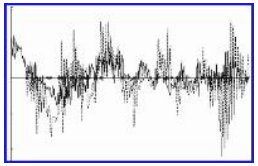
UNIVERSITÄT BAYREUTH

Lehrstuhl
Mathematik VII



# Optimally one-sided bounded influence curves

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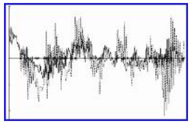


# 1 Introduction

## 1 (a) Motivation

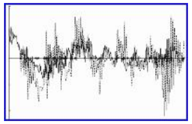
- Situation: estimation of $\theta \in \mathbb{R}$ in smooth parametric model
- common losses/risks like MSE treat upward and downward deviations symmetrically
- not always suitable in applications
  - estimation of mortalities for an insurance company: over-estimation might even be beneficial
  - portfolio optimization: only downside risk is seen as dangerous for the investment

  $\rightsquigarrow$ Markowitz[57]: semivariance

  $\rightsquigarrow$ Bawa & Lindenberg[77], Harlow[91]: lower partial moments

here: overestimation is more serious a problem than underestimation





# 1 (b) Ideal setup

- $L_2$-differentiable model $\mathcal{P} = \{P_\theta \mid \theta \in \mathbb{R}\}$, with derivative $\Lambda_\theta$ and Fisher Information $\mathcal{I}_\theta \in (0, \infty)$
- observations $X_i \overset{\text{i.i.d.}}{\sim} P_\theta$
- consider *asymptotically linear estimators* (ALEs),

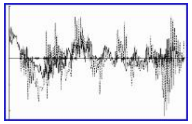
$$\sqrt{n}\,(S_n - \theta) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \psi_\theta(X_i) + \mathrm{o}_{P_\theta^n}(n^0)$$

for some *influence curve* (IC) $\psi_\theta \in L_2(P_\theta)$, that is,

$$\mathrm{E}_{P_\theta}\, \psi_\theta = 0, \qquad \mathrm{E}_{P_\theta}\, \psi_\theta \Lambda_\theta = 1.$$

- in the sequel: $\theta \in \mathbb{R}$ fixed & suppressed from notation; write $\mathrm{E}$ for $\mathrm{E}_{P_\theta}$, $\mathrm{Var}$ for $\mathrm{Var}_{P_\theta}$.





- optimality:
  - in two-sided case (c.f. Rieder[94] or van der Vaart[98]): $\psi_{\mathrm{Hell}} := \mathcal{I}^{-1}\Lambda$ optimal for "bowl-shaped" losses
  - for one-sided criterion $\mathrm{E}(S_n - \theta)_+^2$ ("semivariance"):

    no bias occurs in ideal model

    $\rightsquigarrow$ common argument: asy. risk is $\mathrm{E}\,\psi^2/2$.

  $\Longrightarrow$ $\psi_{\mathrm{Hell}}$ again optimal!

## 1 (c) Robust setup

- following Rieder[94]: infinitesimal neighborhoods $\mathcal{Q}_n = \mathcal{Q}_n(r)$ of types

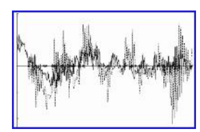
  $(c)$ (convex contam.): $Q_n$ s.t. $Q_n := (1 - \frac{r}{\sqrt{n}})P + \frac{r}{\sqrt{n}}H$

  $(v)$ (total variation) $Q_n$ s.t. $\sup_{A\in\mathbb{B}} \big|P(A) - Q_n(A)\big| \le r/\sqrt{n}$

  $(h)$ (Hellinger) $Q_n$ s.t. $\frac{1}{2}\int \big|\sqrt{dP} - \sqrt{dQ_n}\,\big|^2 \le r^2/n$

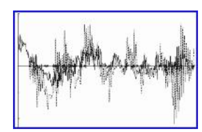
- induce positive and negative bias; only account for positive one





## 1 (d) Relations to other approaches

- compare Rieder[2K]:
  - imposes (one/two-sided) median unbiasedness as regularity condition instead of restriction to ALEs
  - uses one-sided models consisting of alternatives stemming from convex cones
  - (one/two-sided) concentration bounds instead of evaluating particular risks
- both approaches:
  - provide gain w.r.t. the corresp. two-sided formulations
  - induce certain instability when using the one-sided-optimal procedures in a two-sided setup



# 2 One-sided bias terms

- following Rieder[94]:
  *simple perturbations* $dQ_n(q,r) = \left(1 + \frac{r}{\sqrt{n}}\, q\right) dP$ , for $q \in \mathcal{G}_*$,
  where $\mathcal{G}_* = \{q \in L_\infty \colon\ \mathrm{E}\, q = 0,\ (*)\}$ and $(*) =$

$$(c) \quad q \geq -1, \qquad (v) \quad \mathrm{E}\,|q| \leq 2, \qquad (h) \quad \mathrm{E}\, q^2 \leq 8.$$

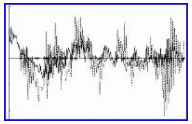
- infinitesimal oscillation terms for an ALE with IC $\psi$

$$\omega_*(\psi) := \sup\{|\, \mathrm{E}\, \psi q|\ \big| q \in \mathcal{G}_*\}, \qquad \omega_*'(\psi) := \sup\{\mathrm{E}\, \psi q\ \big| q \in \mathcal{G}_*\}$$

**Proposition** : *Let $\psi$ be an IC. Then*

$$\begin{aligned}
\omega_c'(\psi) &= \sup\nolimits_P \psi \qquad (\neq \omega_c(\psi) = \sup\nolimits_P |\psi|, \text{ in general}) \\
\omega_v'(\psi) &= \sup\nolimits_P \psi - \inf\nolimits_P \psi = \omega_v(\psi) \\
\omega_h'(\psi) &= \sqrt{8\, \mathrm{E}\, \psi^2} = \omega_h(\psi)
\end{aligned}$$





# 3 One-Sided Hampel problem

- Problem: $\mathrm{E}\,\psi^2 = \min!$ s.t. $\psi$ is IC, $\omega_*'(\psi) \le b\;\;(H')$
  (compare Hampel's[68] Lemma 5, Rieder[94], Thm. 5.5.1)
- $(v)$ and $(h)$ already covered by the corresp. two-sided problems, $\rightsquigarrow$ only $(c)$ leads to new results; let $\hat\omega_c' = \omega_c'(\psi_{\mathrm{Hell}})$
- infimal bias term: $\bar\omega_c' = \inf\big\{b > 0 \mid \exists$ IC $\psi$ and $\omega_c'(\psi) = b\big\}$
- gap condition $(L')$: let $z_0 = \inf_P \Lambda$

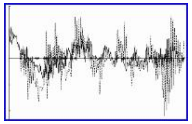
$$\begin{array}{rcl} p_0 & = & P(\Lambda = z_0) > 0 \;\;\wedge \\ \hat\delta & := & \sup\{\delta > 0 \,|\, P\big(\Lambda \in (z_0; z_0 + \delta)\big) = 0\} > 0 \end{array} \qquad (L')$$

**Theorem** (Solution to (H') in case (c)):

- $b \ge \hat\omega_c'$: $\psi_{\mathrm{Hell}}$ *is the unique solution.*
- $\bar\omega_c' < b < \hat\omega_c'$: $\exists!(A, -a) \in \mathbb{R}^2_{>0}$: *unique sol. is of Hampel form*

  $\psi_b' = (A\Lambda - a) \wedge b \qquad (\dagger) \qquad$ and $\omega_c'(\psi_b') = b$
- $b = \bar\omega_c'$: $\bar\omega_c' = -1/z_0$ *and* $\exists$ *IC* $\bar\psi$: $\sup_P(\bar\psi) = b$ *iff (L') holds; under (L'):* $\bar\psi$ *is unique, of Hampel form* $(\dagger)$*, but Lagrange multipliers* $A$ *and* $a$ *are not.*





# 4 One-sided MSE problem

## 4 (a) MSE'

- $\mathrm{MSE}'(S_n) := \mathrm{Var}(S_n) + (\mathrm{E}[S_n] - \theta)_+^2$

$\leadsto$ $\mathrm{MSE}'(\psi) := \mathrm{Var}\,\psi + r^2 {\omega'_*}^2(\psi)$

- Problem: $\mathrm{MSE}'(\psi) = \min!$ $\qquad \psi$ IC $\qquad\qquad (M')_r$

- *one-sided lower case radius*
  $\bar r' := \sqrt{\frac{-z_0/\hat\delta - (1-p_0)}{p_0}}$ $\qquad (< \infty$ under (L'))

**Theorem** (Solution to $(M')_r$ in case (c)):

- *for any* $r \in [0,\infty)$*,* $\exists!$ *solution* $\tilde\psi'_r$
- $\tilde\psi'_r$ *is of Hampel form* $(\dagger)$*, with* $b$ *s.t.* $\mathrm{E}[(A\Lambda - a - b)_+] = r^2 b \;\; (\ddagger)$
- *solution* $b_r$ *to* $(\ddagger)$ *strictly decreases in* $r \in [0, \bar r')$ *from* $\hat\omega'_c$ *to* $\bar\omega'_c$
- *under (L'):* $b_r \equiv -1/z_0$ *for* $r \geq \bar r'$*.*
- *for the* $(M')_r$*-optimal IC* $\tilde\psi'$*:* $\mathrm{MSE}'(\tilde\psi') = A$

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**Optimally one-sided bounded ICs**

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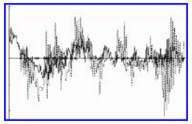

# 4 (b) $E(S_n - \theta)_+^2$

- more in the spirit of semivariance than $\mathrm{MSE}'$: $E(S_n - \theta)_+^2$

$\rightsquigarrow$ asymptotics: $\mathrm{G}(\psi) = G(w, s)$ for $s^2 = \mathrm{E}\,\psi^2$, $w = r\omega_*'(\psi)$ and

$$G(w, s) = \int (sx + w)_+^2\, \Phi(dx) = (w^2 + s^2)\Phi\big(\tfrac{w}{s}\big) + ws\,\varphi\big(\tfrac{w}{s}\big)$$

for $\Phi$, $\varphi$ c.d.f. and density of $\mathcal{N}(0, 1)$

- Problem: $\mathrm{G}(\psi) = \min!$ $\quad \psi$ IC $\qquad (G)_r$

**Theorem** (Solution to $(G)_r$ in case (c)):

- *for any* $r \in [0, \infty)$, $\exists!$ *solution* $\tilde\psi'_{G;r} = \tilde\psi'_G$
- $\tilde\psi'_{G;r}$ *is of Hampel form* $(\dagger)$, *with* $b$ *s.t.*

$$r^2 b + r\varphi\big(\tfrac{rb}{s_G}\big)/\Phi\big(\tfrac{rb}{s_G}\big) = \mathrm{E}[(A\Lambda - a - b)_+], \qquad s_G^2 = \mathrm{Var}\,\tilde\psi'_G. \quad (\ddagger)_G$$

- *to given* $r$, *solution* $b_r'$ *to* $(\ddagger) \geq$ *solution* $b_r^{\mathrm{G}}$ *to* $(\ddagger)_G$

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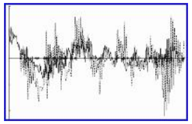


for the proof of the last theorem, we use

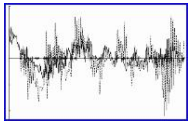
**Theorem**[R.& Rieder:04]

- max.asy.risk $R$ is representable by $G(w,s)$:
  - for $w = r\omega$ asy. (one/two-sided) bias, $s^2$ asy. variance
  - $G$ differentiable, convex and isotone
  - $\liminf_{|w|\to\infty} G(w,s) > \inf G(w,s) \qquad \forall s^2 \geq \mathcal{I}^{-1}$
- partial derivatives denoted by $G_w$, $G_s$ and $G_s > 0$
- THEN
  - infimal risk is attained by some IC $\tilde\psi_G$
  - $\tilde\psi_G$ is necessarily of Hampel form
  - clipping height $b$ s.t. :

$$r\, s_G\, G_w(r\omega_G, s_G) = \mathrm{E}(A\Lambda - a - b)_+\, G_s(r\omega_G, s_G)$$

for $\omega_G = \omega_c'(\tilde\psi_G)$, $s_G^2 = \operatorname{Var}\tilde\psi_G$





# 5 Asymmetric two-sided bias bounds

## 5 (a) Definition and bias terms

- one-sided solutions often instable in two-sided situations

$\rightsquigarrow$ asymmetrically weighted bias:
for $\nu = (\nu', \nu'') \in \mathbb{R}^2_{>0}$, $\max(\nu', \nu'') = 1$ let

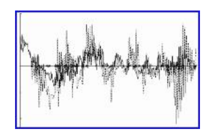
$$\omega_*^\natural(\psi) := \sup \Big\{ \nu'(\mathrm{E}\, \psi q)_+ \vee \nu''(\mathrm{E}\, \psi q)_- \;\big|\; q \in \mathcal{G}_* \Big\}$$

- formally: one-sided formulations as limits for $\nu'[\nu''] \to 0$

**Corollary** *Let $\psi$ be an IC. Then*

$$\begin{aligned}
\omega_c^\natural(\psi) &= \nu' \sup\nolimits_P \psi \;\vee\; (-\nu'' \inf\nolimits_P \psi), \qquad (\neq \omega_c(\psi) \quad \textit{in general})\\
\omega_v^\natural(\psi) &= \sup\nolimits_P \psi - \inf\nolimits_P \psi = \omega_v(\psi)\\
\omega_h^\natural(\psi) &= \sqrt{8\, \mathrm{E}\, \psi^2} = \omega_h(\psi)
\end{aligned}$$





# 5 (b) Asymmetric Hampel problem

- Problem: $\mathrm{E}\,\psi^2 = \min!$  s.t. $\psi$ is IC, $\omega_*^\natural(\psi) \le b\;(H^\natural)$
- let $\hat\omega_c^\natural = \omega_c^\natural(\psi_{\rm Hell})$, $b' = b/\nu'$, $b'' = b/\nu''$ and inf. bias term $\bar\omega_c^\natural$

**Theorem** (Solution to $(H^\natural)$ in case (c)):

- $b \ge \hat\omega_c^\natural$: $\psi_{\rm Hell}$ *is the unique solution.*
- $\bar\omega_c^\natural < b < \hat\omega_c^\natural$: $\exists! A > 0, a \in \mathbb{R}$: *unique sol. is of Hampel form*

$$\psi_b^\natural = (A\Lambda - a)\min\{1, \tfrac{b'}{(A\Lambda-a)_+}, \tfrac{b''}{(A\Lambda-a)_-}\} \qquad \text{and } \omega_c^\natural(\psi_b^\natural) = b$$

- $b = \bar\omega_c^\natural$: let $z_\nu$ be any $\frac{\nu''}{\nu'+\nu''}$-quantile of $\mathcal{L}(\Lambda)$; then

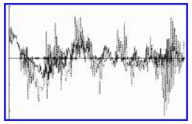
$$\bar\omega_c^\natural = \Big\{\frac{\mathrm{E}(\Lambda - z_\nu)_+}{\nu'} + \frac{\mathrm{E}(\Lambda - z_\nu)_-}{\nu''}\Big\}^{-1}$$

$\exists$ IC $\bar\psi^\natural$: $\omega_c^\natural(\bar\psi^\natural) = b$, and then necessarily

$$\bar\psi^\natural = b'\,\mathrm{I}_{\{\Lambda > z_\nu\}} - b''\,\mathrm{I}_{\{\Lambda < z_\nu\}} + \gamma\,\mathrm{I}_{\{\Lambda = z_\nu\}}$$

for $\gamma$ s.t. $\mathrm{E}\,\bar\psi^\natural = 0$





## 5 (c) Asymmetric MSE problem

- $\mathrm{MSE}^\natural(\psi) := \mathrm{Var}\,\psi + r^2 {\omega_*^\natural}^2$
- Problem: $\mathrm{MSE}^\natural(\psi) = \min!$ $\qquad \psi$ IC $\qquad\qquad (M^\natural)_r$

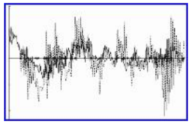
**Theorem** (Solution to $(M^\natural)_r$ in case (c)):

- *for any* $r \in [0,\infty)$, $\exists!$ *solution* $\tilde\psi_r^\natural$
- $\tilde\psi_r^\natural$ *is of Hampel form, with* $b$ *s.t.*

$$\mathrm{E}\left[(A\Lambda - a - b')_+/\nu' + (a - A\Lambda - b'')_+/\nu''\right] = r^2 b.$$

- *for the* $(M^\natural)_r$*-optimal IC* $\tilde\psi^\natural$*:* $\mathrm{MSE}^\natural(\tilde\psi^\natural) = A$

There are analogue statements as to uniqueness for Lagrange multipliers and monotonicity of $b$ in $r$ (using a corresponding lower-case radius) according to a corresponding gap condition



# 6 Comparisons

## 6 (a) Preparations

- solution to two-sided Hampel-Problem (e.g. Hampel et al.[86], Rieder[94])

$$\tilde\psi = \psi_b = (-b) \vee (A\Lambda - a) \wedge b \qquad (\dagger)_2$$

- (two-s.) MSE-optimal IC is of form $(\dagger)_2$ with $b$ determined by

$$\mathrm{E}(|A\Lambda - a| - b)_+ = r^2 b.$$

- assume symmetry: $\mathcal{L}(\Lambda) = \mathcal{L}(-\Lambda) \Longrightarrow \omega_c = \omega_c'$,
  if in addition $\tilde\psi$ is odd then $\mathrm{MSE}'(\psi) = \mathrm{MSE}(\psi)$
- efficiency loss when using $\tilde\psi$ instead of $\tilde\psi'$ in one-sided problem

$$\Delta_{2:1} := \frac{\mathrm{MSE}'(\tilde\psi)}{\mathrm{MSE}'(\tilde\psi')} - 1 = \frac{\mathrm{Var}\,\tilde\psi + r^2 \sup|\tilde\psi|^2}{\mathrm{Var}\,\tilde\psi' + r^2 (\sup\tilde\psi')^2} - 1$$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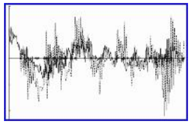
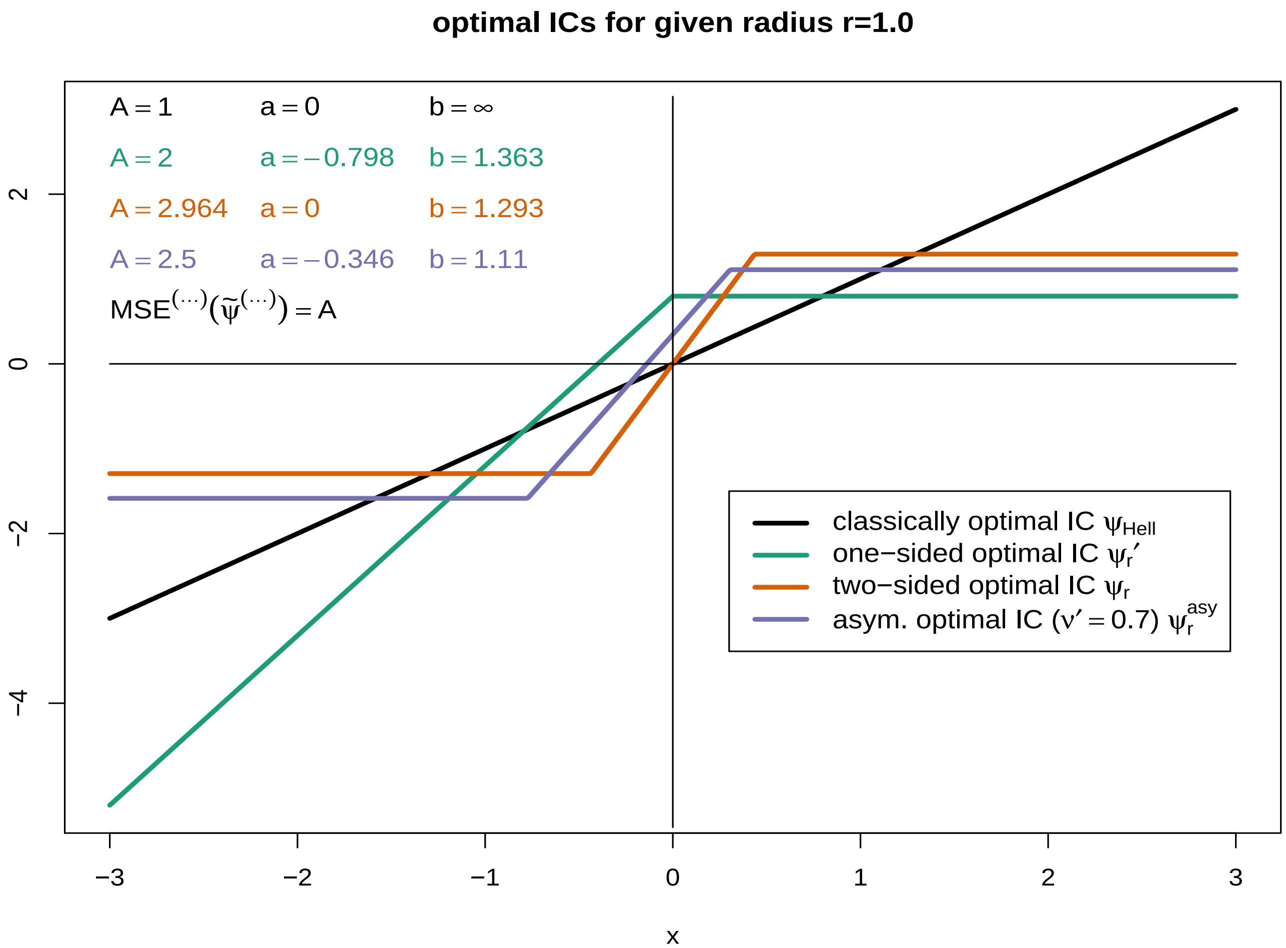


# 6 (b) Numerical Evaluations

- Gaussian location model: $\mathcal{P} = \big\{P_\theta = \mathcal{N}(\theta,1) \mid \theta \in \mathbb{R}\big\}$
- $\Lambda_\theta = x - \theta$, $\mathcal{I} = 1$; w.l.o.g. $\theta = 0$;
- in particular, $\inf_P \Lambda = -\infty$, hence gap-condition (L') fails.

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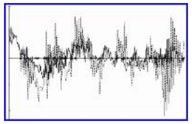
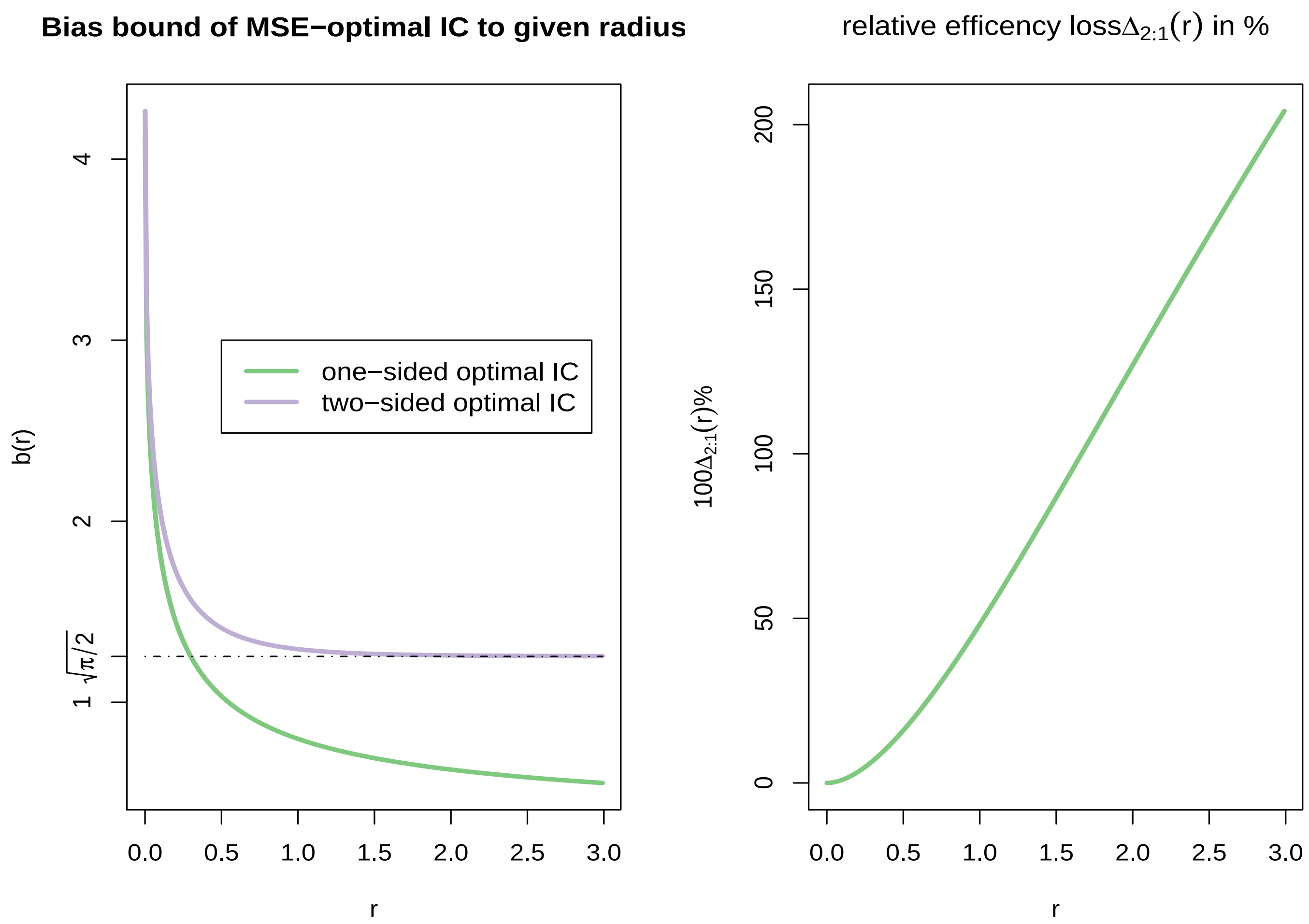





right panel: arbitrarily high relative efficiency loss (in the one-sided setup) when using two-sided clipped IC's instead of one-sided clipped ones

# 7 Least favorable radius and minimax IC

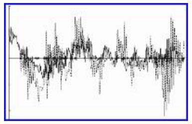
## 7 (a) Definitions

- situation: radius is unknown (to some extent)
- Rieder, Kohl, R.[01]: minimax criterion:
  - to fixed radius $r_0$ define the maximal inefficiency $R(r_0)$ as

$$R(r_0) = \sup_{r \in (r_l, r_u)} \rho(r_0, r), \qquad \rho(r_0, r) := \frac{\mathrm{MSE}(\tilde\psi_{r_0}, r)}{\mathrm{MSE}(\tilde\psi_r, r)}$$

  - $\tilde r_0 = \operatorname{argmin} R(r_0)$ is called *least favorable radius*
  - ALE to corresponding solution $\tilde\psi_{\tilde r_0}$ is a *minimax procedure*
- in Gaussian location model with two-sided MSE for $r_l = 0$, $r_u = \infty$, $\tilde r_0 = 0.621$, $R(\tilde r_0) \doteq 18\%$, $\tilde b = \omega_c(\tilde\psi_{\tilde r_0}) = 1.361$





- One can show (Kohl[05]): for range $(0,\infty)$, $\tilde r_0$ is a zero of

$$H_{0,\infty}(r) = \frac{\operatorname{tr}\operatorname{Cov}(\tilde\psi_r)}{\operatorname{tr}\operatorname{Cov}(\tilde\psi_0)} - \frac{\big(\omega_c(\tilde\psi_r)\big)^2}{\big(\omega_c(\tilde\psi_\infty)\big)^2} = \frac{\operatorname{tr}\operatorname{Cov}(\tilde\psi_r)}{\operatorname{tr}\mathcal{I}^{-1}} - \frac{\big(\omega_c(\tilde\psi_r)\big)^2}{\bar\omega_c^2}$$

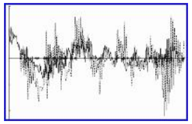
- generalized to one-sided setup:
  - here: if $\bar\omega_c' = 0$, $r_u < \infty$!
  - define $\tilde r_0'$ as the minimizer of

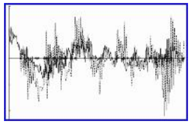
$$R'(r_0) = \sup_{r\in(r_l,r_u)} \rho'(r_0,r), \qquad \rho'(r_0,r) := \frac{\operatorname{MSE}'(\tilde\psi_{r_0}',r)}{\operatorname{MSE}'(\tilde\psi_r',r)}.$$

  - $\tilde r_0'$ may be obtained as zero of

$$H_{r_l,r_u}'(r) = \frac{\operatorname{Var}(\tilde\psi_r')}{\operatorname{Var}(\tilde\psi_{r_l}')} - \frac{\big(\omega_c'(\tilde\psi_r')\big)^2}{\big(\omega_c'(\tilde\psi_{r_u}')\big)^2}.$$







## 7 (b) Numerical Evaluations

Table 1: Least favorable radii, minimax efficiency loss and bias bound for one-sided bounded ICs

| $[r_l, r_u]$ | $\tilde{r}_0'$ | $R'(\tilde{r}_0')$ | $\Delta_{2:1}(\tilde{r}_0')$ in % | $b'_{\tilde{r}_0'}$ |
|---|---|---|---|---|
| $[0.0, \ 1.0]$ | 0.463 | 1.130 | 14.0 | 1.064 |
| $[0.0, \ 3.0]$ | 1.013 | 1.370 | 49.1 | 0.794 |
| $[0.0, \ 5.0]$ | 1.364 | 1.536 | 76.0 | 0.714 |
| $[0.0, 10.0]$ | 1.913 | 1.809 | 119.9 | 0.637 |

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