V and that the Hessian matrix $V''(0)$ is non-degenerate. Denote by $\mu_1, \dots, \mu_d$ the eigenvalues of $V''(0)$ and consider the disjoint union*

$$\mathbb{E} := \bigsqcup_{(n_1,\dots,n_d)\in\mathbb{N}^d} \Big\{ \sum_{j=1}^d (2n_j - 1)\sqrt{\frac{\mu_j}{2}} \Big\},$$

then denote by ε_n the n-th element of $\mathbb{E}$. Let $n \in \mathbb{N}$, then for $\lambda \to +\infty$ there holds

$$E_n(-\Delta + \lambda V) = V(0)\lambda + \varepsilon_n\lambda^{\frac{1}{2}} + O(\lambda^{\frac{2}{5}}).$$

Proof. Denote $A := \frac{1}{2}V''(0)$, which is a positive definite matrix. The Taylor expansion $V(x) = V(0) + x\cdot(Ax) + O(|x|^3)$, $x \to 0$, shows that we are in the setting of Theorem 8.14 with $k = 2$, $U(x) = x \cdot (Ax)$ and $m = 3$. Therefore,

$$E_n(-\Delta + \lambda V) = \lambda V(0) + \lambda^{\frac{1}{2}} E_n(L) + O(\lambda^{\frac{2}{5}}), \quad L := -\Delta + U,$$

and it remains to compute the eigenvalues of L.

First, the eigenvalues of the harmonic oscillator $T = -d^2/dx^2 + x^2$ are $2n-1$, $n \in \mathbb{N}$ (see Example 3.11). As x^2 is 2-homogeneous, it follows (Proposition 8.13) that for any $\omega > 0$ the eigenvalues of $T_\omega = -d^2/dx^2 + \omega x^2$ are $E_n(T_\omega) = (2n-1)\sqrt{\omega}$, $n \in \mathbb{N}$. We denote by $\psi_{n,\omega}$ the associated eigenfunctions forming an orthonormal basis in $L^2(\mathbb{R})$. Now let $\omega_1, \dots, \omega_d > 0$, then the functions

$$\Phi_{n_1,\dots,n_d}(x_1, \dots, x_d) = \psi_{n_1,\omega_1}(x_1) \cdot \dots \cdot \psi_{n_d,\omega_d}(x_d), \quad (n_1, \dots, n_d) \in \mathbb{N}^d,$$

form an orthonormal basis in $L^2(\mathbb{R}^d)$ and are eigenfunctions of

$$L_\omega := -\Delta + \omega_1 x_1^2 + \dots + \omega_d^2 x_d^2$$

in $L^2(\mathbb{R}^d)$ with eigenvalues $E_{n_1,\dots,n_d} := \sum_{j=1}^d (2n_j - 1)\sqrt{\omega_j}$, and these eigenvalues exhaust the whole spectrum of L_ω.

In order to reduce L to L_ω we use the fact that there exists an orthogonal matrix θ (i.e. $\theta^t = \theta^{-1}$) such that

$$\theta^{-1} A\theta x = \frac{1}{2}\operatorname{diag}(\mu_1, \dots, \mu_d).$$

Consider the unitary transform $U : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ defined by $(\Theta u)(x) = u(\theta x)$, then it is an easy exercise to show that

$$\Theta^{-1} L\Theta = -\Delta + \frac{1}{2}\sum_{j=1}^d \mu_j x_j^2 \quad (= L_\omega \text{ with } \omega_j = \tfrac{1}{2}\mu_j),$$

hence, the spectrum of L is exactly the above set $\mathbb{E}$. $\square$

We remark that the strong coupling asymptotics can be transformed to the so-called *semiclassical* asymptotics for the eigenvalues of $-h^2\Delta + V$ with $h \to 0^+$: one denotes $h := \lambda^{-1/2}$, then

$$E_n(-\Delta + \lambda V) = \lambda E_n(-h^2\Delta + V).$$

Corollary 8.17, if rewritten with this new parameter h, is usually referred to as the *WKB asymptotics* for the eigenvalues. We remark that the remainders in the above asymptotics are not optimal and can be improved with the help of different approaches.

The exercises below are to fill (very small) gaps in the proofs of Subsection 8.4. They refer to the most basic questions.

Exercise 23. Let $V \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$ be real-valued and semibounded from below. Define

$$V_0 := \operatorname*{ess\,inf}_{x\in\mathbb{R}^d} V(x) \equiv \sup\big\{a \in \mathbb{R} : \big|\{x : V(x) < a\}\big| = 0\big\}.$$

Let M_V be the self-adjoint operator of multilpication by V.

1. We want to show that $V_0 = \inf \operatorname{spec}_{\mathrm{ess}} M_V$.
   (a) Show that $V_0 > -\infty$ and that $\operatorname{spec} M_V \subset [V_0, +\infty)$.
   (b) Assume that $|\{x : V(x) = V_0\}| > 0$. Show that $V_0 \in \operatorname{spec}_{\mathrm{ess}} M_V$.
   (c) Now assume $|\{x : V(x) = V_0\}| = 0$. For $\varepsilon > 0$ consider the set
   $$X_\varepsilon := \big\{x : V(x) < V_0 + \varepsilon\big\}.$$
      i. Show that one can find a monotonically decresing sequence (ε_n) with $\lim_{n\to+\infty} \varepsilon_n = 0$ such that $|X_{\varepsilon_n} \setminus X_{\varepsilon_{n+1}}| > 0$ for all n.
      ii. Let Y_n be a subset of $X_{\varepsilon_n} \setminus X_{\varepsilon_{n+1}}$ with positive finite measure and consider the functions $\varphi_n := \frac{1_{Y_n}}{\sqrt{|Y_n|}}$, where 1_{Y_n} is the indicator function of Y_n. Show that (φ_n) is a singular Weyl sequence for M_V (see Theorem 5.22).
      iii. Show that $V_0 \in \operatorname{spec}_{\mathrm{ess}} M_V$.
   (d) Show the claim.

2. Now we want to show that $C_c^\infty(\mathbb{R}^d)$ is dense in $D(m_V)$ (which is viewed as a Hilbert space with the norm $\|\cdot\|_{m_V}$, see Subsection 2.1).
   (a) Show that $L^2_c(\mathbb{R}^d) = \{u \in L^2(\mathbb{R}^d) : u \text{ has compact support}\}$ is a dense subspace of $D(m_V)$.
   (b) Let $u \in L^2_c(\mathbb{R}^d)$. Show that there exists a ball B containing the support of u and a sequence $(u_n) \subset C_c^\infty(\mathbb{R}^d)$ with supports in B satisfying
   $$\|u - u_n\|_{L^2(\mathbb{R}^d)} \to 0.$$
   (c) Show that $m_V(u - u_n, u - u_n) \to 0$.
   (d) Show the claim.

Exercise 24. Let T_1 and T_2 be lower semibounded self-adjoint operators in Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, and let t_1, t_2 be their closed sesquilinear forms (Subsection 2.1). Define a sesqulinear form t in the Hilbert space $\mathcal{H}_1 \times \mathcal{H}_2$ by

$$D(t) = D(t_1) \times D(t_2), \quad t\big((u_1, u_2), (v_1, v_2)\big) = t_1(u_1, v_1) + t_2(u_2, v_2).$$

1. Show that t is closed.
2. Show that the operator T generated by t is the direct sum $T = T_1 \oplus T_2$, i.e.
$$D(T) = D(T_1) \times D(T_2), \quad T(u_1, u_2) = (T_1u_1, T_2u_2).$$
3. Let $n \in \mathbb{N}$ and assume that $\Lambda_n(T_1) < \Lambda_1(T_2)$. Show that $\Lambda_n(T) = \Lambda_n(T_1)$.

9 Laplacians in unbounded domains

9.1 Bottom of the essential spectrum, decay of eigenfunctions

In the present subsection, let $\Omega \subset \mathbb{R}^d$ be an *unbounded* domain. By T^Ω we denote the Dirichlet Laplacian in Ω, which is the self-adjoint operator in $L^2(\Omega)$ generated by the closed sesquilinear form

$$t^\Omega(u,v) = \int_\Omega \overline{\nabla u} \cdot \nabla v \, dx, \quad D(t^\Omega) = H_0^1(\Omega).$$

For the moment we do not make any additional assumptions on the geometry of Ω (but some particular classes of unbounded domains will considered later). As Ω is unbounded, the previous discussion involving the compactness of the resolvent does not apply, and the operator T^Ω can have a non-empty essential spectrum. We will first discuss some similarities (if expressed in a suitable language) between the Laplacians in unbounded domains and the Schrödinger operators discussed in the preceding chapter.

We recall that the principle of the domain monotonicity:

Theorem 9.1. *For any open domains $\Omega' \subset \Omega$ there holds*

$$\Lambda_n(T^\Omega) \leq \Lambda_n(T^{\Omega'}) \text{ for all } n \in \mathbb{N}. \tag{9.1}$$

Proof. Let $J : H_0^1(\Omega') \to H_0^1(\Omega)$ be the extension by zero. In fact, if $u \in C_c^\infty(\Omega')$, then clearly $Ju \in H_0^1(\Omega)$ with $\|Ju\|_{H_0^1(\Omega)} = \|u\|_{H^1(\Omega')}$, which then extends by density to the whole of $H_0^1(\Omega')$. We further have $J\partial_j u = \partial_j Ju$, which shows that

$$t^\Omega(Ju, Ju) = t^{\Omega'}(u,u), \quad \|Ju\|_{L^2(\Omega)} = \|u\|_{L^2(\Omega')} \quad \text{for all } u \in H_0^1(\Omega').$$

Now we are in the situation of Proposition 6.6, and the claim follows. □

As our constructions involve a number of truncations, we will introduce some associated notation. For $R > 0$ we denote

$$\Omega_R := \Omega \cap \big\{|x| < R\big\}, \quad \Omega_R^c := \Omega \cap \big\{x : |x| > R\big\},$$
$$T := T^\Omega, \quad T_R := T^{\Omega_R}, \quad T_R^c := T^{\Omega_R^c}.$$

Due to $\Omega_R \cap \Omega_R^c \subset \Omega$, the domain monotonicity (Theorem 9.1) implies the inequalities

$$\Lambda_n(T) \leq \Lambda_n(T_R), \quad \Lambda_n(T) \leq \Lambda_n(T_R^c), \tag{9.2}$$

and each of them will be of use in what follows.

We will first discuss the position of the essential spectrum, and then continue with the study of the discrete eigenvalues lying below the essential spectrum;

Theorem 9.2. *There exists $C > 0$ such that for any unbounded open domain Ω any $R > 0$ and any $n \in \mathbb{N}$ there holds*

$$\Lambda_n(T) \geq \Lambda_n(T_{2R} \oplus T_R^c) - \frac{C}{R^2}. \tag{9.3}$$

As a consequence,

$$\inf \operatorname{spec}_{\text{ess}} T = \lim_{R \to +\infty} \inf \operatorname{spec}_{\text{ess}} T_R^c. \tag{9.4}$$

Proof. We first show (9.4) under the assumption that (9.3) holds.

Recall that for any lower semibounded self-adjoint operator A one has $\inf \operatorname{spec}_{\text{ess}} A = \lim_{n \to +\infty} \Lambda_n(A)$. By combining the lower bound (9.3) with the upper bound (9.2) we obtain

$$\Lambda_n(T_{2R} \oplus T_R^c) - \frac{C}{R^2} \leq \Lambda_n(T) \leq \Lambda_n(T_R^c). \tag{9.5}$$

Remark that the number $\Lambda_n(T_{2R} \oplus T_R^c)$ is the nth element of the disjoint union

$$\bigsqcup_{j \in \mathbb{N}} \{\Lambda_j(T_{2R})\} \sqcup \bigsqcup_{j \in \mathbb{N}} \{\Lambda_j(T_R^c)\}.$$

The domain Ω_{2R} is bounded, hence, the operator T_{2R} has compact resolvent and $\lim_{n \to +\infty} \Lambda_n(T_{2R}) = +\infty$, and then

$$\lim_{n \to +\infty} \Lambda_n(T_{2R} \oplus T_R^c) = \lim_{n \to +\infty} \Lambda_n(T_R^c).$$

Hence by sending n to ∞ in (9.5) one obtains

$$\inf \operatorname{spec}_{\text{ess}} T_R^c - \frac{C}{R^2} \leq \inf \operatorname{spec}_{\text{ess}} T \leq \inf \operatorname{spec}_{\text{ess}} T_R^c,$$

and by sending $R \to +\infty$ one arrives at (9.4). One remarks that the existence of the limit in (9.4) can also be deduced from the domain monotonicity: the function $R \mapsto \Lambda_n(T_R^c)$ is non-decreasing, hence, $R \mapsto \inf \operatorname{spec}_{\text{ess}} T_R^c$ is non-decreasing too.

Now let us prove (9.3). In fact, it is almost the same proof as for Lemma 8.16, but we prefer to repeat the main steps, as the procedure is important. Let $\psi_1, \psi_2 \in C^\infty(\mathbb{R})$ with

$$0 \leq \psi_j \leq 1, \quad \psi_1^2 + \psi_2^2 = 1, \quad \psi_1(t) = 1 \text{ for } t \leq 1, \quad \psi_1(t) = 0 \text{ for } t \geq 2,$$

and for $R > 0$ define $\chi_j : \mathbb{R}^d \to \mathbb{R}$ by $\chi_j(x) = \psi_j\big(\frac{|x|}{R}\big)$, $j = 1, 2$. For any $u \in C_c^\infty(\Omega)$ one obtains as in the proof of Lemma 8.16:

$$\int_\Omega \Big(\big|\nabla(\chi_1 u)\big|^2 + \big|\nabla(\chi_2 u)\big|^2 \Big)\, dx \leq \int_\Omega |\nabla u|^2\, dx + \frac{C}{R^2} \, \|u\|^2_{L^2(\Omega)}, \tag{9.6}$$

with $C := \big|(\psi_1')^2 + (\psi_2')^2\big|_\infty$, which extends by density to all $u \in H_0^1(\Omega)$.

Now consider the closed sesquilinear form t' in $L^2(\Omega_{2R}) \oplus L^2(\Omega_R^c)$ given by

$$t'\big((u_1, u_2), (v_1, v_2)\big) = \int_{\Omega_{2R}} \overline{\nabla u_1} \cdot \nabla v_1 \, dx + \int_{\Omega_R^c} \overline{\nabla u_2} \cdot \nabla v_2 \, dx, \quad D(t') = Q(T_{2R}) \oplus Q(T_R^c),$$

the closed sesquilinear form s in $L^2(\Omega)$ given by

$$s(u,v)=\int_\Omega \overline{\nabla u}\cdot\nabla v\,dx+\frac{C}{R^2}\langle u,v\rangle_{L^2(\Omega)},\quad D(t)=H_0^1(\Omega),$$

and the linear map $J: D(t)\ni u\mapsto(\chi_1 u,\chi_2 u)\in D(t')$. Then the inequality (9.6) can be rewritten as $t'(Ju,Ju)\le t(u,u)$, and due to $\chi_1^2+\chi_2^2=1$ one also has

$$\begin{aligned}\|Ju\|^2_{L^2(\Omega_{2R})\oplus L^2(\Omega_R^c)}&=\int_{\Omega_{2R}}|\chi_1 u|^2\,dx+\int_{\Omega_R^c}|\chi_2 u|^2\,dx\\&=\int_\Omega\big(|\chi_1 u|^2+|\chi_2 u|^2\big)\,dx=\int_{\mathbb{R}^d}|u|^2\,dx=\|u\|^2_{L^2(\Omega)}.\end{aligned}$$

Therefore, if T' and S are the self-adjoint operators generated by t' and s, one has $\Lambda_n(T')\le\Lambda_n(S)$ for all n (see Proposition 6.6), and we have

$$T'=T_{2R}\oplus T_R^c,\quad S=T+\frac{C}{R^2},\quad \Lambda_n(S)=\Lambda_n(T)+\frac{C}{R^2}.\qquad\square$$

Example 9.3. Let $\omega\subset\mathbb{R}^{d-1}$ be a bounded domain. Consider

$$\Omega:=\omega\times(0,\infty),$$

which is a half-infinite cylinder with cross-section ω_j. For any $u\in C_c^\infty(\Omega)$ and any $x_d\in(0,\infty)$ one has $u(\cdot,t)\in C_c^\infty(\omega)$, and

$$\begin{aligned}\int_\Omega|\nabla u|^2\,dx&=\int_0^\infty\int_\omega\sum_{j=1}^d|\partial_j u(x',x_d)|^2dx'\,dx_d\\&\ge\int_0^\infty\int_\omega\sum_{j=1}^{d-1}|\partial_j u(x',x_d)|^2dx'\,dx_d\\&\ge\int_0^\infty\lambda_1^D(\omega)\int_\omega|u(x',x_d)|^2dx'\,dx_d\\&=\lambda_1^D(\omega)\|u\|^2_{L^2(\Omega)},\end{aligned}$$

i.e. $\operatorname{spec}T^\Omega\subset\big[\lambda_1^D(\omega),\infty\big)$ and

$$\inf\operatorname{spec}_{\mathrm{ess}}T^\Omega\ge\lambda_1^D(\omega).$$

This results in the following observation:

Theorem 9.4. *Assume that there exists $r>0$ and bounded domains $\omega_1,\dots\omega_m\subset\mathbb{R}^{d-1}$ such that $\Omega_r^c\subset\bigcup_{j=1}^m U_j$, where U_j are non-intersecting domains such that each U_j is isometric to $\omega_j\times(0,+\infty)$, then*

$$\inf\operatorname{spec}_{\mathrm{ess}}T\ge\min\big\{\lambda_1^D(\omega_j):\ j=1,\dots,m\big\}.$$

Proof. Let S_j be the Dirichlet Laplacian in U_j, then $\operatorname{spec}_{\mathrm{ess}} S_j \subset \big[\lambda_1^D(\omega_j),\infty\big)$ (Example 9.3). Due to $\Omega_R^c \subset \bigcup_{j=1}^m U_j$ for all $R \geq r$ one has, using the domain monotonicity,

$$\begin{aligned}\inf\operatorname{spec} T_R^c \geq \Lambda_1(T_R^c) &\geq \Lambda_1(S_1\oplus\cdots\oplus S_m)\\ &= \min\{\Lambda_1(S_j): j=1,\dots,m\} = \min\big\{\lambda_1^D(\omega_j):\, j=1,\dots,m\big\},\end{aligned}$$

and the result follows by Theorem 9.2. □

A typical example of a domain satisfying the assumptions of Theorem 9.4 looks as a bounded core to which one glues several half-infinite cylinders. Such unbounded domains are often called *waveguides*, and a more detailed control of their spectral properties is possible (see below).

The preceding result can be transformed into a sufficient condition to have a purely discrete spectrum:

Corollary 9.5. *Denote $B_\varepsilon := \big\{x\in\mathbb{R}^{d-1}: |x|<\varepsilon\big\}$. Assume that for any $\varepsilon>0$ one can find $r>0$ and $m\in\mathbb{N}$ satisfying $\Omega_r^c\subset\bigcup_{j=1}^m U_j$, where U_j are non-intersecting domains and each U_j is isometric to of $B_\varepsilon\times(0,+\infty)$. Then T has compact resolvent.*

Proof. By Theorem 9.4 one has $\inf\operatorname{spec}_{\mathrm{ess}} T \geq \lambda_1^D(B_\varepsilon)$ for any $\varepsilon>0$. On the other hand, $\lambda_1^D(B_\varepsilon) = \varepsilon^{-1}\lambda_1^D(B_1)\to+\infty$ as $\varepsilon>0$, hence, $\operatorname{spec}_{\mathrm{ess}} T=\emptyset$. □

Example 9.6. Let $f:\mathbb{R}\to\mathbb{R}$ be such that $f>0$ and $\lim_{|x|\to+\infty} f(x)=0$. Consider

$$\Omega := \big\{(x_1,x_2): 0<x_2<f(x_1)\big\}\subset\mathbb{R}^2,$$

then the associated Dirichlet Laplacian T has compact resolvent: this case is covered by Corollary 9.5.

Similarly to Schrödinger operators we can show a decay estimate for the eigenfunctions of T corresponding to the discrete eigenvalues below the essential spectrum (Agmon estimate).

Theorem 9.7. *Let u be an eigenfunction of T with eigenvalue E. Assume that there exists $r>0$ such that $\Lambda_1(T_r^c)>E$, then there exists $A>0$ such that*

$$\int_\Omega e^{A|x|}\big(|\nabla u|^2+|u|^2\big)\,dx<\infty.$$

Proof. The proof employs a number of constructions from the proof of Theorem 8.8, but we prefer to repeat them for completeness.

Denote

$$V_R := \Lambda_1(T_R),\quad V_R^c := \Lambda_1(T_R^c),$$

then the initial assumption and the domain monotonicity imply

$$V_R^c \geq V_r^c > E \text{ for all } R\geq r. \tag{9.7}$$

Consider $\phi : x \mapsto \sqrt{|x|^2+1}$: this is a C^∞ function with $|\nabla\phi| \le 1$. Let us pick a non-decreasing C^1 function $\theta : \mathbb{R} \to \mathbb{R}$ with the following properties:

$$\theta(t) = t \text{ for } t \le 0, \quad \theta(t) = 1 \text{ for } t \ge 2, \quad \theta\,'(t) \le 1 \text{ for all } t,$$

and for $L > 0$ consider $\psi(x) = L + \theta\big(\phi(x) - L\big)$, then $\psi \in C^1$ with

$$0 \le \psi \le L+1, \quad |\nabla\psi| = |(\theta' \circ \phi)\nabla\phi| \le |\nabla\phi| \le 1,$$
$$\psi(x) = \phi(x) \text{ for } \phi(x) \le L.$$

For $a > 0$, to be chosen later, consider the function $\Phi : x \mapsto a\psi(x)$.
Using the same computations as in Lemma 8.7 we obtain first

$$\int_\Omega e^{2\Phi}\big(E + |\nabla\Phi|^2\big)|u|^2 dx = \int_\Omega \big|\nabla(e^\Phi u)\big|^2\, dx.$$

On the other hand, using the equality (9.6),

$$\int_\Omega |\nabla(e^\Phi u)|^2\, dx \ge \int_\Omega \Big(\big|\nabla(\chi_1 e^\Phi u)\big|^2 + \big|\nabla(\chi_2 e^\Phi u)\big|^2\Big)\, dx - \frac{C}{R^2}\, \|e^\Phi u\|^2_{L^2(\Omega)}.$$

Let $\delta_0 \in (0,1)$ (whose value will be chosen later), then we can combine the last two inequalities as

$$\begin{aligned}\int_\Omega e^{2\Phi}\big(E + |\nabla\Phi|^2\big)|u|^2 dx &= \delta_0 \int_\Omega |\nabla(e^\Phi u)|^2\, dx + (1-\delta_0)\int_\Omega |\nabla(e^\Phi u)|^2\, dx\\ &\ge \delta_0 \int_\Omega |\nabla(e^\Phi u)|^2\, dx + (1-\delta_0)\int_\Omega \Big(\big|\nabla(\chi_1 e^\Phi u)\big|^2 - \frac{C}{R^2}|\chi_1 e^\Phi u|^2\Big)\, dx\\ &\qquad + (1-\delta_0)\int_\Omega \Big(\big|\nabla(\chi_2 e^\Phi u)\big|^2 - \frac{C}{R^2}|\chi_2 e^\Phi u|^2\Big)\, dx.\end{aligned}$$

Now represent

$$\int_\Omega e^{2\Phi}\big(E + |\nabla\Phi|^2\big)|u|^2 dx = \int_\Omega \big(E + |\nabla\Phi|^2\big)|\chi_1 e^\Phi u|^2 dx + \int_\Omega \big(E + |\nabla\Phi|^2\big)|\chi_2 e^\Phi u|^2 dx,$$

then, using $(1-\delta_0)C \le C$,

$$\begin{aligned}\int_\Omega \big(E + |\nabla\Phi|^2\big)|\chi_1 e^\Phi u|^2 dx &+ \int_\Omega \big(E + |\nabla\Phi|^2\big)|\chi_2 e^\Phi u|^2 dx\\ &\ge \delta_0 \int_\Omega \big|\nabla(e^\Phi u)\big|^2\, dx + \int_\Omega \Big((1-\delta_0)\big|\nabla(\chi_1 e^\Phi u)\big|^2 - \frac{C}{R^2}|\chi_1 e^\Phi u|^2\Big)\, dx\\ &\qquad + \int_\Omega \Big((1-\delta_0)\big|\nabla(\chi_2 e^\Phi u)\big|^2 - \frac{C}{R^2}|\chi_2 e^\Phi u|^2\Big)\, dx \qquad (9.8)\end{aligned}$$

By construction one has $\chi_1 e^\Phi u \in H^1_0(\Omega_{2R})$ and $\chi_2 e^\Phi u \in H^1_0(\Omega^c_R)$, therefore,

$$\int_\Omega \Big(\big|\nabla(\chi_1 e^\Phi u)\big|^2\, dx \ge V_{2R}\|\chi e^\Phi u\|^2_{L^2(\Omega_{2R})},$$

$$\int_\Omega \Big(\big|\nabla(\chi_2 e^\Phi u)\big|^2\,dx \geq V_R^c \|\chi e^\Phi u\|^2_{L^2(\Omega_R^c)},$$

and then it follows from (9.8) that

$$\begin{aligned}
\int_{\Omega_{2R}} \big(E+|\nabla\Phi|^2\big)|\chi_1 e^\Phi u|^2 dx + \int_{\Omega_R^c} \big(E+|\nabla\Phi|^2\big)|\chi_2 e^\Phi u|^2 dx&\\
\geq \delta_0 \int_\Omega \big|\nabla(e^\Phi u)\big|^2\,dx + \int_{\Omega_{2R}} \Big(\Big((1-\delta_0)V_{2R}-\frac{C}{R^2}\Big)|\chi_1 e^\Phi u|^2\Big)\,dx&\\
+\int_{\Omega_R^c} \Big(\Big((1-\delta_0)V_R^c-\frac{C}{R^2}\Big)|\chi_2 e^\Phi u|^2\Big)\,dx.&
\end{aligned}$$

and, by regrouping the terms,

$$\begin{aligned}
\delta_0 \int_\Omega \big|\nabla(e^\Phi u)\big|^2\,dx + \int_{\Omega_R^c} \Big((1-\delta_0)V_R^c - E - |\nabla\Phi|^2 - \frac{C}{R^2}\Big)|\chi_2 e^\Phi u|^2\,dx&\\
\leq \int_{\Omega_{2R}} \Big(E+|\nabla\Phi|^2-(1-\delta_0)V_{2R}+\frac{C}{R^2}\Big)|\chi_1 e^\Phi u|^2\,dx.&\quad (9.9)
\end{aligned}$$

As $V_r^c > E$, one can choose $\delta_0>0$ sufficiently small to have $b:=(1-\delta_0)V_r^c - E>0$. Due to (9.7), one can choose $R>r$ sufficiently large to have

$$(1-\delta_0)V_R^c - E - \frac{C}{R^2} \geq (1-\delta_0)V_r^c - E - \frac{C}{R^2} \geq b - \frac{C}{R^2} \geq \frac{b}{2}.$$

Now recall that $|\nabla\Phi|\leq a$, so one can choose a sufficiently small to have $a^2<\frac{b}{4}$. For this choice of δ_0, R, a one obtains

$$\delta := (1-\delta_0)V_R^c - E - |\nabla\Phi|^2 - \frac{C}{R^2} \geq \frac{b}{2} - a^2 \geq \frac{b}{4} > 0,$$

and (9.9) gives

$$\begin{gathered}
\delta_0 \int_\Omega \big|\nabla(e^\Phi u)\big|^2\,dx + \delta \int_{\Omega_R^c} |\chi_2 e^\Phi u|^2\,dx \leq B \int_{\Omega_{2R}} |\chi_1 e^\Phi u|^2\,dx, \qquad (9.10)\\
B := E + a^2 - (1-\delta_0)V_{2R} + \frac{C}{R^2},
\end{gathered}$$

and then

$$\begin{aligned}
\int_\Omega \Big(\big|\nabla(e^\Phi u)\big|^2 + |e^\Phi u|^2\Big)\,dx = \int_\Omega \big|\nabla(e^\Phi u)\big|^2\,dx + \int_{\Omega_R^c} |\chi_2 e^\Phi u|^2\,dx + \int_{\Omega_{2R}} |\chi_1 e^\Phi u|^2\,dx&\\
\leq \big(B(\delta_0^{-1}+\delta^{-1})+1\big)\int_{\Omega_{2R}} |\chi_1 e^\Phi u|^2\,dx.&
\end{aligned}$$

For $x\in\Omega_{2R}$ one has $\Phi(x)\leq a\psi(\sqrt{4R^2+1})$. One can assume that $L>\sqrt{4R^2+1}$, then $\psi(\sqrt{4R^2+1})=\sqrt{4R^2+1}$, and $\Phi(x)\leq a\sqrt{4R^2+1}$ for all $x\in\Omega_{2R}$, and

$$\int_{\Omega_{2R}} |\chi_1 e^\Phi u|^2\,dx \leq e^{2a\sqrt{4R^2+1}},$$

implying

$$\int_\Omega \Big(\big|\nabla(e^\Phi u)\big|^2 + |e^\Phi u|^2\Big)\,dx \le B_1 := \big(B(\delta_0^{-1}+\delta^{-1})+1\big)e^{2a\sqrt{4R^2+1}}\int_{\Omega_{2R}} |\chi_1 u|^2\,dx.$$

Proceeding literally as in the proof of Theorem 8.8 we obtain

$$\int_\Omega e^{2\Phi}\big(|\nabla u|^2+|u|^2\big)\,dx \le (4a^2+3)B_1 =: B_2,$$

which holds for all sufficiently large L. In a more detailed form (using the explicit expression of Φ),

$$\int_\Omega \exp\Big[2a\Big(L+\theta\big(\phi(x)-L\big)\Big)\Big]\big(|\nabla u|^2+|u|^2\big)\,dx \le B_2.$$

The constant B_2 is independent of L, and $L+\theta\big(\phi(x)-L\big)$ converges monotonically (as $\theta' \le 1$) to $\phi(x)=\sqrt{x^2+1}$ for any x as $L\to+\infty$. Hence, by the monotone convergence theorem,

$$\int_\Omega e^{2a\sqrt{x^2+1}}\big(|\nabla u|^2+|u|^2\big)\,dx \le B_2.$$

Using $e^{2a|x|}\le e^{2a\sqrt{x^2+1}}$ one obtains the claim with $A=2a$. $\square$

Exercise 25. We continue using the same notation as in the preceding subsection. Let $\Omega \subset \mathbb{R}^d$ be an unbounded domain.

1. Let $R > 0$ and $\varepsilon > 0$. Show that the Dirichlet Laplacian T has at most finitely many eigenvalues below $\Lambda_1(T_R^c) - \varepsilon$.

2. Let $R > 0$ and $N \in \mathbb{N}$ such that $\Lambda_N(T_R) < \Lambda_1(T_R^c)$. Show that T has at least N eigenvalues below $\inf \operatorname{spec}_{\text{ess}} T$.

3. Let R and N be as in the previous question. Show that there exists $a > 0$ such that for any $n = 1, \dots, N$ one has $\Lambda_n(T_r) = \Lambda_n(T) + O(e^{-ar})$ as $r \to +\infty$. Hint: Use Theorem 9.7 and mimic the proof of Theorem 8.9.

9.2 Domains with cylindrical ends

Let $\Omega \subset \mathbb{R}^d$ be a domain with lipschitzian boundary and having the following property: there exist $R > 0$ and bounded lipschitzian domains $\omega_1, \dots \omega_m \subset \mathbb{R}^{d-1}$ such that

$$\Omega \cap \{|x| > R\} = \Big(\bigcup_{j=1}^m U_j\Big) \cap \{|x| > R\}, \tag{9.11}$$

where U_j are disjoint domains such that each U_j is isometric to $\omega_j \times (0, +\infty)$. Such domains are often called *domains with cylindrical ends with cross-sections* ω_j (each U_j is then referred to as a cylindrical end).

Theorem 9.8. *There holds* $\operatorname{spec}_{\text{ess}} T^\Omega = [\lambda, +\infty)$*, where* $\lambda := \min_{j=1}^m \lambda_1^D(\omega_j)$.

Proof. The inclusion $\operatorname{spec}_{\text{ess}} T^\Omega \subset [\lambda, +\infty)$ is already known (Theorem 9.4). Now we are going to show that $\lambda + k^2 \in \operatorname{spec} T^\Omega$ for any $k \ge 0$: then it follows that $[\lambda, +\infty) \subset \operatorname{spec} T^\Omega$, and automatically $[\lambda, +\infty) \subset \operatorname{spec}_{\text{ess}} T^\Omega$ as the set $[\lambda, +\infty)$ has no isolated points.

Let us choose j with $\lambda_1^D(\omega_j) = \lambda$ and let φ be an eigenfunction of $-\Delta_D^{\omega_j}$ for the eigenvalue $\lambda = \lambda_1^D(\omega_j)$, which we assume normalized, $\|\varphi\|_{L^2(\omega_j)} = 1$. Without loss of generality we may assume that $U_j = \omega_j \times (0, \infty)$.

We write $x \in \mathbb{R}^d$ as $x = (x', x_d)$ with $x' = (x_1, \dots, x_{d-1})$. Let us choose $\chi \in C_c^\infty(\mathbb{R})$ with $0 \le \chi \le 1$, $\chi(t) = 1$ for $2 \le t \le 3$, $\chi(t) = 0$ for $x \le 1$ and for $x \ge 4$. Assume that R is chosen sufficiently large to have (9.11). For $n \ge R$ introduce $u_n : \Omega \to \mathbb{C}$ by

$$u_n(x', x_d) = \begin{cases} \varphi(x') e^{ikx_d} \chi\big(\frac{x_d}{n}\big), & (x', x_d) \in U_j \text{ with } x_d \ge R, \\ 0, & \text{for all other } x \in \Omega, \end{cases}$$

then $u_n \in D(T^\Omega)$ with

$$\|u_n\|^2_{L^2(\Omega)} \ge \int_{(x',x_d)\in U_j : 2n \le x_d \le 3n} \big|\varphi(x') e^{ikx_d}\big|^2 dx = \int_{2n}^{3n} \int_{\omega_j} \big|\varphi(x')\big|^2 dx' dx_d = n \tag{9.12}$$

$$\begin{aligned}(T^\Omega u_n)(x', x_d) &= (-\Delta u_n)(x', x_d) = -\Delta_{x'} \varphi(x') \chi\big(\tfrac{x_d}{n}\big) \\ &\quad + \varphi(x') \Big(k^2 e^{ikx_d} \chi\big(\tfrac{x_d}{n}\big) - \tfrac{2ik}{n} e^{ikx_d} \chi'\big(\tfrac{x_d}{n}\big) - \tfrac{1}{n^2} e^{ikx_d} \chi''\big(\tfrac{x_d}{n}\big)\Big)\Big) \\ &= (\lambda + k^2) \varphi(x') \chi\big(\tfrac{x_d}{n}\big) - \varphi(x') \Big(\tfrac{2ik}{n} e^{ikx_d} \chi'\big(\tfrac{x_d}{n}\big) + \tfrac{1}{n^2} e^{ikx_d} \chi''\big(\tfrac{x_d}{n}\big)\Big).\end{aligned}$$

Therefore, $\big(T^\Omega - (\lambda + k^2)\big) u_n = -\varphi(x') \Big(\frac{2ik}{n} e^{ikx_d} \chi'\big(\frac{x_d}{n}\big) + \frac{1}{n^2} e^{ikx_d} \chi''\big(\frac{x_d}{n}\big)\Big)$,

$$\begin{aligned}\big\|\varphi(x') \chi'\big(\tfrac{x_d}{n}\big)\big\|^2_{L^2(\Omega)} &\le \|\chi'\|^2_\infty \int_{x \in U_j,\, n \le x_d \le 4n} \big|\varphi(x')\big|^2 dx \\ &= \|\chi'\|^2_\infty \int_n^{4n} \int_{\omega_j} \big|\varphi(x')\big|^2 dx' dx_d = 3n \|\chi'\|^2_\infty,\end{aligned}$$

$$\left\|\varphi(x')\chi''\left(\tfrac{x_d}{n}\right)\right\|^2_{L^2(\Omega)} \le \|\chi''\|^2_\infty \int_{x\in U_j,\, n\le x_d\le 4n} \left|\varphi(x')\right|^2 dx = 3n\|\chi''\|^2_\infty,$$

which shows that $\left\|\left(T^\Omega - (\lambda + k^2)\right)u_n\right\|_{L^2(\Omega)} = O(1)$ for large n. Using the norm estimate (9.12) for u_n one obtains $\lim_{n\to\infty} \left\|\left(T^\Omega - (\lambda+k^2)\right)u_n\right\|_{L^2(\Omega)}/\|u_n\|_{L^2(\Omega)} = 0$, which implies $\lambda + k^2 \in \operatorname{spec} T^\Omega$. $\square$

In order continue we will need some additional manipulations with Sobolev spaces. For any open subset $U \subset \Omega$ we denote by $P_U : L^2(\Omega) \to L^2(U)$ the operator of restriction to U, i.e. $P_U u(x) = u(x)$ for $x \in U$. Furthermore, consider the sesquilinear form r^U defined by

$$r^U(u,v) = \int_U \overline{\nabla u}\cdot \nabla v\, dx, \quad D(r^U) = P_U H^1_0(\Omega).$$

Remark that by construction one has $P_U H^1_0(\Omega) \subset H^1(U)$, which means that r^U is the restriction of the sesquilinear form for the Neumann Laplacian. As the sesqulinear form for the Neumann Laplacian is closed, it follows that r^U is closable, and we denote by $\widehat{H}^1_0(U)$ the domain of its closure $\overline{r^U}$. Then $\widehat{H}^1_0(U)$ is a closed subspace of $H^1(U)$ in the H^1-norm with

$$\overline{r^U}(u,v) = \int_U \overline{\nabla u}\cdot\nabla v\, dx.$$

Moreover, the density of $C^\infty_c(\Omega)$ in $H^1_0(\Omega)$ implies the density of $P_U C^\infty_c(\Omega)$ in $\widehat{H}^1_0(U)$ in the H^1-norm. We denote

$$R^U := \text{the self-adjoint operator in } L^2(U) \text{ generated by the form } \overline{r^U}.$$

If U has Lipschitz boundary, then then R^U has compact resolvent (as $H^1(U)$ is compactly embedded into $L^2(U)$, see Theorem 3.32).

Theorem 9.9. *The discrete spectrum of T^Ω is at most finite.*

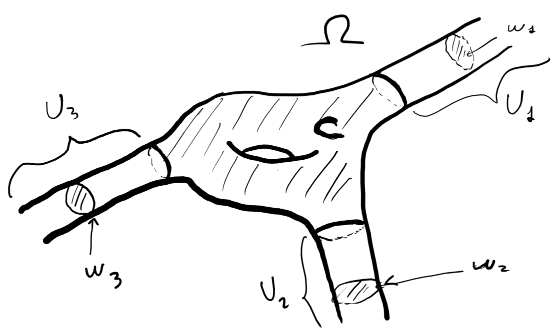


Figure 1: Construction of C in the proof of Theorem 9.9

Proof. Due to the assumption on Ω one can find a bounded lipschitzian subdomain $C\subset\Omega$ such that $\Omega\setminus\overline{C}$ is the union of half-infinite cylinders $U_1,\dots,U_m$ (i.e. each U_j is isometric to $\omega_j\times(0,\infty)$): see Figure 1 for an illustration. Denote $\Omega' := C\cup\bigcup_{j=1}^m U_j$ and let S be the self-adjoint operator in $L^2(\Omega')\equiv L^2(\Omega)$ generated by the closed sesquilinear form

$$s(u,v)=\int_{\Omega'}\overline{\nabla u}\cdot\nabla v\,dx,\quad D(s)=\widehat{H}^1_0(\Omega')\equiv\widehat{H}^1_0(C)\oplus\bigoplus_{j=1}^m\widehat{H}^1_0(U_j).$$

The sesquilinear form s is an extension of the sesquilinear form of T^Ω, i.e. $S\le T^\Omega$ (see Definition 6.7), therefore, $\Lambda_n(S)\le\Lambda_n(T^\Omega)$ for all $n\in\mathbb{N}$ (Corollary 6.8). At the same time we have the representation $S=R^C\oplus R^{U_1}\oplus\dots\oplus R^{U_m}$. As C is chosen bounded with Lipschitz boundary, the operator R^C has compact resolvent. On the other hand, one can easily show that $\Lambda_1(R^{U_j})\ge\lambda_1(\omega_j)$: one can assume without loss of generality that the orthogonal coordinates are chosen in such a way that $U_j=\omega_j\times(0,\infty)$ and take any $u\in C_c^\infty(\Omega)$, then the same computations as in Example 9.3 show that

$$\int_{U_j}|\nabla u|^2\,dx\ge\lambda_1^D(\omega_j)\int_{U_j}|u|^2\,dx,$$

which extends then by density to all $u\in\widehat{H}^1_0(\Omega)$.

Recall that $\lambda:=\min_j\lambda_1^D(\omega_j)$ is the bottom of the essential spectrum of T^Ω, hence, $U_j\ge\lambda$ for all j, and the condition

$$\Lambda_n(S)\equiv\Lambda_n(R^C\oplus R^{U_1}\oplus\dots\oplus R^{U_m})<\lambda$$

is equivalent to $\Lambda_n(R^C)<\lambda$. As R^C has compact resolvent, the latter condition only holds for finitely many n, and then there exists $N\in\mathbb{N}$ with $\Lambda_N(S)\ge\lambda$. Then $\Lambda_N(T^\Omega)\ge\Lambda_N(S)\ge\lambda$, i.e. T^Ω has at most N eigenvalues below λ. □

Remark 9.10. As seen from the proof, one has the estimate,

$$\#\big\{n:\Lambda_n(T^\Omega)<\lambda\big\}\le\#\big\{n:\Lambda_n(R^C)<\lambda\big\}.$$

One can show that the operator R^C is in fact associated with the sesquilinear form

$$r^C(u,v)=\int_C\overline{\nabla u}\cdot\nabla v\,dx$$

with domain $D(r^C)=\big\{u\in H^1(C):u=0\text{ on }\partial\Omega\cap\partial C\big\}$. The condition "$u=0$ on $\partial\Omega\cap\partial C$" should be understood in the spirit of the trace theorem (Theorem 7.1): for $u\in H^1(C)$ its restriction on ∂C is defined as a function from $L^2(\partial C)$, an we require that this function vanishes of the subset $\partial C\cap\partial\Omega$. In some cases one can give a rather precise estimate for the eigenvalues of R^C, which then results in a lower bound for the number of discrete eigenvalues of T^Ω.

Remark that due to $C\subset\Omega$ one also has $\Lambda_n(T^\Omega)\le\Lambda_n(T^C)$ for all n (domain monotonicity). In particular, if $\Lambda_n(T^C)<\lambda$ for some n, then T^Ω has at least n eigenvalues below λ. □

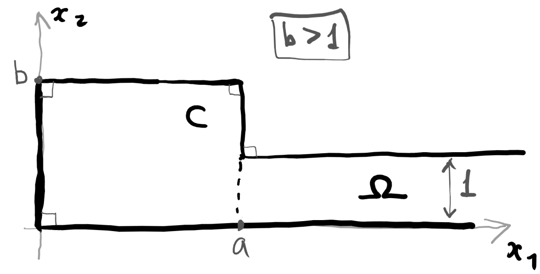


Figure 2: Example 9.11

Example 9.11. Consider $\Omega \subset \mathbb{R}^2$ shown in Figure 2: It is a domain with a single cylindrical end $(a,\infty)\times(0,1)$ whose cross-section is $\omega=(0,1)$ and $\lambda_1^D(\omega)=\pi^2$, so we obtain $\operatorname{spec}_{\text{ess}} T^\Omega=[\pi^2,+\infty)$. An obvious candidate for C is the rectangle $(0,a)\times(0,b)$, and the operator R^C is generated by the sesqulinear form

$$\begin{gathered}
r^c(u,v)=\int_C \overline{\nabla u}\cdot\nabla v\,dx,\\
D(r^c)=\Big\{u\in H^1(C): u=0 \text{ on } \partial C\setminus\big(\{a\}\times(0,1)\big)\Big\}.
\end{gathered}$$

This operator is difficult to deal with, and in order to estimate its eigenvalues we consider first the extension r' of r^C given by

$$\begin{gathered}
r'(u,v)=\int_C \overline{\nabla u}\cdot\nabla v\,dx,\\
D(r')=\Big\{u\in H^1(C): u=0 \text{ on } \partial C\setminus\big(\{a\}\times(0,b)\big)\Big\}.
\end{gathered}$$

If R' is the self-adjoint operator generated by r', then $\Lambda_n(R')\le\Lambda_n(R^C)$ for all $n\in\mathbb{N}$. The operator R' appears to be with separated variables: one has (Exercise!) $R'=A\otimes 1+1\otimes B$, where B is just the Dirichlet Laplacian on $(0,b)$, while A acts as $f\mapsto -f''$ on the functions $f\in H^2(0,a)$ with $f(0)=f'(a)=0$: the sesquilinear form t_A for A is

$$t_A(f,g)=\int_0^a \overline{f'}g'\,dx,\quad D(t_A)=\big\{f\in H^1(0,a): f(0)=0\big\}.$$

The eigenvalues of B are $\frac{\pi^2k^2}{b^2}$, $k\in\mathbb{N}$, and the eigenvalues of A can be easily computed as well, they are $\frac{\pi^2(2j-1)^2}{4a^2}$, $j\in\mathbb{N}$. Hence, the eigenvalues of R' are $\frac{\pi^2(2j-1)^2}{4a^2}+\frac{\pi^2k^2}{b^2}$ with $(j,k)\in\mathbb{N}\times\mathbb{N}$. It follows that the number of eigenvalues of T^Ω below π^2 does not exceed the number of eigenvalues of R' below π^2, i.e. the number of pairs $(j,k)\in\mathbb{N}\times\mathbb{N}$ with $\frac{\pi^2(2j-1)^2}{4a^2}+\frac{\pi^2k^2}{b^2}<\pi^2$.

On the other hand, the eigenvalues of the Dirichlet Laplacian T^C in C are $\frac{\pi^2j^2}{a^2}+\frac{\pi^2k^2}{b^2}$ with $(j,k)\in\mathbb{N}$. For example, if $\Lambda_1(T^C)=\pi^2\big(\frac{1}{a^2}+\frac{1}{b^2}\big)<\pi^2$, then T^Ω has at least one eigenvalue below π^2. We conclude that for $\frac{1}{a^2}+\frac{1}{b^2}<1$ the operator T^Ω hat at least one discrete eigenvalue.

Now pick any $N \in \mathbb{N}$ and $a > 1$, then one can choose a sufficiently large b for which

$$\Lambda_N(T^c) = \tfrac{\pi^2}{a^2} + \tfrac{\pi^2 N^2}{b^2} < \pi^2,$$

then $\Lambda_N(T^\Omega) < \pi^2$, and T^Ω has at least N eigenvalues below π^2. $\square$

Now we would like to discuss in greater detail the discrete spectrum of T^Ω: while we know that it is (at most) finite, we only have very rough conditions (in terms of the Dirichlet eigenvalues of C) guaranteeing the existence of at least one discrete eigenvalue. In order to simplify geometric constructions and to avoid using the formalism of differential geometry, in the rest of this subsection we will restrict ourselves by considering two-dimensional domains with cylindrical ends. In two dimensions, the only possible cross-sections ω_j are intervals: without loss of generality one assumes $\omega_j = (0, \ell_j)$ with some $\ell_j > 0$, which severely reduces the technical complexity of the analysis. Furthermore, in this case $\lambda_1^D(\omega_j) = \frac{\pi^2}{\ell_j^2}$, and $\lambda = \frac{\pi^2}{\ell^2}$ with $\ell := \max \ell_j$.

Many results on the existence of eigenvalues are based on comparisons of Ω with infinite strips. Without loss of generality one can assume that the strip has unit width and that it is horizontal, i.e. we consider

$$\Pi := \mathbb{R} \times (0,1).$$

The preceding discussion implies (Exercise!) that $\operatorname{spec} T^\Pi = [\pi^2, \infty)$.

The following theorem states that if Ω is obtained from Π by removing a bounded set, then the Dirichlet Laplacians in Π and Ω have the same spectrum.

Theorem 9.12. *Assume that $\Omega \subset \Pi$ and that $\Pi \setminus \Omega$ is a bounded set. Then $\operatorname{spec} T^\Omega = \operatorname{spec} T^\Pi = [\pi^2, \infty)$.*

Proof. Due to the assumption, the sets Ω and Π coincide outide a ball, then by Theorem 9.8 we obtain $\operatorname{spec}_{\text{ess}} T^\Omega = [\pi^2, \infty)$. The domain monotonicity implies $\pi^2 = \Lambda_1(T^\Pi) \le \Lambda_1(T^\Omega) \le \inf \operatorname{spec}_{\text{ess}} T^\Omega = \pi^2$, which shows that $\Lambda_1(T^\Omega) = \pi^2$: this means that T^Ω has no spectrum below π^2. $\square$

Now we consider the opposite situation: assume that Ω is strictly larger that Π. It appears that this necessarily influences the bottom of the spectrum (even without assuming that the perturbation is localized in a ball):

Theorem 9.13. *Let $\Omega \subset \mathbb{R}^2$ be a connected domain such that $\Omega \supset \Pi$ and $\Omega \neq \Pi$, then $\Lambda_1(T^\Omega) < \pi^2$.*

Proof. Remark first that the inclusion $\Omega \supset \Pi$ implies, due to the domain monotonicity, $\Lambda_1(\Omega) \le \Lambda_1(\Pi) = \pi^2$, and the meaning of the result is that the inequality is strict.

Due to the assumption, some boundary points of Π are interior points of Ω. Without loss of generality we assume that the point $M = (0,1)$ is an interior point of Ω, then there exists $\delta > 0$ such that $|x - M| < \delta$ implies $x \in \Omega$. Let $f \in C_c^\infty(\mathbb{R})$ with

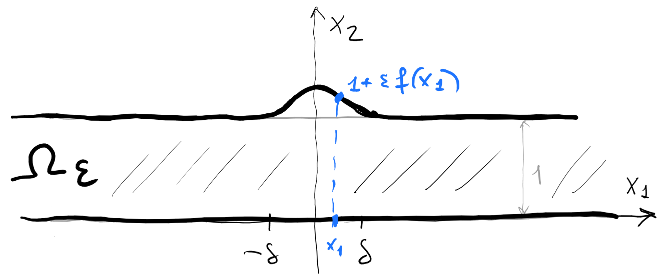


Figure 3: The domain Ω_ε in the proof of Theorem 9.13

$\operatorname{supp} f \subset (-\delta, \delta)$, $f \geq 0$ and $f(0) > 0$, then for sufficiently small $\varepsilon > 0$ all points (x_1, x_2) with $1 \leq x_1 < 1 + \varepsilon f(x_1)$ belong to Ω. Therefore, if one denotes

$$\Omega_\varepsilon = \Big\{(x_1, x_2) : 0 < x_1 < 1 + \varepsilon f(x_2)\Big\},$$

then $\Omega_\varepsilon \subset \Omega$ for all sufficiently small $\varepsilon > 0$, and due to the domain monotonicity one has $\Lambda_1(\Omega) \leq \Lambda_1(\Omega_\varepsilon)$, see Figure 3. In order to obtain the claim it is sufficient to show that $\Lambda_1(\Omega_\varepsilon) < \pi^2$ for all sufficiently small $\varepsilon > 0$, and for that it is sufficient to show that there exists a function $u_\varepsilon \in H_0^1(\Omega_\varepsilon)$ with $\|\nabla u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} - \pi^2 \|u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} < 0$. Let $\chi : \mathbb{R} \to \mathbb{R}$ such that $0 \leq \chi \leq 1$, $\chi(t) = 1$ for $|t| \leq 1$, and $\chi(t) = 0$ for $|t| \geq 2$. Consider the function

$$u_{\varepsilon,n}(x_1, x_2) = \sin\Big(\frac{\pi x_2}{1 + \varepsilon f(x_1)}\Big)\chi\Big(\frac{x_1}{n}\Big), \quad (x_1, x_2) \in \Omega_\varepsilon,$$

which clearly belongs to $H_0^1(\Omega_\varepsilon)$ for any $n > 0$. The choice is motivated by the fact that the function $x_2 \mapsto \sin\big(\frac{\pi x_2}{1+\varepsilon f(x_1)}\big)$ is the eigenfunction for the first eigenvalue of the Dirichlet Laplacian on $\big(0, 1 + \varepsilon f(x_1)\big)$, and $1 + \varepsilon f(x_1)$ is exactly the "height" of Ω_ε at x_1. Using the integration by parts we have

$$\begin{aligned}
\int_{\Omega_\varepsilon} |\partial_2 u_{\varepsilon,n}|^2\, dx &= \int_{\mathbb{R}} \int_0^{1+\varepsilon f(x_1)} \Big[\tfrac{\partial}{\partial x_2} \sin\big(\tfrac{\pi x_2}{1+\varepsilon f(x_1)}\big)\Big]^2 dx_2\, \chi\big(\tfrac{x_1}{n}\big)^2\, dx_1 \\
&= \int_{\mathbb{R}} \int_0^{1+\varepsilon f(x_1)} \sin\big(\tfrac{\pi x_2}{1+\varepsilon f(x_1)}\big)\Big[-\tfrac{\partial^2}{\partial x_2^2} \sin\big(\tfrac{\pi x_2}{1+\varepsilon f(x_1)}\big)\Big] dx_2\, \chi\big(\tfrac{x_1}{n}\big)^2\, dx_1 \\
&= \int_{\mathbb{R}} \int_0^{1+\varepsilon f(x_1)} \tfrac{\pi^2}{(1+\varepsilon f(x_1))^2} \sin\big(\tfrac{\pi x_2}{1+\varepsilon f(x_1)}\big)^2 \chi\big(\tfrac{x_1}{n}\big)^2\, dx_1\, dx_2.
\end{aligned}$$

and

$$\begin{aligned}
I_{\varepsilon,n} &:= \int_{\Omega_\varepsilon} |\partial_2 u_{\varepsilon,n}|^2\, dx - \pi^2 \int_{\Omega_\varepsilon} |u_{\varepsilon,n}|^2\, dx \\
&= \int_{\mathbb{R}} \Big(\tfrac{\pi^2}{(1+\varepsilon f(x_1))^2} - \pi^2\Big)\chi\big(\tfrac{x_1}{n}\big)^2 \int_0^{1+\varepsilon f(x_1)} \sin\big(\tfrac{\pi x_2}{1+\varepsilon f(x_1)}\big)^2 dx_2\, dx_1 \\
&= \pi^2 \int_{\mathbb{R}} \big(\tfrac{1}{(1+\varepsilon f(x_1))^2} - 1\big)\, \tfrac{1+\varepsilon f(x_1)}{2}\, \chi\big(\tfrac{x_1}{n}\big)^2 dx_1
\end{aligned}$$

Remark that the first factor in the subintegral expression vanishes for $|x_1| \geq \delta$, while $\chi\big(\frac{x_1}{n}\big) = 1$ for $|x_1| \leq \delta$ if $n \geq \delta$. Therefore, for all $n \geq \delta$ one has

$$I_{\varepsilon,n} = \frac{\pi^2}{2} \int_{\mathbb{R}} \Big(\frac{1}{(1+\varepsilon f(x_1))^2} - 1 \Big) \tfrac{1+\varepsilon f(x_1)}{2}\, dx_1 = \frac{\pi^2}{2} \int_{\mathbb{R}} \frac{-2\varepsilon f(x_1) - \varepsilon^2 f(x_1)^2}{1+\varepsilon f(x_1)}\, dx_1,$$

and for $\varepsilon \in (0, \varepsilon_0)$ and $n \geq \delta$ we have

$$I_{\varepsilon,n} \leq -a\varepsilon \quad \text{with} \quad a := \pi^2 \int_{\mathbb{R}} \frac{f(x_1)}{1+\varepsilon_0 f(x_1)}\, dx_1 > 0.$$

Now let us control the term

$$J_{\varepsilon,n} := \int_{\Omega_\varepsilon} |\partial_1 u_{\varepsilon,n}|^2\, dx.$$

Assuming again $n \geq \delta$ and $\varepsilon \in (0, \varepsilon_0)$ one first estimates

$$\begin{aligned}\big|\partial_1 u_{\varepsilon,n}(x_1,x_2)\big|^2 &= \Big| -\varepsilon\, \tfrac{\pi x_2 f'(x_1)}{(1+\varepsilon f(x_1))^2} \cos\big(\tfrac{\pi x_2}{1+\varepsilon f(x_1)}\big)\, \chi\big(\tfrac{x_1}{n}\big) + \tfrac{1}{n} \sin\big(\tfrac{\pi x_2}{1+\varepsilon f(x_1)}\big) \chi'\big(\tfrac{x_1}{n}\big)\Big|^2 \\ &\leq 2\varepsilon^2 \Big| \tfrac{\pi x_2 f'(x_1)}{(1+\varepsilon f(x_1))^2} \cos\big(\tfrac{\pi x_2}{1+\varepsilon f(x_2)}\big)\, \chi\big(\tfrac{x_1}{n}\big)\Big|^2 + \tfrac{2}{n^2} \Big| \sin\big(\tfrac{\pi x_2}{1+\varepsilon f(x_1)}\big) \chi'\big(\tfrac{x_1}{n}\big)\Big|^2,\end{aligned}$$

and then

$$\begin{aligned}\int_{\mathbb{R}} \int_0^{1+\varepsilon f(x_1)} &\Big| \tfrac{\pi x_2 f'(x_1)}{(1+\varepsilon f(x_1))^2} \cos\big(\tfrac{\pi x_2}{1+\varepsilon f(x_1)}\big)\, \chi\big(\tfrac{x_1}{n}\big)\Big|^2 dx_2\, dx_1 \\ &\leq \int_{-2n}^{2n} \big(1+\varepsilon f(x_1)\big) \cdot \pi^2 \big(1+\varepsilon f(x_1)\big)^2 \big|f'(x_1)\big|^2\, dx_1 \\ &\qquad\qquad \leq \pi^2 \int_{-\delta}^{\delta} \big(1+\varepsilon_0 f(x_1)\big)^3 \big|f'(x_1)\big|^2\, dx_1 =: b,\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}} \int_0^{1+\varepsilon f(x_1)} &\Big| \sin\big(\tfrac{\pi x_2}{1+\varepsilon f(x_1)}\big) \chi'\big(\tfrac{x_1}{n}\big)\Big|^2 dx_2\, dx_1 \\ &\leq \|\chi'\|_\infty \int_{-2n}^{2n} \big(1+\varepsilon f(x_1)\big)\, dx_1 \leq cn, \quad c := 4\|\chi'\|_\infty \|1+\varepsilon_0 f\|_\infty.\end{aligned}$$

This gives the estimate $J_{\varepsilon,n} \leq 2b\varepsilon^2 + \dfrac{2c}{n}$. It follows that, for all $n \geq \delta$ and $\varepsilon \in (0, \varepsilon_0)$,

$$A_{\varepsilon,n} := \int_{\Omega_\varepsilon} \big(|\nabla u_{\varepsilon,n}|^2 - \pi^2 |u_{\varepsilon,n}|^2\big)\, dx = I_{\varepsilon,n} + J_{\varepsilon,n} \leq -a\varepsilon + 2b\varepsilon^2 + \frac{2c}{n}.$$

We can choose $\varepsilon_1 \in (0, \varepsilon_0)$ such that $-a\varepsilon + 2b\varepsilon^2 < 0$ for any $\varepsilon \in (0, \varepsilon_1)$. Then for any fixed $\varepsilon \in (0, \varepsilon_1)$ one can find a sufficiently large n for which $A_{\varepsilon,n} < 0$. This shows that $\Lambda_1(T_{\Omega_\varepsilon}) < \pi^2$ for any $\varepsilon \in (0, \varepsilon_1)$ and proves the claim. $\square$

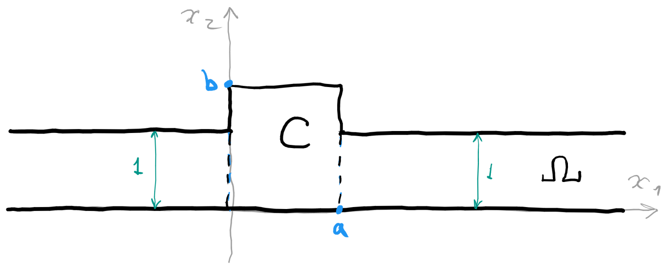


Figure 4: Domain Ω from Example 9.15.

Corollary 9.14. *Let $\Omega \subset \mathbb{R}^2$ be a connected domain with $\Omega \supset \Pi$ but $\Omega \neq \Pi$, and Ω coincides with Π outside a ball, then $\operatorname{spec}_{\mathrm{ess}} T^\Omega = [\pi^2, +\infty)$, and T^Ω has at least one and at most finitely many eigenvalues below π^2.*

Proof. The equality $\operatorname{spec}_{\mathrm{ess}} T^\Omega = [\pi^2, +\infty)$ follows from Theorem 9.8, and the finiteness of the discrete spectrum follows from Theorem 9.9. Finally, by Theorem 9.13 one has $\Lambda_1(T^\Omega) < \pi^2 = \inf \operatorname{spec}_{\mathrm{ess}} T^\Omega$, hence, $\Lambda_1(T^\Omega)$ is an eigenvalue of T^Ω. $\square$

Example 9.15. Consider the domain Ω shown in Figure 4. We are in the situation of Corollary 9.14, hence, the essential spectrum of T^Ω is $[\pi^2, +\infty)$ and T^Ω has at least one eigenvalue below π^2. On the other hand, if one uses the above Remark 9.10 one concludes that the number of eigenvalues does not exceed the number of eigenvalues below π^2 for the self-adjoint operator R in $L^2(C)$, $C := (0, a) \times (0, b)$, given by the sesquilinear form

$$r(u, v) = \int_C \overline{\nabla u} \cdot \nabla v \, dx$$
$$D(r) = \Big\{ u \in H^1(\Omega) : u = 0 \text{ on } \partial C \setminus \Big(\big(\{0\} \times (0,1)\big) \cup \big(\{a\} \times (0,1)\big) \Big) \Big\}.$$

Let r' be the extension of r given by

$$r'(u, v) = \int_C \overline{\nabla u} \cdot \nabla v \, dx$$
$$D(r') = \Big\{ u \in H^1(\Omega) : u = 0 \text{ on } \partial C \setminus \Big(\big(\{0\} \times (0,b)\big) \cup \big(\{a\} \times (0,b)\big) \Big) \Big\},$$

and R' be the associated self-adjoint operator, then $\Lambda_n(R') \leq \Lambda_n(R)$ for all $n \in \mathbb{N}$. The operator R' can be written as $R' = A \otimes 1 + 1 \otimes B$ (Exercise!), where A is the Neumann Laplacian on $(0, a)$ and B is the Dirichlet Laplacian on $(0, b)$, and the eigenvalues of R' are $\frac{\pi^2 j^2}{a^2} + \frac{\pi^2 k^2}{b^2}$, $j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $k \in \mathbb{N}$. As a result, the number of discrete eigenvalues of T^Ω cannot exceed the number of eigenvalues of R' below π^2, i.e. the number of pairs $(j, k) \in \mathbb{N}_0 \times \mathbb{N}$ for which one has $\frac{\pi^2 j^2}{a^2} + \frac{\pi^2 k^2}{b^2} < \pi^2$. For example, assume that $a \in (0, 1)$ and $b \in (1, 4)$, then the inequality is satisfied for the unique pair $(j, k) = (0, 1)$, which shows that T^Ω has a unique discrete eigenvalue for such a and b.

Exercise 26. Let $\Omega \subset \mathbb{R}^d$ be an open set. Assume that there exists an open set $\omega \subset \mathbb{R}^d$ such that Ω contains infinitely many disjoint copies of ω. Show that the essential spectrum of T^Ω is non-empty and that $\inf \operatorname{spec}_{\text{ess}} T^\Omega \le \lambda_1^D(\omega)$.

Exercise 27. Let $f : \mathbb{R} \to 0$ be a C^∞ function satisfying $\lim_{|x|\to\infty} f^{(k)}(x) = 0$ for all $k \in \mathbb{N} \cup \{0\}$. Consider the domain

$$\Omega := \big\{(x_1, x_2) : \, x_1 \in \mathbb{R}, \, 0 < x_2 < 1 + f(x_1)\big\}.$$

1. Let $\varepsilon > 0$. Show that there exists $R > 0$ such that $\Omega_R^c \subset (0, 1+\varepsilon) \times \mathbb{R}$ and deduce that $\operatorname{spec}_{\text{ess}} T^\Omega \subset \big[\frac{\pi^2}{(1+\varepsilon)^2}, \infty\big)$.
2. Show that $\operatorname{spec}_{\text{ess}} T = [\pi^2, +\infty)$. Hint: consider the functions

   $$u_n(x_1, x_2) = \sin\Big(\frac{\pi x_2}{1 + f(x_1)}\Big) e^{ikx_1} \chi_n(x_1)$$

   with $k > 0$ and suitable cut-off functions χ_n.
3. Assume that $f \ge 0$ and $f \not\equiv 0$. Show that T^Ω has at least one discrete eigenvalue.

Exercise 28. Consider again the example 9.15. Find an explicit range of a and b for which the operator T^Ω has (1) at most two discrete eignvalues, (2) at least two discrete eigenvalues?

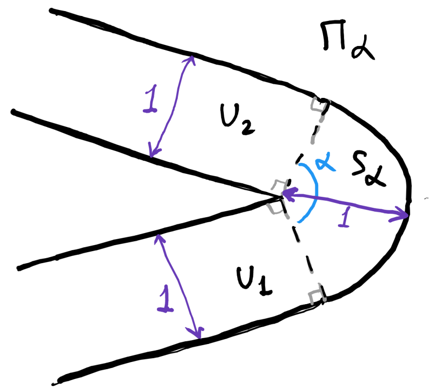


Figure 5: Domain Π_α.

As a much more involved class of domains with cylindrical ends we consider so-called "bent" strips. The general theory of such objects uses some computations from the differential geometry: due to this reason we will only consider a kind of minimalistic example.

Let $\alpha \in (0,\pi)$ and consider the domain Π_α obtained by gluing two copies (denoted U_1 and U_2) of $((0,\infty)\times(0,1)$ to the straight sides of a sector S_α of unit radius and of opening angle α, see Figure 5.

The preceding discussion implies that $\operatorname{spec}_{\text{ess}} T^{\Pi_\alpha} = [\pi^2,+\infty)$ for any choice of α. We are going to prove the following result:

Theorem 9.16. *For any $\alpha \in (0,\pi)$ the Dirichlet Laplacian T^{Π_α} in Π_α has at least one eigenvalue below π^2.*

Proof. We simply need to show that $\Lambda_1(T^{\Pi_\alpha}) < \pi^2$, and for that we need to show the existence of a function $u \in H^1_0(\Pi_\alpha)$ satisfying the strict inequality

$$\int_{\Pi_\alpha} \big(|\nabla u|^2 - \pi^2 |u|^2\big) < 0.$$

This function will be constructed using a limiting procedure involving several parameters.

Pick a C^∞ function $\chi : \mathbb{R} \to \mathbb{R}$ with $0 \le \chi \le 1$, $\chi(t) = 1$ for $|t| \le 1$ and $\chi(t) = 0$ for $|t| \ge 2$. Without loss of generality we assume that the sector S_α is described in the standard polar coordinates (r,θ) by $(r,\theta) \in (0,1)\times(0,\alpha)$. In addition, choose the orthogonal coordinates (y_1,y_2) such that the half-strip U_1 corresponds to $(y_1,y_2) \in (0,\infty)\times(0,1)$, and the orthogonal coordinates (z_2,z_2) such that the half-strip U_2 corresponds to $(z_1,z_2) \in (0,\infty)\times(0,1)$, see Figure 6 for illustration.

For $n > 0$ (to be chosen later) we define $u_n \in H^1_0(\Pi_\alpha)$ as follows:

$$\text{in } S_\alpha: \qquad\qquad u(x_1,x_2) = \sin(\pi r),$$

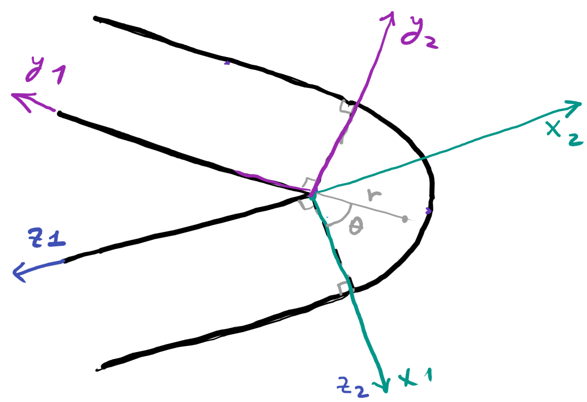


Figure 6: Coordinates in Π_α.

$$\text{in } U_1: \qquad u(x_1,x_2) = \sin(\pi y_2)\chi\big(\frac{y_1}{n}\big),$$
$$\text{in } U_2: \qquad u(x_1,x_2) = \sin(\pi z_2)\chi\big(\frac{z_1}{n}\big).$$

We have

$$\begin{aligned}
I &:= \int_{U_1} \big(|\nabla u_n|^2 - \pi^2|u_n|^2\big)\,dx \\
&= \int_{U_1} \big(|\partial_{y_1} u_n|^2 + |\partial_{y_2} u_n|^2 - \pi^2|u_n|^2\big)\,dx \\
&= \int_0^\infty \int_0^1 \Big[\tfrac{1}{n^2}\Big|\sin(\pi y_2)\chi'\big(\tfrac{y_1}{n}\big)\Big|^2 + \pi^2\Big|\cos(\pi y_2)\chi\big(\tfrac{y_1}{n}\big)\Big|^2 \\
&\qquad - \pi^2\Big|\sin(\pi y_2)\chi\big(\tfrac{y_1}{n}\big)\Big|^2\Big]\,dy_2\,dy_1
\end{aligned}$$

Due to

$$\int_0^1 \big(\cos^2(\pi y_2) - \sin^2(\pi y_2)\big)dy_2 = \int 0^2 \cos(2\pi y_2)\,dy_2 = 0$$

we have

$$\begin{aligned}
I_n &= \tfrac{1}{n^2}\int_0^\infty \int_0^1 \Big|\sin(\pi y_2)\chi'\big(\tfrac{y_1}{n}\big)\Big|^2\,dy_2\,dy_1 \\
&\le \tfrac{1}{n^2}\int_n^{2n}\int_0^1 \|\chi'\|_\infty^2 dy_2\,dy_1 = \frac{a}{n}, \quad a = \|\chi'\|_\infty^2.
\end{aligned}$$

By construction we have

$$\int_{U_2} \big(|\nabla u_n|^2 - \pi^2|u_n|^2\big)\,dx = \int_{U_1} \big(|\nabla u_n|^2 - \pi^2|u_n|^2\big)\,dx$$

Therefore,

$$\int_{\Pi_\alpha\setminus S_\alpha} \big(|\nabla u_n|^2 - \pi^2|u_n|^2\big)\,dx = \int_{U_1} + \int_{U_2} = \frac{2a}{n}.$$

The integral in S_α will be computed using polar coordinates. Recall that for the standard change of variables $(x_1, x_2) = (r\cos\theta, r\sin\theta)$ one has

$$\partial_r u = \cos\theta\, \partial_{x_1} u + \sin\theta\, \partial_{x_2} u, \quad \partial_\theta u = -r\sin\theta\, \partial_{x_1} u + r\cos\theta\, \partial_{x_2} u$$

implying

$$\partial_{x_1} u = \cos\theta\, \partial_r u - \frac{\sin\theta}{r}\, \partial_\theta u, \quad \partial_{x_2} u = \sin\theta\, \partial_r u + \frac{\cos\theta}{r}\, \partial_\theta u, \quad |\nabla u|^2 = |\partial_r u|^2 + \frac{1}{r^2}\, |\partial_\theta u|^2.$$

Therefore, using $\partial_\theta u_n = 0$ the change of variables in the integral,

$$\begin{aligned}
\int_{S_\alpha} \big(|\nabla u_n|^2 - \pi^2 |u_n|^2\big)\, dx &= \int_0^\alpha \int_0^1 \big(|\partial_r u_n|^2 - \pi^2 |u|^2\big)\, r dr\, d\theta \\
&= \pi^2 \int_0^\alpha \int_0^1 r\big(\cos^2(\pi r) - \sin^2(\pi r)^2\big)\, dr\, d\theta \\
&= \pi^2 \int_0^\alpha \underbrace{\int_0^1 r\cos(2\pi r)\, dr}_{=0}\, d\theta = 0.
\end{aligned}$$

At this point the result is insatisfactory: we have

$$I_n := \int_{\Pi_\alpha} \big(|\nabla u_n|^2 - \pi^2 |u_n|^2\big)\, dx = \int_{\Pi_\alpha \setminus S_\alpha} + \int_{S_\alpha} \le \frac{2a}{n},$$

and this upper bound does not help us to show that the integral can be negative.

We now proceed with a clever trick consisting in a local perturbation of u_n. Namely, let $v \in C_c^\infty(\Pi_\alpha)$ be real-valued with $\operatorname{supp} v \subset S_\alpha$. For $\varepsilon > 0$ consider the function $u_{n,\varepsilon} = u_n + \varepsilon v$ and

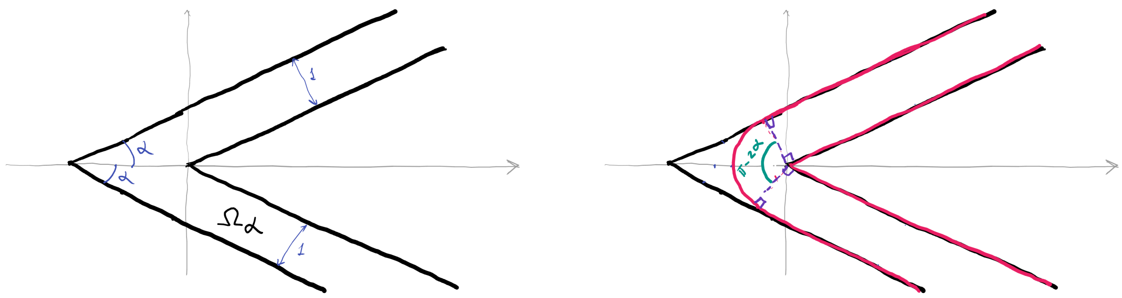
$$J_n(\varepsilon) := \int_{\Pi_\alpha} \big(|\nabla u_{n,\varepsilon}|^2 - \pi^2 |u_{n,\varepsilon}|^2\big)\, dx.$$

One has $J_n(\varepsilon) = I_n + 2A\varepsilon + B\varepsilon^2$ with

$$A := \int_{\Pi_\alpha} (\nabla u_n \cdot \nabla v - \pi^2 u_n v)\, dx, \quad B := \int_{\Pi_\alpha} \big(|\nabla v|^2 - \pi^2 v^2\big)\, dx.$$

Using the fact $\operatorname{supp} v \subset S_\alpha$ we compute

$$\begin{aligned}
A &= \int_{\Pi_\alpha} (\nabla u_n \cdot \nabla v - \pi^2 u_n v)\, dx \\
&= \int_{S_\alpha} (\nabla u_n \cdot \nabla v - \pi^2 u_n v)\, dx \\
&= \int_{S_\alpha} (-\Delta u_n - \pi^2 u_n) v\, dx
\end{aligned}$$



(a) (b)

Figure 7: "Broken" strip Ω_α from Example 9.19.

(we have used the integration by parts in the last step). Now remark that

$$\begin{aligned}
\Delta u_n &= \frac{1}{r}\partial_r\big(r\partial_r u_n\big) + \frac{1}{r^2}\,\partial_\theta^2 u_n\\
&= \frac{\pi}{r}\partial_r\big(r\cos(\pi r)\big) = \pi\frac{\cos(\pi r) - \pi r\sin(\pi r)}{r},\\
-\Delta u_n - \pi^2 u_n &= -\frac{\pi\cos(\pi r)}{r} \not\equiv 0.
\end{aligned}$$

It follows that $(-\Delta u_n - \pi^2 u_n)$ is a continuous function in S_α and does not vanish identically, and then the function v can be chosen to have $A < 0$, which then gives some value for B: we remark that both A and B are independent of n. Therefore,

$$J_n(\varepsilon) = I_n + 2A\varepsilon + B\varepsilon^2 \le \frac{2a}{n} - 2|A|\varepsilon + B\varepsilon^2.$$

We first choose $\varepsilon > 0$ sufficiently small to have $-2|A|\varepsilon + B\varepsilon^2 < 0$, then one can choose n sufficiently large to obtain $J_n(\varepsilon) < 0$. □

Remark 9.17. In fact using some additional computations based on Bessel functions one can show[5] that T^{Π_α} has *exactly* one eigenvalue below π^2 for all $\alpha \in (0, \pi)$.

The domain comparision gives the following useful result:

Corollary 9.18. *If Ω is a two-dimensional domain with $\Pi_\alpha \subset \Omega$, then $\Lambda_1(\Omega) < \pi^2$.*

Proof. One has $\Lambda_1(T^\Omega) \le \Lambda_1(T^{\Pi_\alpha})$ due to the domain monotonicity, and then $\Lambda_1(T^{\Pi_\alpha}) < \pi^2$ due to Theorem 9.16. □

Example 9.19. For $\alpha \in (0, \frac{\pi}{2})$ consider the domain Ω_α show in Figure 7(a). Clearly, $\operatorname{spec}_{\mathrm{ess}} T^{\Omega_\alpha} = [\pi^2, \infty)$. On the other hand, one easily observes that Ω_α contains (a copy of) $\Pi_{\pi-2\alpha}$, see Figure 7(b), and then it follows by Corollary 9.18 that we have $\Lambda_1(T^{\Omega_\alpha}) < \pi^2$, i.e. that T^{Ω_α} has at least one discrete eigenvalue below π^2 for any $\alpha \in (0, \frac{\pi}{2})$.

[5] See Subsection 3.1 in the paper K. Pankrashkin: Eigenvalue inequalities and absence of threshold resonances for waveguide junctions. J. Math. Anal. Appl. 449 (2017) 907–925, Preprint `https://arxiv.org/abs/1606.09620`

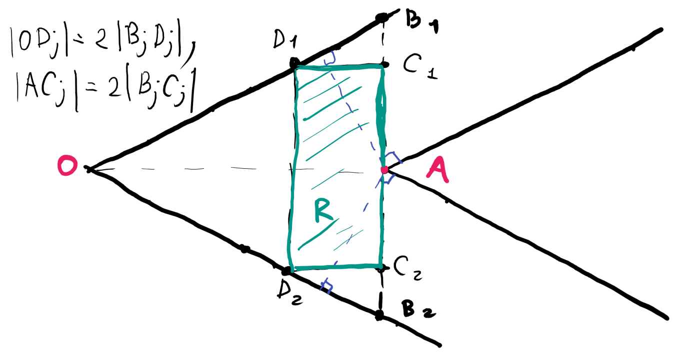


Figure 8: Rectangle R from from Example 9.19. The points C_j and D_j are chosen to have $|AC_j| = 2|B_jC_j|$, and $|OD_j| = 2|B_jD_j|$.

We will show in addition that the number of discrete eigenvalues can be made arbitrarily big if one chooses α sufficiently small. For that, consider the rectangle R shown in Figure 8. As R is contained in Ω_α, one has $\Lambda_n(T^{\Omega_\alpha}) \le \Lambda(T^R)$ for all $n \in \mathbb{N}$. The eigenvalues of the Dirichlet Laplacian T^R in R can be computed explicitly if we compute the side lengths of R. One clearly has

$$\begin{aligned}
|OA| &= \frac{1}{\sin\alpha}, & |C_1D_1| &= |C_2D_2| = \frac{1}{3}\,|OA| = \frac{1}{3\sin\alpha},\\
|AB_j| &= \frac{1}{\cos\alpha}, & |D_1D_2| &= |C_1C_2| = \frac{2}{3}\,|B_1B_2| = \frac{4}{3}\,|AB_1| = \frac{4}{3\cos\alpha}.
\end{aligned}$$

Then the eigenvalues of R are

$$E_{j,k} = \frac{9\pi^2}{16}(\cos\alpha)^2 j^2 + 9\pi^2(\sin\alpha)^2 k^2, \quad (j,k) \in \mathbb{N}\times\mathbb{N}.$$

In particular,

$$\begin{aligned}
E_{1,k} &= \frac{9\pi^2}{16}(\cos\alpha)^2 + 9\pi^2(\sin\alpha)^2 k^2 \le \frac{9\pi^2}{16} + 9\pi^2(\sin\alpha)^2\,k^2\\
&= \pi^2 - \pi^2\Big(\frac{7}{16} - 9(\sin\alpha)^2\,k^2\Big).
\end{aligned}$$

Therefore, if

$$k < \sqrt{\frac{7}{144}}\,\frac{1}{\sin\alpha},$$

then $E_{1,k} < \pi^2$. For any given $k \in \mathbb{N}$, the condition is satisfied for all sufficiently small α. It means that for any k one can find $\alpha_k > 0$ with $\Lambda_k(T^R) < \pi^2$ for all $\alpha \in (\alpha_k)$, and then $\Lambda_k(T^{\Omega_\alpha}) < \pi^2$ for the same α, which shows that T^{Ω_α} has at least k eigenvalues below π^2 for any $\alpha \in (0, \alpha_k)$[6].

[6]This proof method is borrowed from the paper S. A. Nazarov, A. V. Shanin: Trapped modes in angular joints of waveguides. Appl. Anal. 93 (2014) 572–582.

In fact, the dependence of the spectrum of T^{Ω_α} on the angle α has attracted a lot of attention[7]. It can be shown that the eigenvalues are monotonically increasing in α, and it is known that there is a single discrete eigenvalue for $\alpha \in (\arctan \frac{\sqrt{3}}{4}, \frac{\pi}{2})$ (the approximate value $\arctan \frac{\sqrt{3}}{4} \simeq 23,4°$). Numerical simulations suggest that this value is not optimal and that one has a single eigenvalue for $\alpha \geq 14°$.

[7]See e.g. Subsection 3.2 in the paper K. Pankrashkin: Eigenvalue inequalities and absence of threshold resonances for waveguide junctions. J. Math. Anal. Appl. 449 (2017) 907–925, Preprint `https://arxiv.org/abs/1606.09620`

Exercise 29. Discuss the spectral properties of the Dirichlet Laplacians in the following domains: indicate the essential spectrum, the existence of discrete eigenvalues, number of discrete eigenvalues. What can be said about the eigenfunctions associated with the discrete eigenvalues?