On the Role of Frustration and Noise in Dynamical Systems

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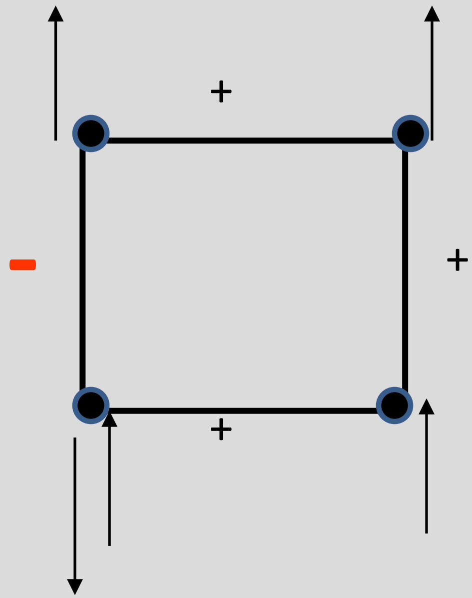
- Concept and Impact of Frustration in Dynamical Systems
- Frustration in Oscillatory and Excitable systems
- Frustration in the Context of Social Science and Computer Science

Colloquium at the University of Oldenburg, January 2017

Frustration in spin systems

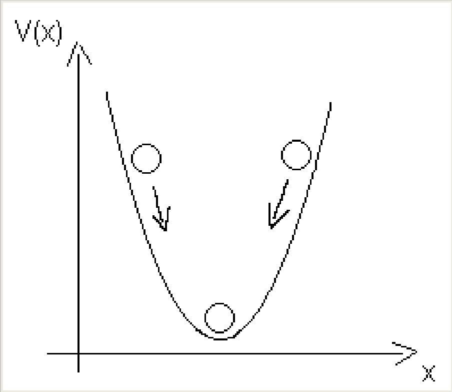


+ ferromagnetic coupling

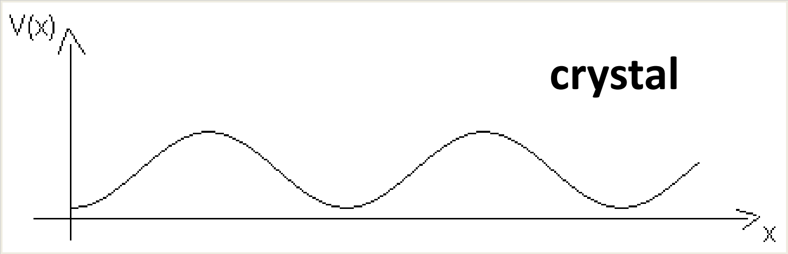
- antiferromagnetic coupling

Whatever position this spin takes, one of the neighbor bonds is frustrated .

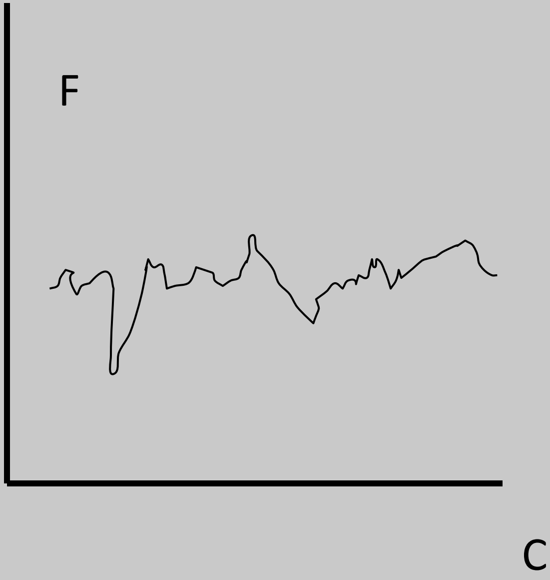
IMPACT OF FRUSTRATION ON "ENERGY" LANDSCAPES



Particle in an attractive potential



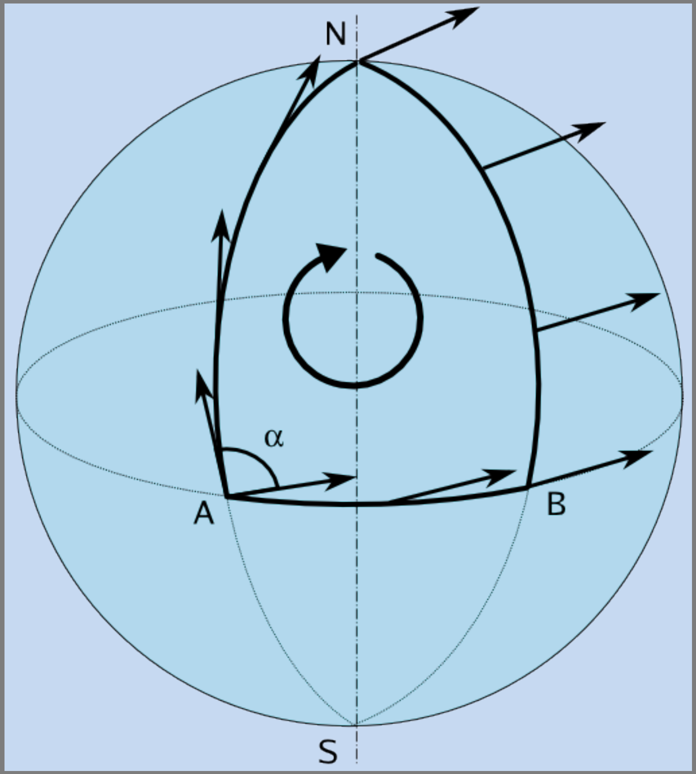
Spin glass



Funnel energy landscape
over configuration space. Are such shapes relevant for proteins or cell states?

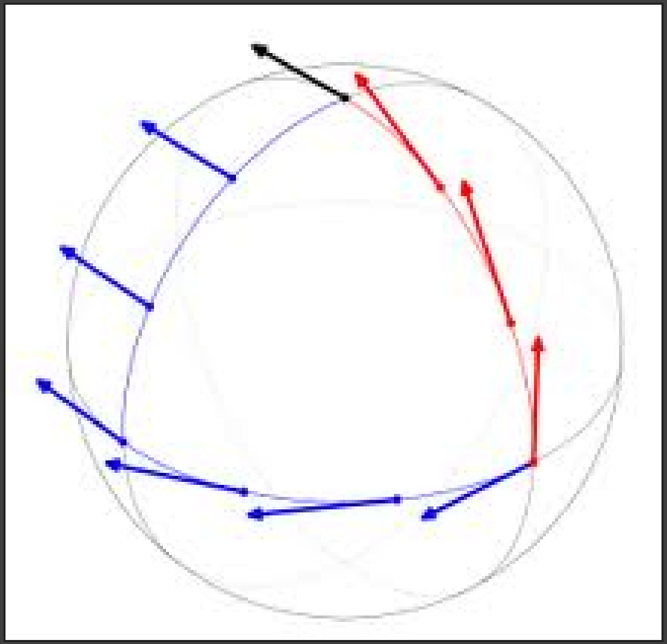
WHAT IS FRUSTRATION IN SPIN SYSTEMS IS CURVATURE IN GENERAL RELATIVITY.

THE GENERIC CONCEPT OF PARALLEL TRANSPORT



Figures from Wikipedia

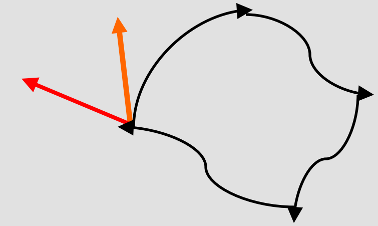
And in the context of gauge theories:

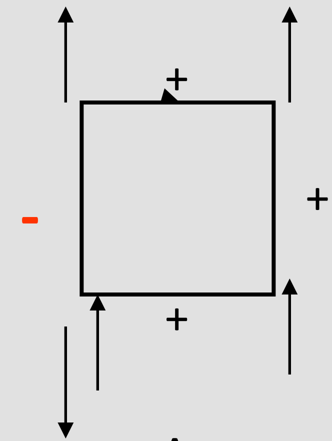


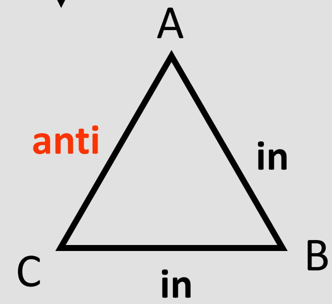
Parallel transport is needed to compare vectors at different points in curved space, here to compare matter fields at different space points due to the curvature of internal space.
The parallel transport is provided by the connection Γ or the gauge field A, respectively.
The derivative which is sensitive to the intrinsic changes of vectors or matter fields is the covariant derivative, it vanishes if the vector is only parallel transported.

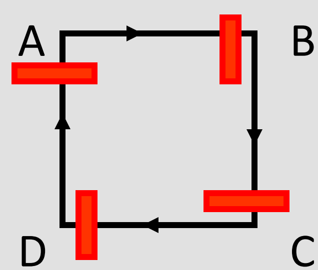
CONCEPT AND IMPACT OF FRUSTRATION IN DYNAMICAL SYSTEMS

Physics	Gauge theories	Field strength
	General relativity	Curvature
	Spin glasses Oscillatory systems: phase oscillators, excitable systems	Frustration
Social systems	Approach to balance	Imbalance
Communication	Assigning a meaning	MIsunderstanding
Economics	Financial markets	Arbitrage
Psychology	"daily life"	Frustration









G. Mack, Commun.Math.Phys.219, 141 (2001).
& Fortschritte der Physik, 81: 135-185 (1981): Physical Principles, Geometrical Aspects and Locality of Gauge Field Theories.

Conjecture concerning the impact of frustration:

Tuning an appropriate degree of frustration:

- not too low so that the dynamics is flexible enough and therefore functionally stable
- not too high, so that the dynamics is stable against noise

One of the “design principles” in evolutionary processes?

Criterion for frustration in coupled oscillatory and excitable systems:

- Consider a loop with **interaction bonds** and couplings that can be **antagonistic:** either ferromagnetic or antiferromagnetic, attractive or repulsive, excitatory or inhibitory, repressive or supportive.
- Consider a path from A to B along the shortest connection and along a complementary path in the loop from B to A.

For undirected couplings:

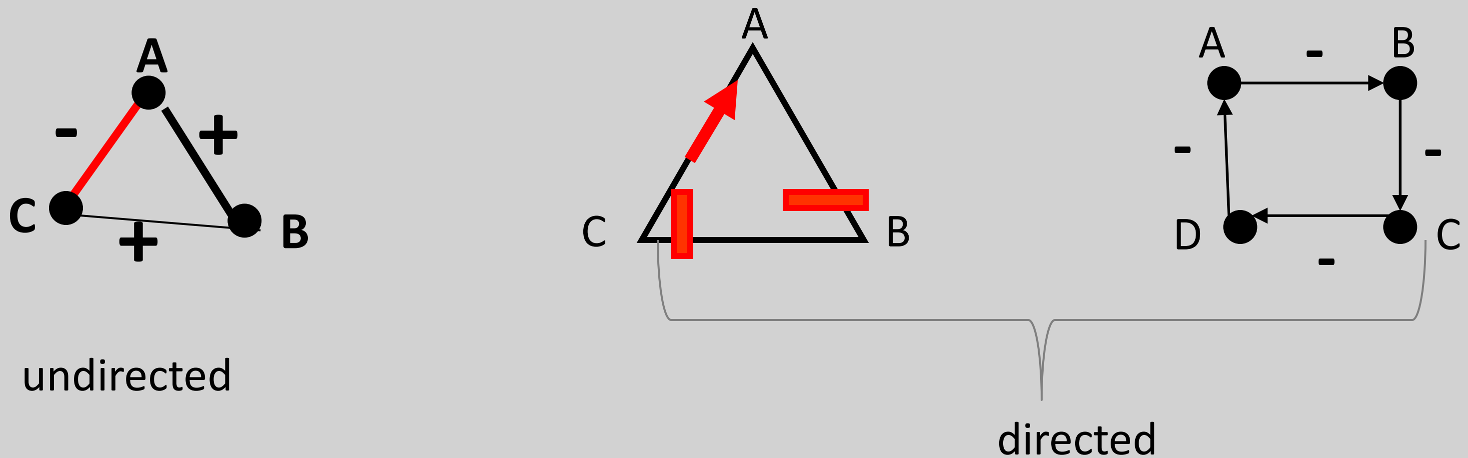
The bond from A to B is satisfied if A acts upon B in the same way as B upon A (e.g. attractive), otherwise it is not. If it is not, the loop from A to A is then frustrated.

For directed couplings:

The bond from A to B is satisfied if A acts upon B in the opposite way as B upon A (e.g. A to B activating, B to A via C and D repressing), otherwise it is not satisfied and the loop from A to A is frustrated.

Different realizations of frustration

via the combination of antagonistic couplings or the topology



e.g. A in phase with B, B with C → C with A, but if C wants to be antiphase with A, the link CA or CB is unsatisfied and the triangle is frustrated.

Quantitative criterion for frustration in case of phase oscillators

$$\dot{\phi}_i = \omega + \frac{1}{N} \sum_{j=1}^{N} w_{ij} \sin(\phi_j - \phi_i) \quad i \in 1, \ldots, N$$

Frustration in phase oscillators in larger systems to study synchronization behavior
(H. Daido, PRL (1992); Progr. Theor. Phys. (1987); D. H. Zanette, EPL 72, 190 (2005).)

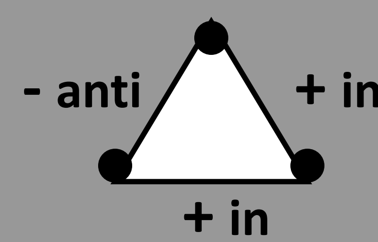
$$F = -\frac{1}{2N} \sum_{i,j=1}^{N} w_{ij} \cos(\phi_j - \phi_i)$$

Quantitative measure

$$\dot{F} = \sum_{i=1}^{N} \frac{\partial F}{\partial \phi_i} \dot{\phi}_i = -\sum_{i=1}^{N} \left(\frac{\partial F}{\partial \phi_i}\right)^2 \leq 0$$

Qualitative measure:

w(i j) =+-1



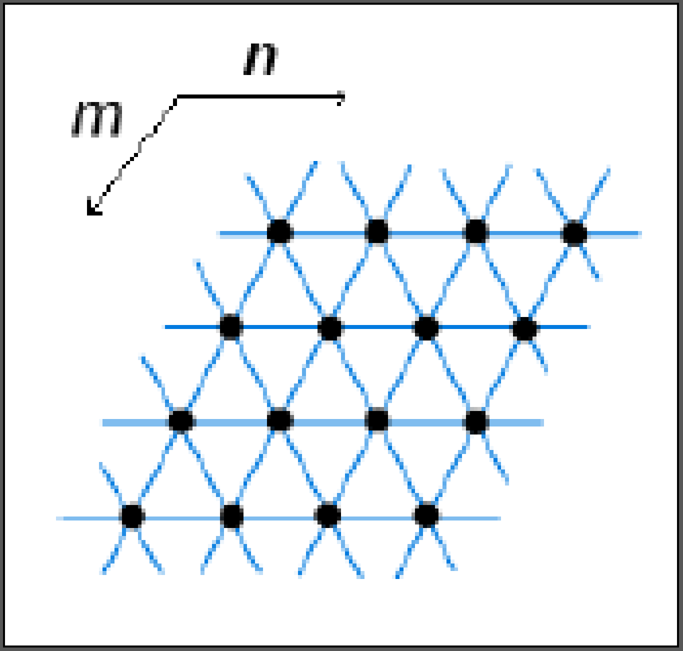
Kuramoto dynamics drives the system in local minima of the frustration landscape, depending on the initial conditions

FRUSTRATION IN OSCILLATORY AND EXCITABLE SYSTEMS

Individual dynamics here

- **Phase oscillators: Kuramoto oscillators and classical rotators**
- **Motifs from genetic circuits: Bistable frustrated units,**

coupled on regular grids



Frustrated Sets of Classical Rotators and Kuramoto Oscillators

Without noise:

- Extremely rich attractor space
- Spontaneous onset of global couplings
- The Watanabe-Strogatz phenomenon, dynamically generated
- Spontaneous generation of slow dynamics for small perturbations
- Physical aging for natural frequency distributions

Under the action of noise:

- Order-by-disorder phenomenon
- Hints on hierarchies in the potential landscape
- Noise-driven migration of oscillator phases through the attractor landscape
- Physical aging

Outline: In analogy to spin systems we shall see how

frustration

⇓

multistability

⇓

rough “energy” landscape with a hierarchy of barriers

⇓

multitude of inherent time scales

⇓

aging

here in excitable and oscillatory systems with and without noise

CLASSICAL ROTATORS WITH FRUSTRATION

The model: N active rotators with repressive coupling

$$\frac{d\varphi_i}{dt} = \omega_i - b\sin\varphi_i + \sigma_A \xi_i(t) + \frac{(\kappa + \sigma_M \eta_i(t))}{\mathcal{N}_i} \sum_j A_{ij} \sin(\varphi_j - \varphi_i).$$

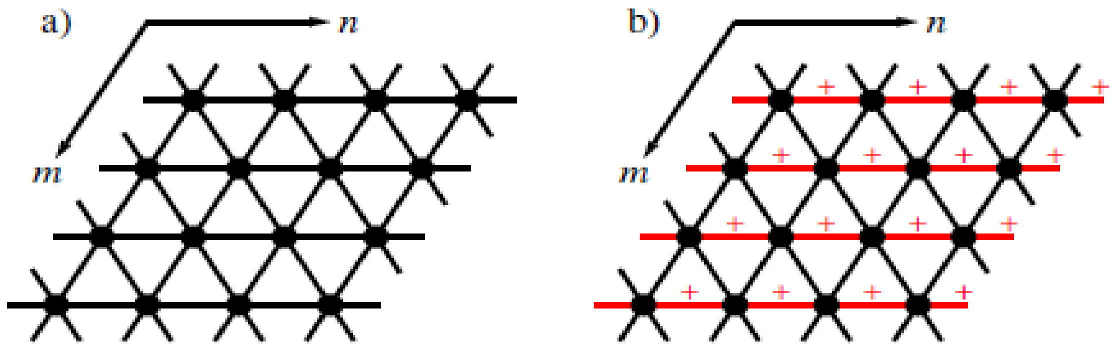


Fig. 1: Hexagonal lattice with all triangles frustrated for all couplings being negative (a) and not frustrated (b) for positive couplings along the horizontal links and negative ones otherwise.

The versatility of attractors in comparison to spin systems

In spin glasses:

fixed points

In these systems:

various combinations of fixed points

a variety of limit cycles differing by their

correlation between individual phases
frequency
pattern of phase-locked motion
basin of attraction
stability
symmetry

quasiperiodic solutions

chaotic solutions

as a combined effect of frustration, lattice size and lattice symmetry.

Examples for Phase-Locked Motion in particular plane waves and spherical waves:

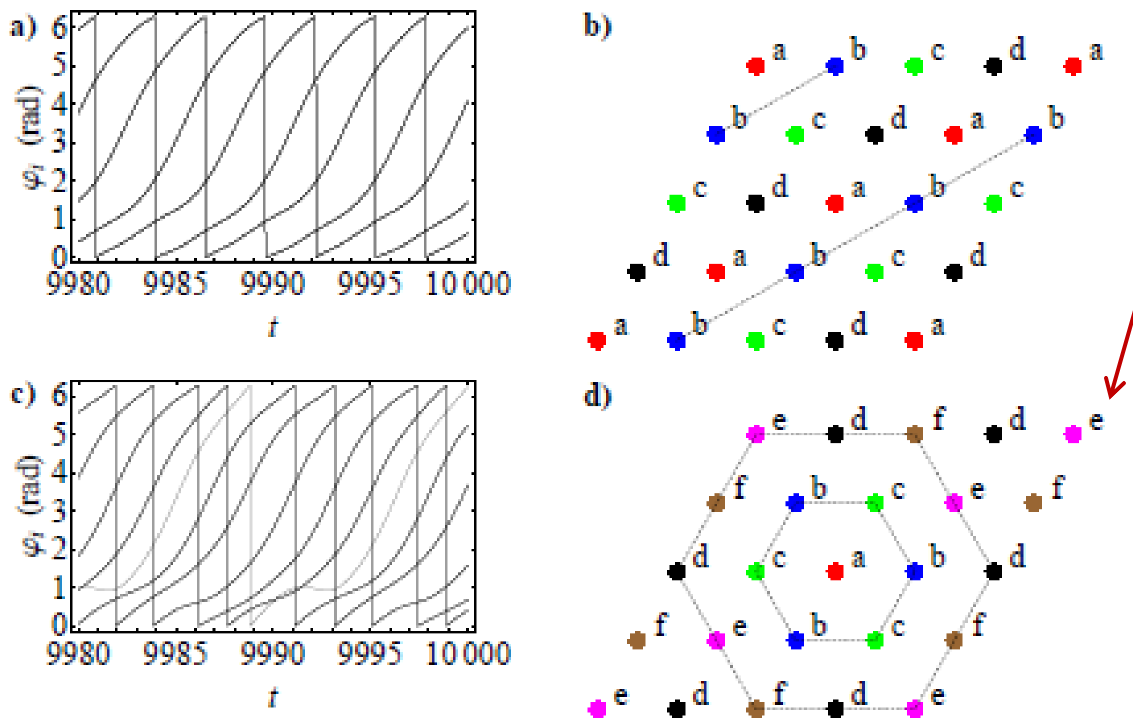


Fig. 2: Solutions of eq. (1) with 4 and 6 clusters on a 4×4 lattice for $\omega = 0.7$, $b = 1$, $\kappa = -2$ and $\sigma_M = \sigma_A = 0$.

We expected to see order-by-disorder phenomena in analogy to spin systems.

Phase Space without Noise:

Self-organized Strogatz-Watanabe phenomenon (M. Zaks, P. Tomov)

Network of N *identical* one-dimensional elements.

$$\frac{d\varphi_i}{dt} = f(\varphi_i) + \kappa \sum_j A_{ij}\, g(\varphi_i, \varphi_j)$$

with adjacency matrix A_{ij}.

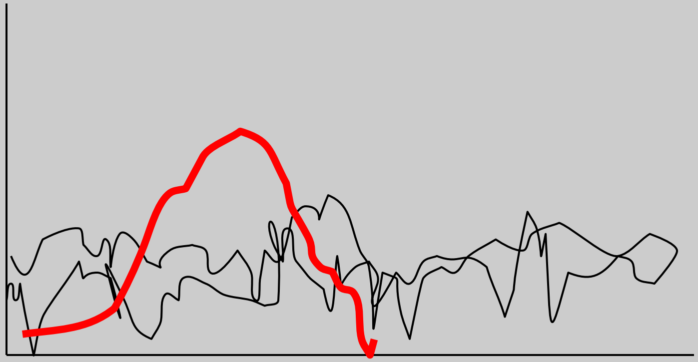
- Global coupling: synchronization,
  in case of coupling through the first Fourier harmonics:
  Strogatz-Watanabe phenomenon
  (existence of N-3 conserved quantities).

Here N=4 ⇒ frequency continuum of 4-cluster states of 16 oscillators, globally coupled

In view of the application of noise: Short excursion from

The Impact of Stochastic Fluctuations in Physics and Biology

The role of noise in dynamical systems



Destructive:

- if bad signal-to-noise ratio
- loss of synchrony in circadian clocks
- lower precision of signals

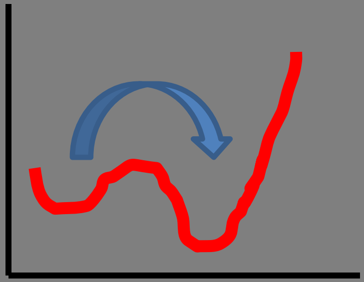
Noise in nonlinear systems and out-of-equilibrium can act in a way that appears as counterintuitive or is not foreseeable.

Constructive:

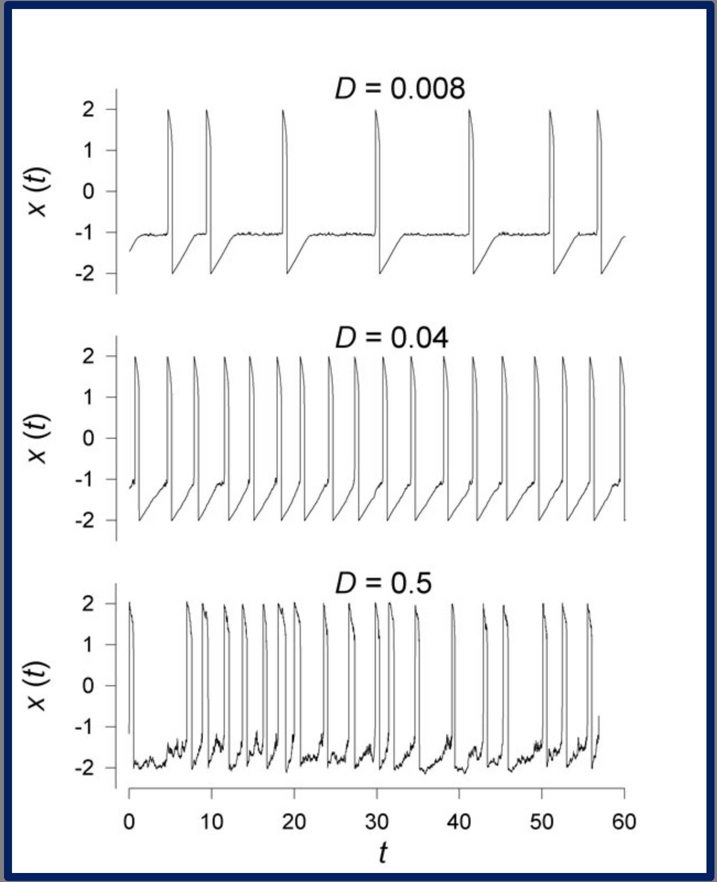
if it improves the signal-to-noise ratio such as in

- stochastic resonance,
- coherence resonance and
- system size resonance

Stochastic resonance:
Under the application of a deterministic periodic external field an optimal signal-to-noise ratio is achieved for an intermediate noise level and



Coherence resonance:
For an intermediate strength of noise, oscillatory response in an excitable system is most coherent (here no external field)



System size resonance
For an optimal size of the system, the system's response is most regular, illustrated in a system of coupled nonlinear noisy oscillators (ensemble averages fluctuate with $(D/N)^{1/2}$)

D the (effective) noise intensity

Results for Classical Oscillators with Noise:

The Order-by-Disorder Phenomenon

Usually: **Order-by-disorder is considered in spin systems**.

Generic: The ground state is degenerate due to competitions among the interactions.
The degeneracy is lifted due to disorder.

The lifting can be
- temperature driven (Villain et al. J. Phys. 1980, Bergman et al., Nature Physics 2007)
- or quantum driven (Chubukov, PRL (1992), Reimers et al. PRB (1993))
- or due to dilution (Henley PRL 1989)

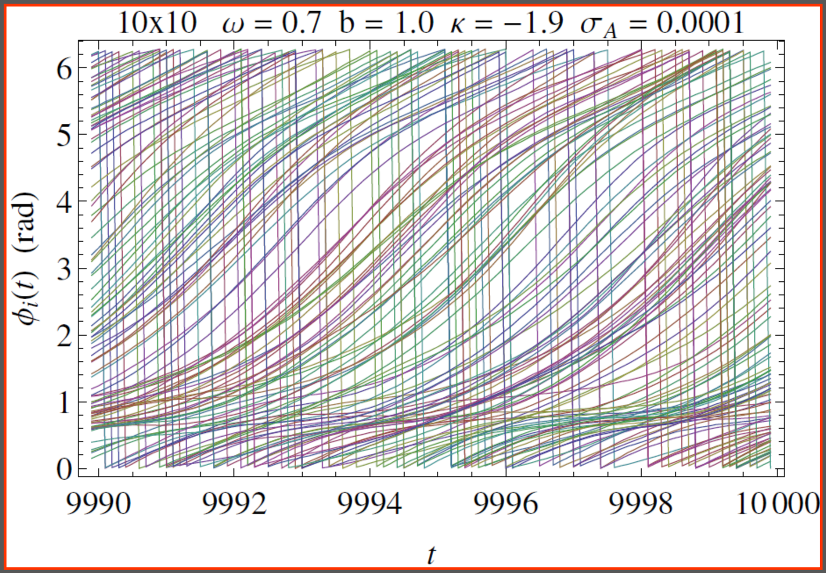
Here: the more oscillator phases coalesce, the higher the order, disorder introduced by additive or multiplicative noise

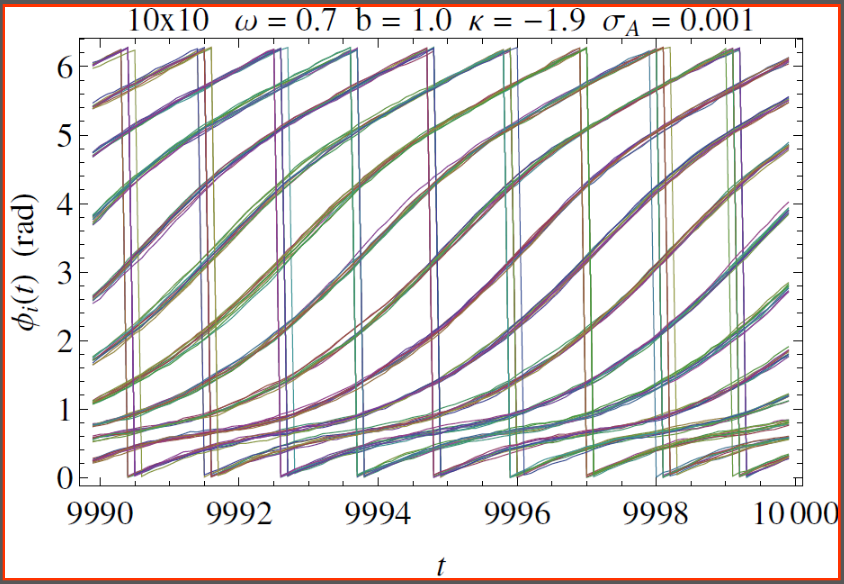
In view of the order-by-disorder phenomenon:

The classification according to p_n-patterns with n denoting the number of clusters of coalescing phases is **not unique**, but suited for our notion of order.

- **p4-solutions:** 4 clusters, originally only 75 such solutions could be identified, differing by their frequency, in particular plane waves (without counting the degeneracy due to the lattice symmetries)
- **p6-solutions**: spherical waves (degeneracy 96 due to 16 sites for the center and 6 rotations about 60 degrees due to the lattice symmetry)
- **p16-solutions** of individual limit cycles
- **quasiperiodic solutions**

using the numerically obtained Poincaré-mapping on the hypersurface Φ_1 = const.

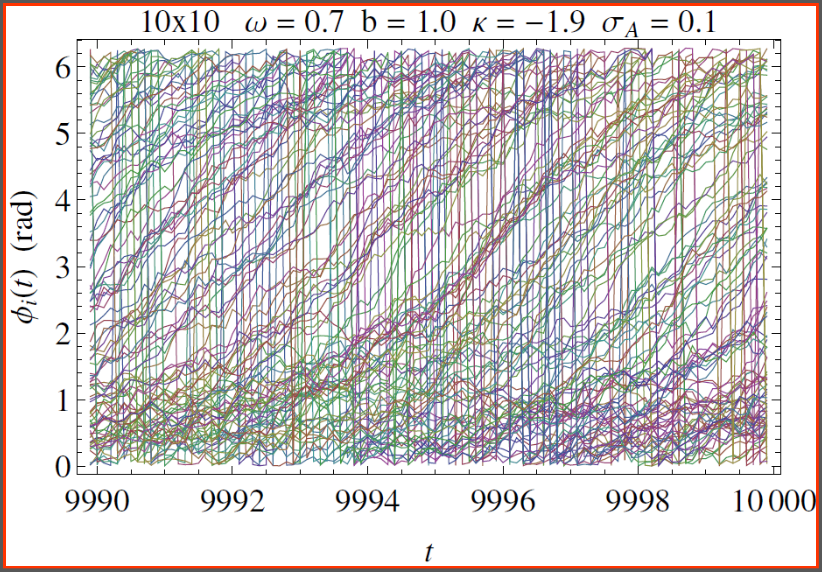




Low noise

Intermediate noise

zoom



lattice of 10x10 active rotators
clusters 100—10--100

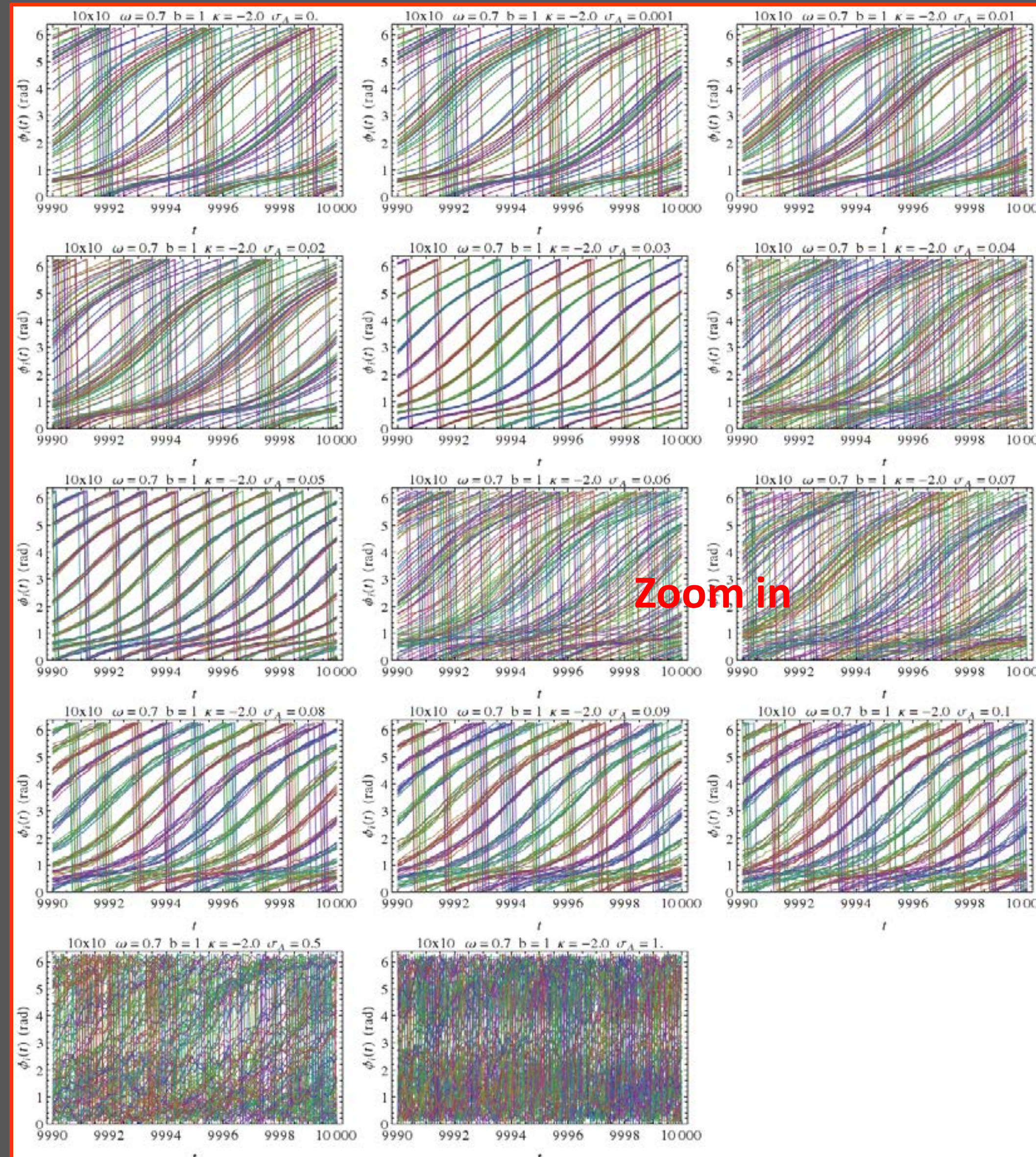
Varying the noise intensity between 0.0001-0.1

Strong noise

Zoom into the noise intensity:

Increasing monotonically the noise intensity from left to right and top to bottom one observes disorder (d) and order (o) as the sequence:

d o d o d o d



The system is very sensitive to the initial condition.

Varying the noise intensity between 0.001-0.1 in steps of 0.01

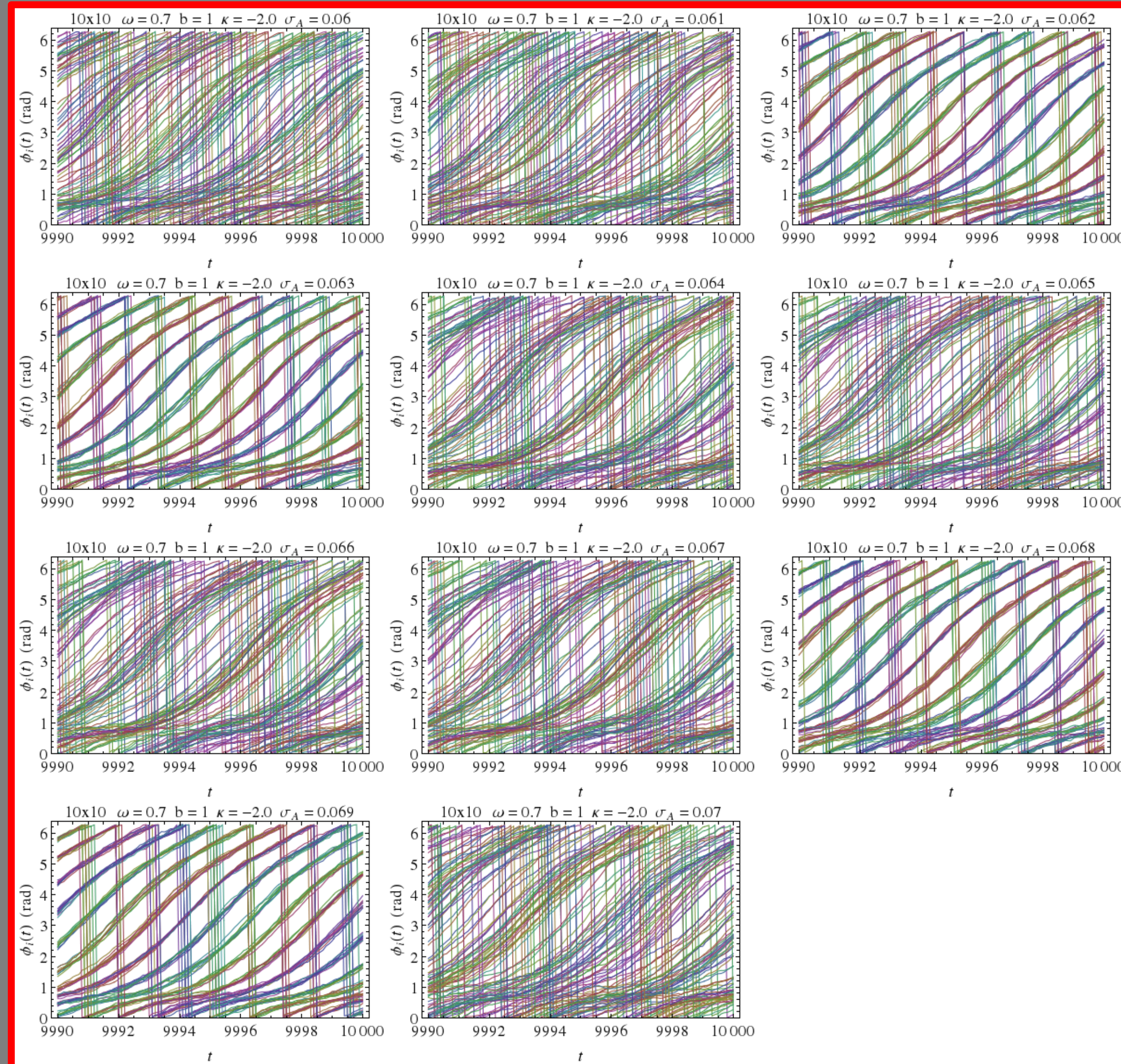
lattice of 10x10 active rotators

clusters 100—10—100—10—100—10---100

Zoom further into the noise intensity:

Increasing monotonically the noise intensity from left to right and top to bottom, one observes disorder (d) and order (o) as the sequence:

d o d o d



lattice of 10x10 active rotators clusters 100—10—100—10—100

Varying the noise intensity between 0.06-0.07 in steps of 0.001

How representative are these plots? Similar plots are obtained

- **if the noise realization is varied**
- **the additive noise is replaced by multiplicative one**
- **the oscillators are reduced to Kuramoto oscillators**
- **the lattice size is increased**
- **the time windows for snapshots over 200 time units are varied.**

What is the explanation for the repeatedly increasing order when the noise strength is monotonically varied?

The system obeys a gradient dynamics with potential V.
The resolution of its shape depends on the noise intensity

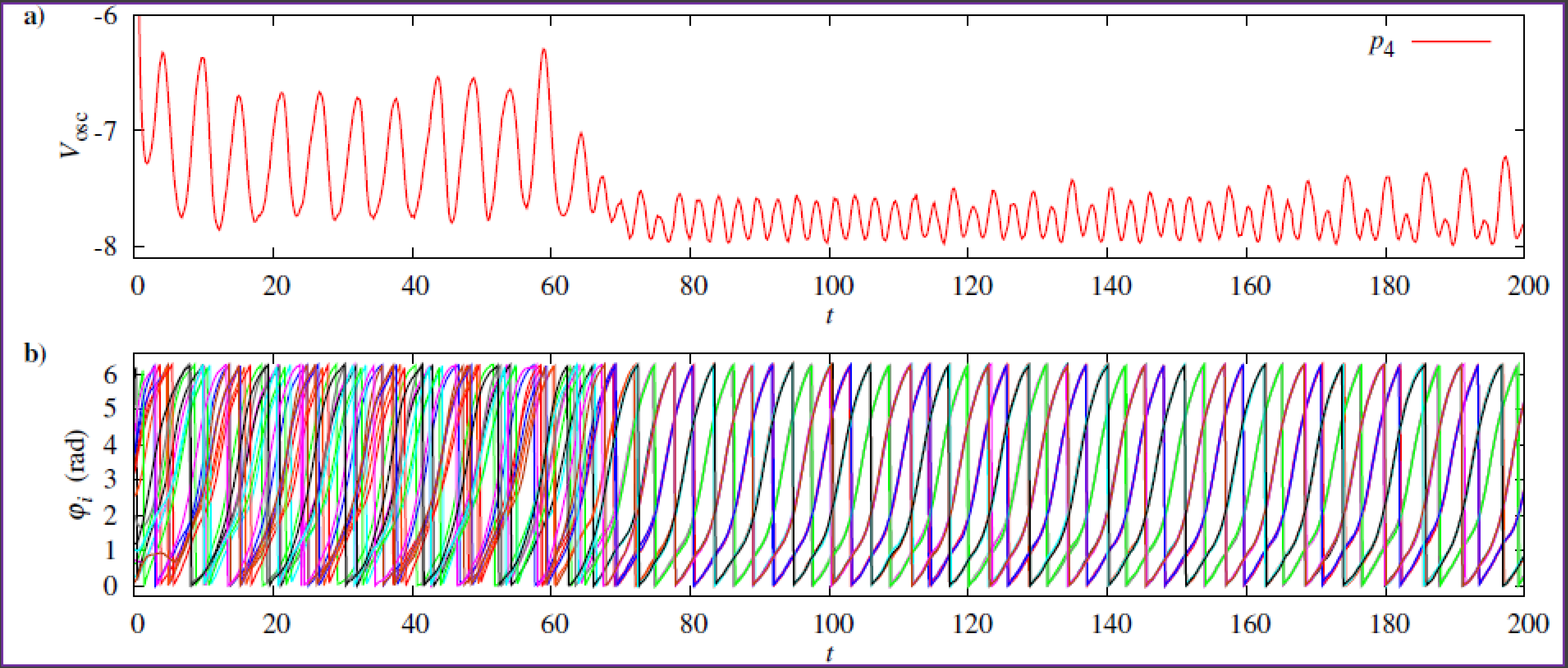
-> hierarchy in potential barriers

$$V = -\omega \sum_i \varphi_i - b \sum_i \cos \varphi_i - \frac{\kappa}{2N} \sum_{i,j} A_{ij} \cos(\varphi_j - \varphi_i).$$

NOISE-DRIVEN MIGRATION OF OSCILLATOR PHASES THROUGH A COMPLEX LANDSCAPE:

Keeping the noise intensity fixed we see the following kind of "state"

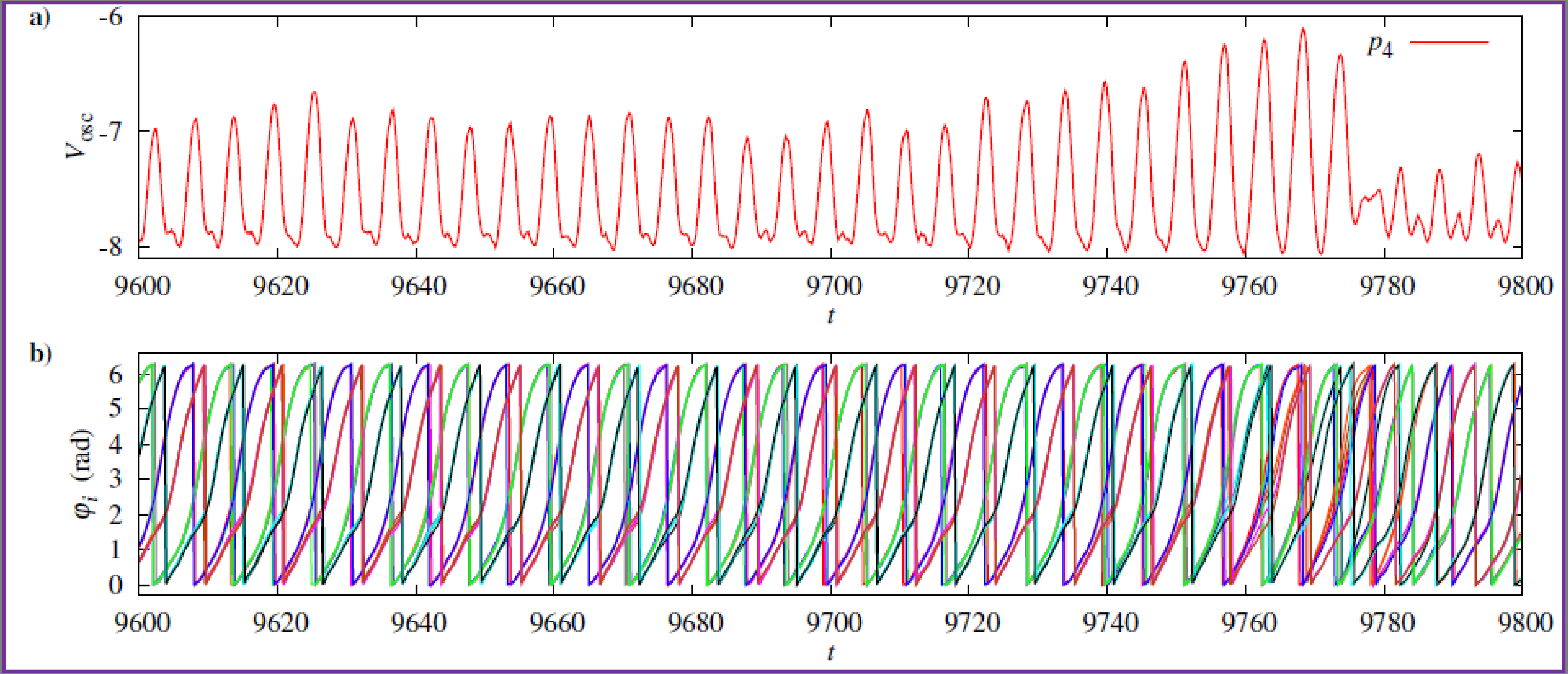
V_{osc}



$\Phi_i(t)$

0-200 time units

The state is characterized by ongoing transitions between the different pattern of phase locked motion, characterized as p4, p6,p16, disordered, or transient p3 states.



9600-9800 and so on for ever

We see a multitude of escape times between the metastable states.

SUMMARY SO FAR

Active rotators and Kuramoto oscillators on a hexagonal lattice with frustrated bonds

- **show a large number and a variety of coexisting attractors.**
- **can exhibit a spontaneous generation of global couplings and the W.-S.-phenomenon with a continuum of solutions**
- **Under noise we see the ongoing migration of phases through the potential landscape.**
- **The escape times between the metastable states define a multitude of time scales.**

⟹ **We expect to see aging of these oscillators.**

DEFINING CRITERIA FOR PHYSICAL AGING (IN CONTRAST TO BIOLOGICAL AGING):

Do relaxation processes towards the stationary state show

- **slow dynamics**
- **breaking of time-translation invariance**
- **dynamical scaling?**

MEASURE AGING OF ACTIVE ROTATORS AND KURAMOTO OSCILLATORS

via the autocorrelation functions:

- **Prepare the system in the vicinity of the unique fixed point at $\kappa > 0$.**
- **Quench the system towards $\kappa < 0$ in the regime of coexisting synchronized oscillations to push it "out-of-equilibrium".**
- **Wait and let it evolve under the action of additive noise.**
- **Perform a first measurement of the autocorrelation function at time t_w.**
- **Perform a second measurement at time $t > t_w$**

for two lattice sizes (32x32 and 4x4) and three noise intensities $\sigma = 0.01, 0.1, 0.5$ with and without frustration.

Here the coupling provides the bifurcation parameter from one phase into the other and the noise creates the fluctuations.

The state of the system at time t is specified by the vector of all phases

$\vec{\phi}$ = (Φ_1(t), Φ_2(t), …..,Φ_N(t)).

We compute the two-time autocorrelation function defined as

$$C(t,t_w) := \frac{\langle \vec{\phi}(t)\vec{\phi}(t_w)\rangle - \langle \vec{\phi}(t)\rangle\langle \vec{\phi}(t_w)\rangle}{\sigma_t \sigma_{t_w}}$$

with standard deviations $\sigma_t^2 = \langle \vec{\phi}(t)\vec{\phi}(t)\rangle - \langle \vec{\phi}(t)\rangle\langle \vec{\phi}(t)\rangle$,

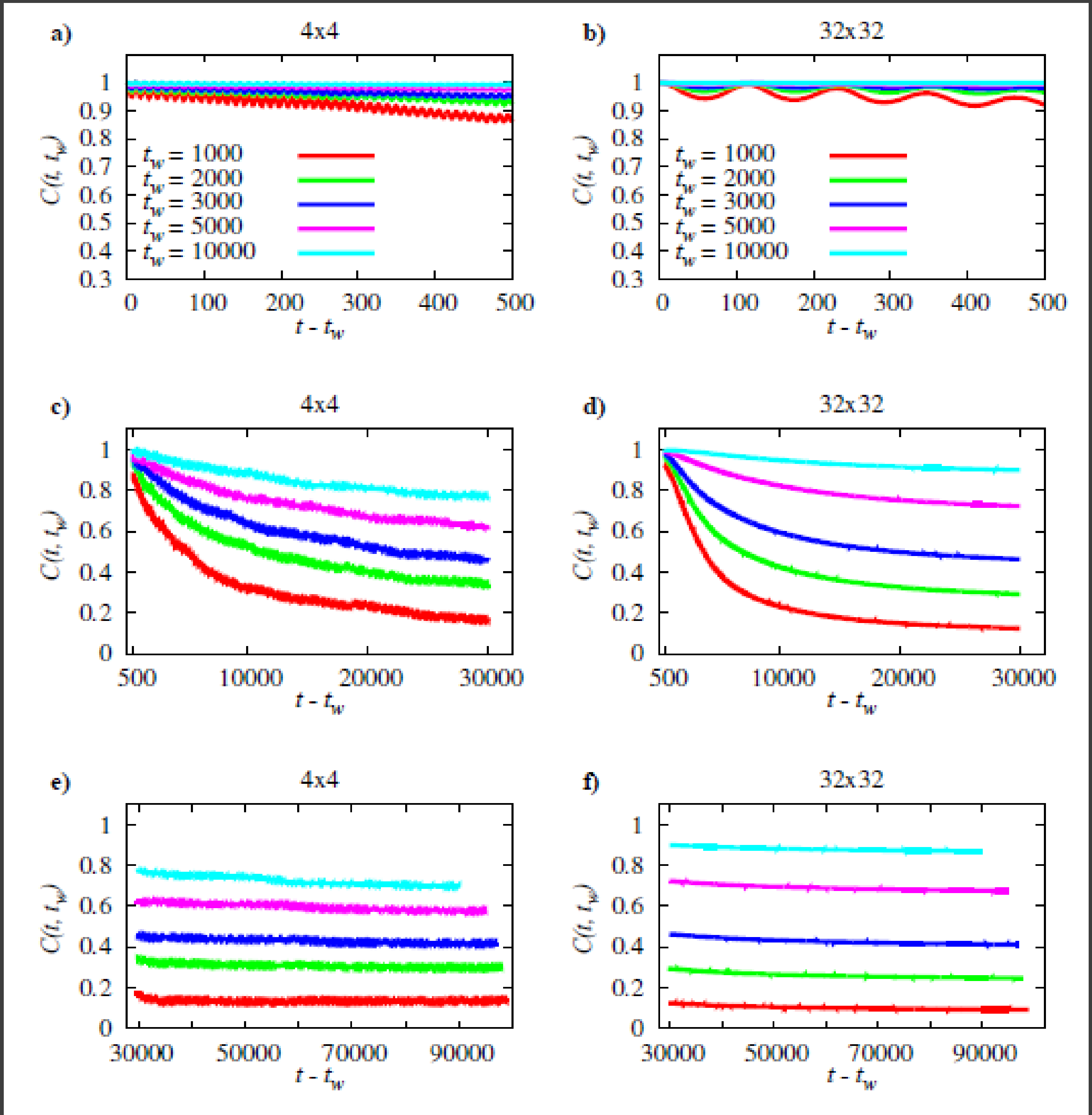
where the averages are calculated over a sufficient number of noise realizations.

AGING OF ACTIVE ROTATORS: 4X4 AND 32X32 WITH NOISE

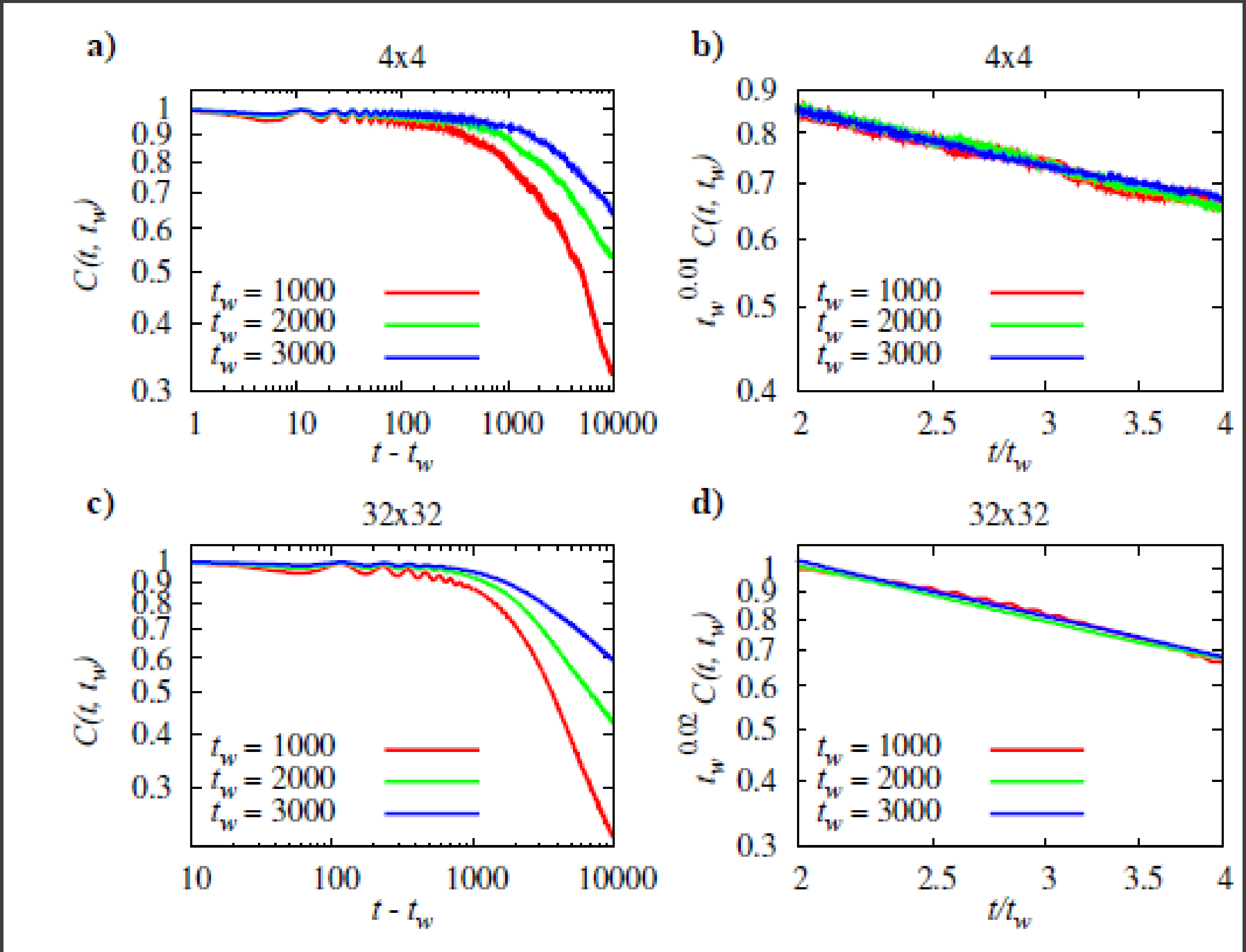
Regime (i) Quasiequi-librium

Regime (ii) Drop off

Regime (iii) Slow saturation



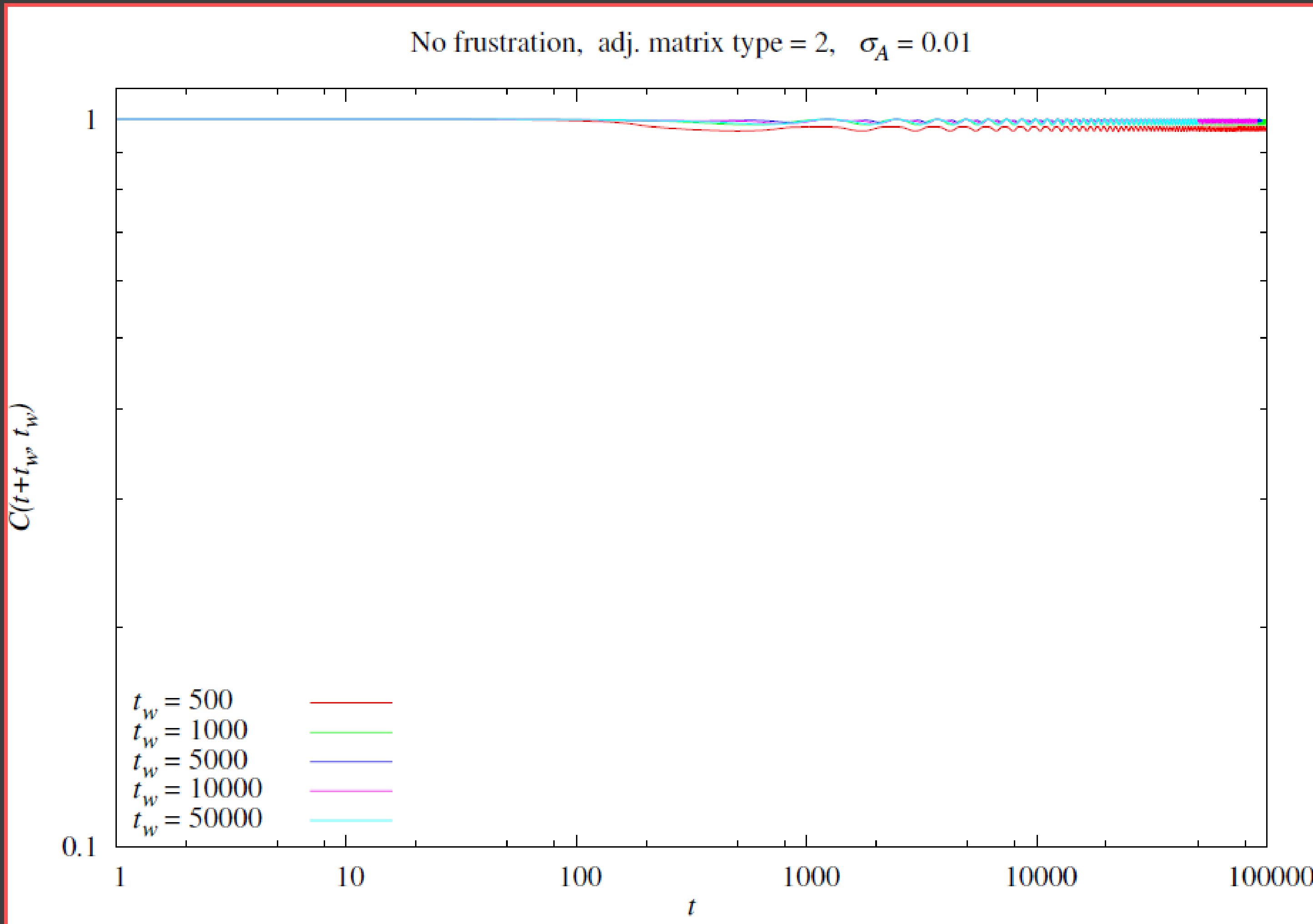
Dynamical Scaling of Active Rotators for 4x4 and 32x32



$$t_w^{|b|} C(t, t_w) = f\left(\frac{t}{t_w}\right)$$

$b \in [0, 0.02]$ (4×4) and $b \in [0, 0.03]$ (32×32)

ACTIVE ROTATORS ON A 32X32 LATTICE WITHOUT FRUSTRATION BUT WITH NOISE



So no dependence on the waiting time t_W, neither for the 4x4 case.

The mechanism seems to be the same as for spin glasses, but the attractor landscape is much more versatile, in which the phases continue to move from one metastable state to another, kicked by noise.

- **What are different aging mechanisms?**

- **What is the role of noise in aging?**(Montemurro et al. PRE67,031106(2003))

- Replace noise by "**disorder" in the natural frequencies** for Kuramoto oscillators with repulsive coupling with deterministic or random realizations of the frequency distributions (ongoing work with D. Labavic)

Aging in Deterministic Systems

Consider again the Kuramoto model on a hexagonal lattice,

$$\frac{d\phi_i}{dt} = \omega_i + \frac{\kappa}{6}\sum_j A_{ij}\sin(\phi_j - \phi_i)$$

but now with a distribution in natural frequencies according to

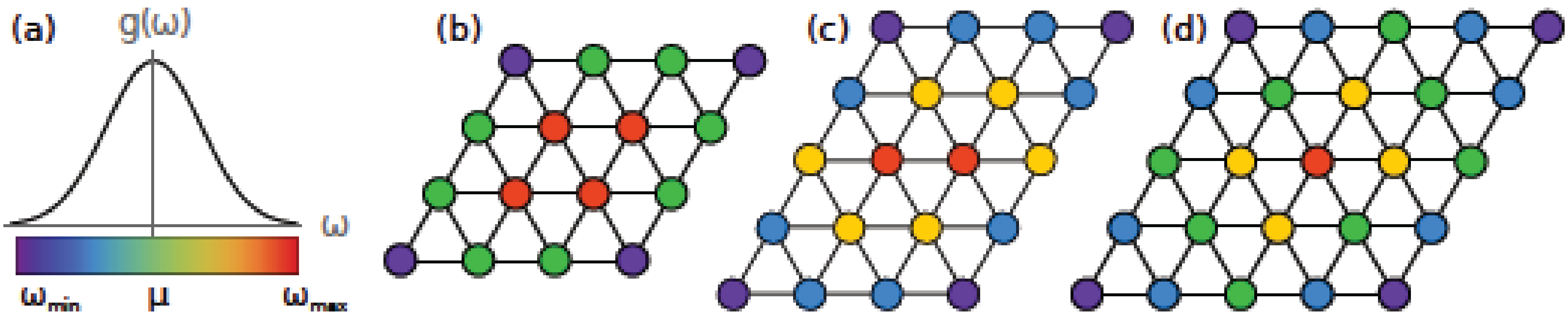
$$g(\omega) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\omega-\mu)^2}{2\sigma^2}}$$

where we either

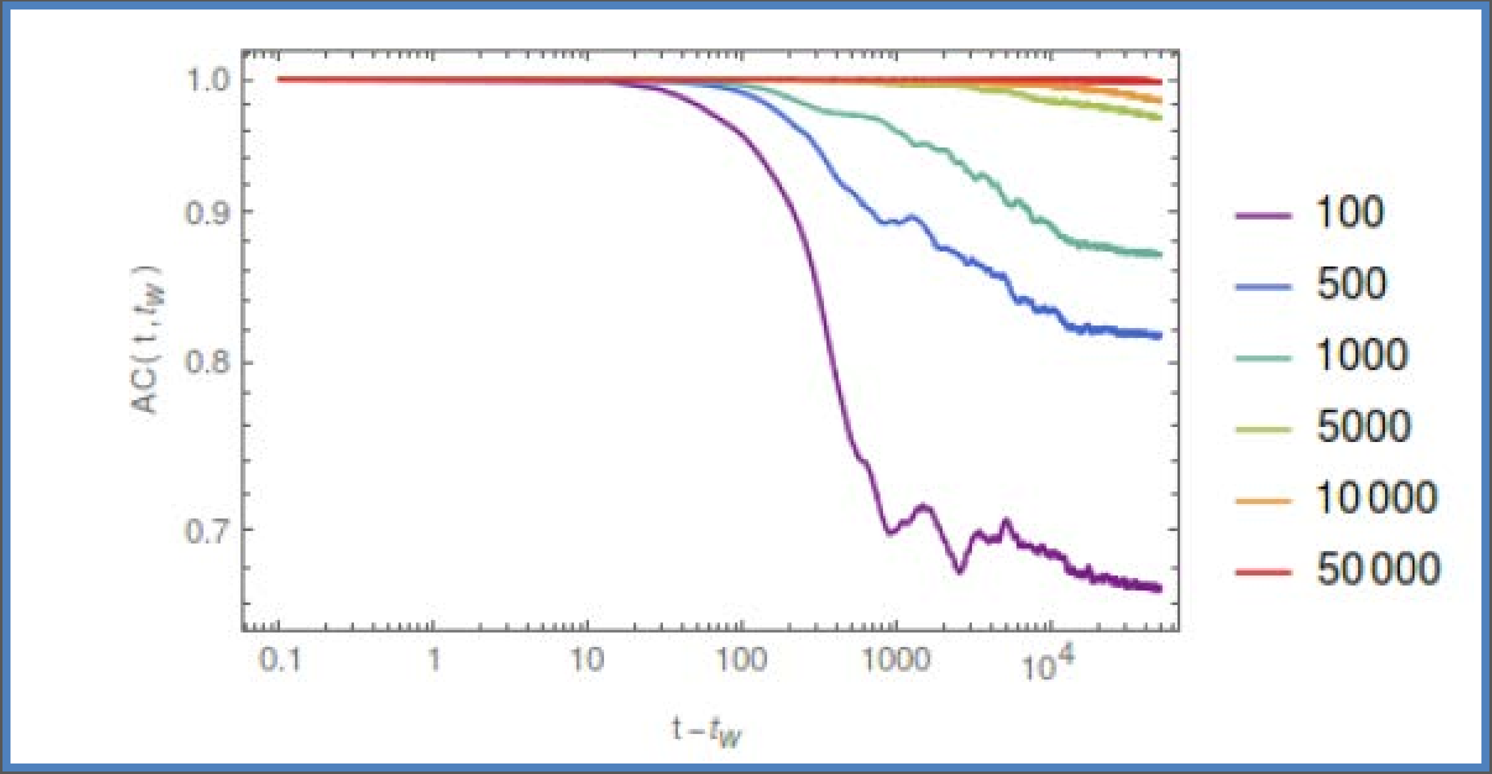
- **I. choose g_i randomly from $g(\omega)$ for all sites, or**
- **II. deterministically such that the maximal ω_i is in the center of the grid and the minimal ones at the boundaries,**

otherwise $\mu = 1$ and $\sigma = 0.01$ comparable to Kuramoto with $g(\omega) = \omega$.

Regular frequency distribution with the largest in the center



I. Choosing the protocol as before by starting from a state, where all phases are the same and then choosing the coupling deeply in the multistable regime, measuring the autocorrelation time as before, we observe

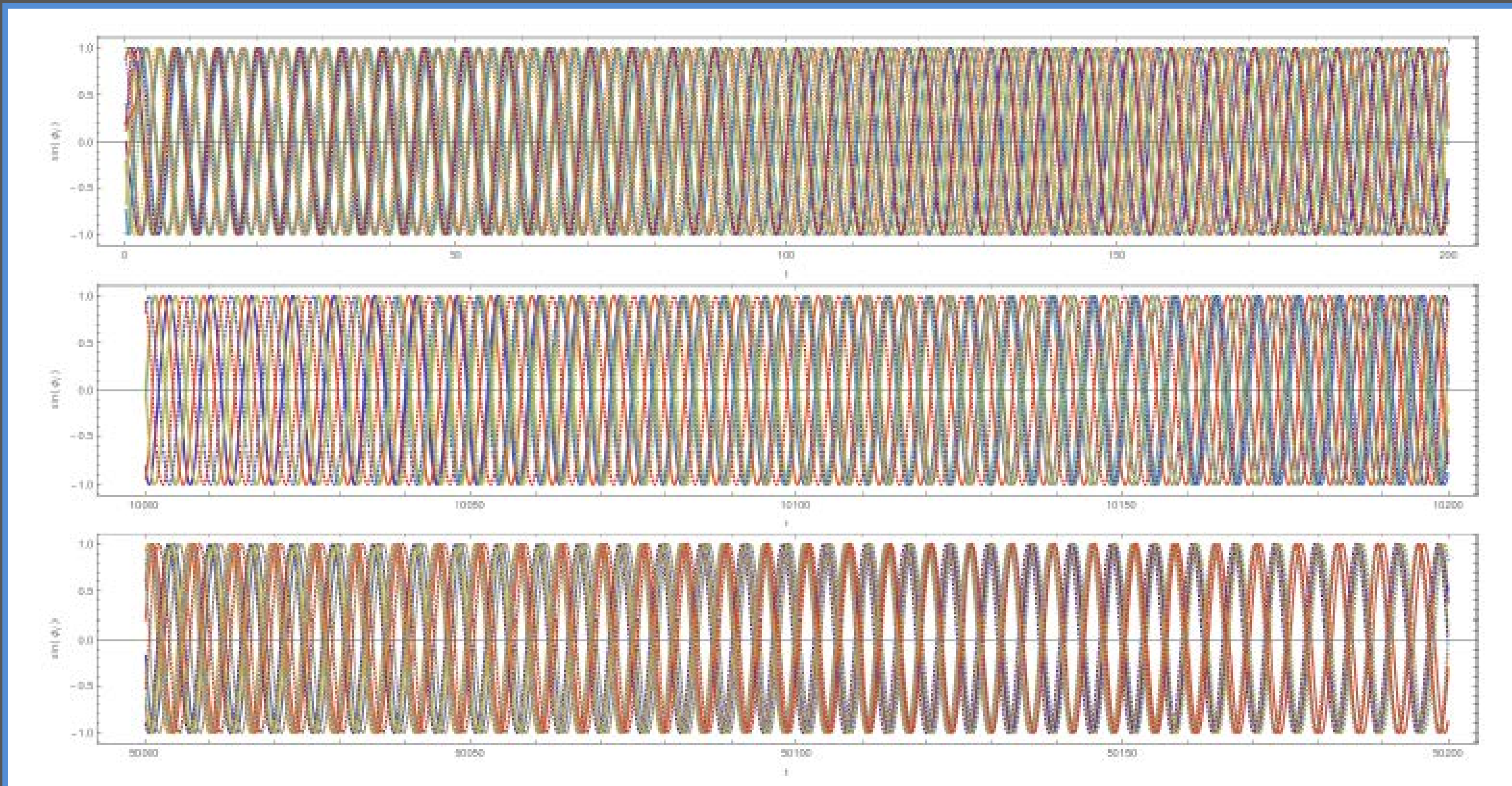


4x4-lattice with I. random selection of ω according to g(ω), averaged over different realizations of g(ω)

The role of disorder is similar to the role of noise w. r. t. the phase evolution in time: the synchronization patterns change as if the system explores a rich landscape

4x4 lattice random distribution of ω with $\mu = 1$ and $\sigma = 0.01$

$\sin \Phi_i$

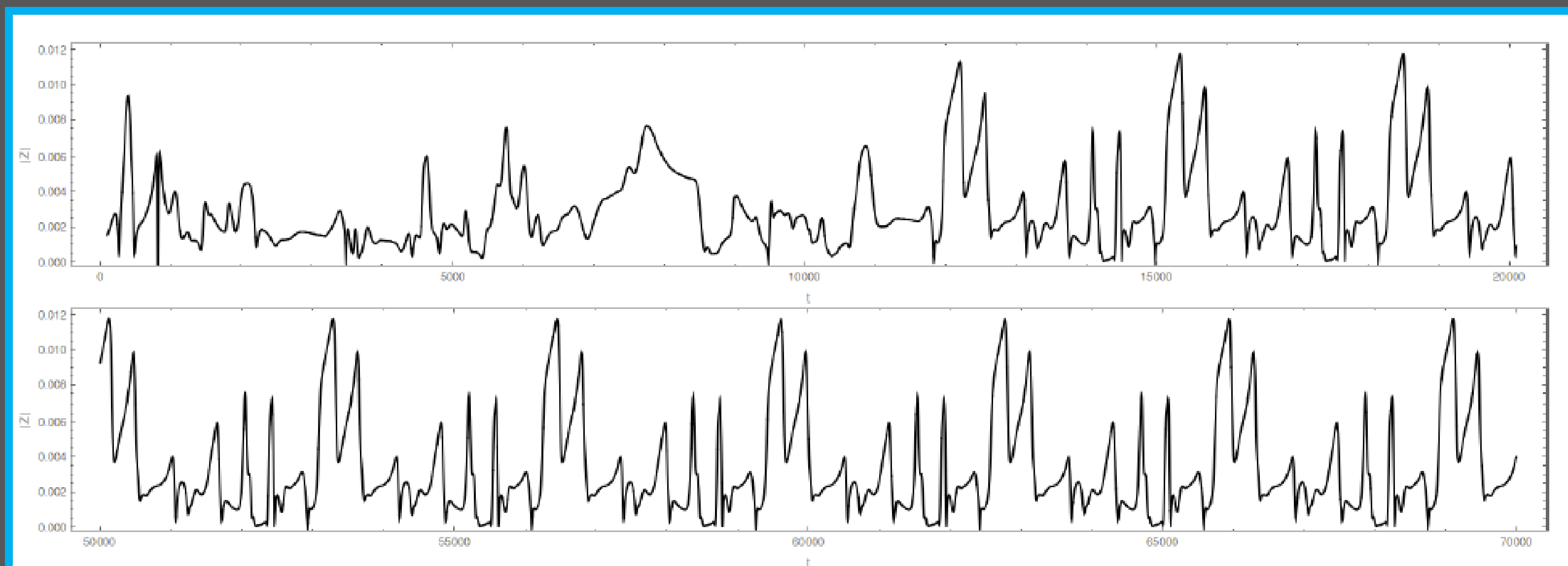


t

i)0-200, ii) 10000-10200, iii)50000-50200 t.u.

but in contrast to being randomly kicked from one metastable state to the next, here the different intermediate metastable states are visited periodically after very long periods, as it can be seen from the order parameter:

|Z|



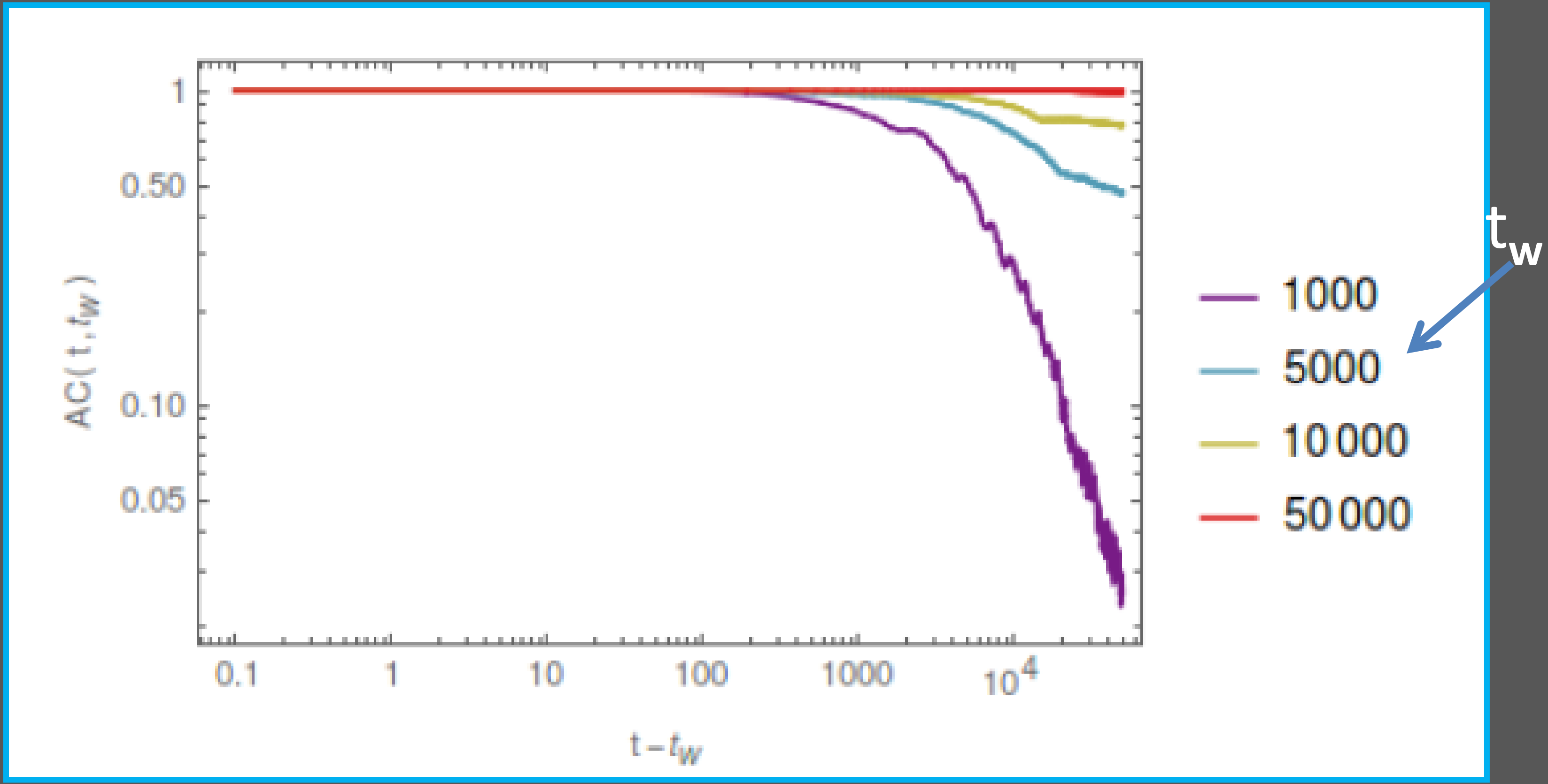
t

2000 t.u. in contrast to a period of 10 t.u. in the period of oscillator phases

$$Z(t) := |Z|e^{i\theta} = \frac{1}{N}\sum_{j=1}^{N} e^{i\phi_j}$$

II. Deterministic realization of frequencies for a 4x4 lattice:

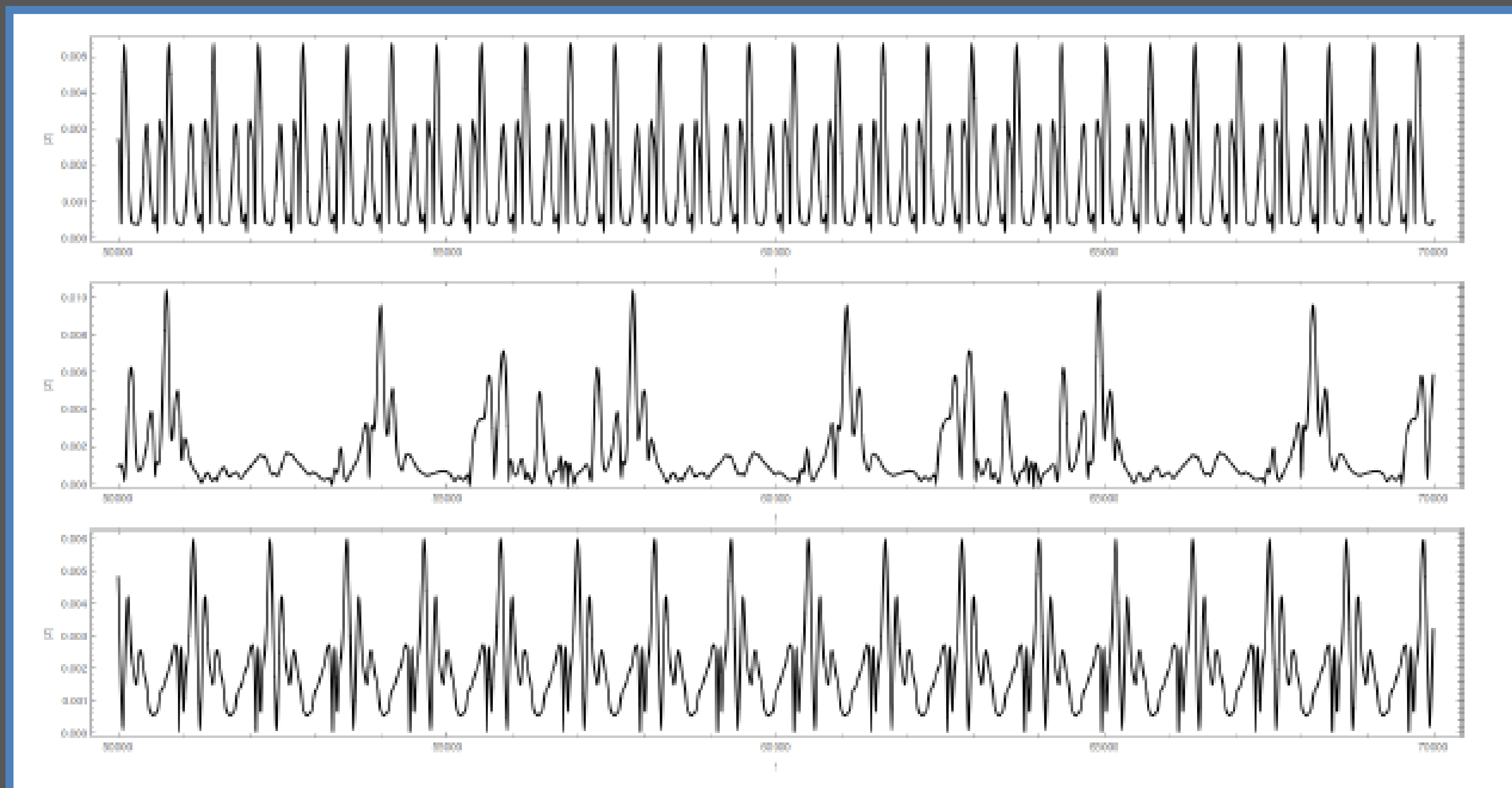
Aging in the autocorrelation function with averages over different initial conditions



4x4-lattice with II. deterministic selection of ω according to g(ω), averaged over different initial conditions only

Deterministic implementation of the natural frequency distribution:
Order parameter for 3 slightly different initial conditions after 50000 t.u

|Z|

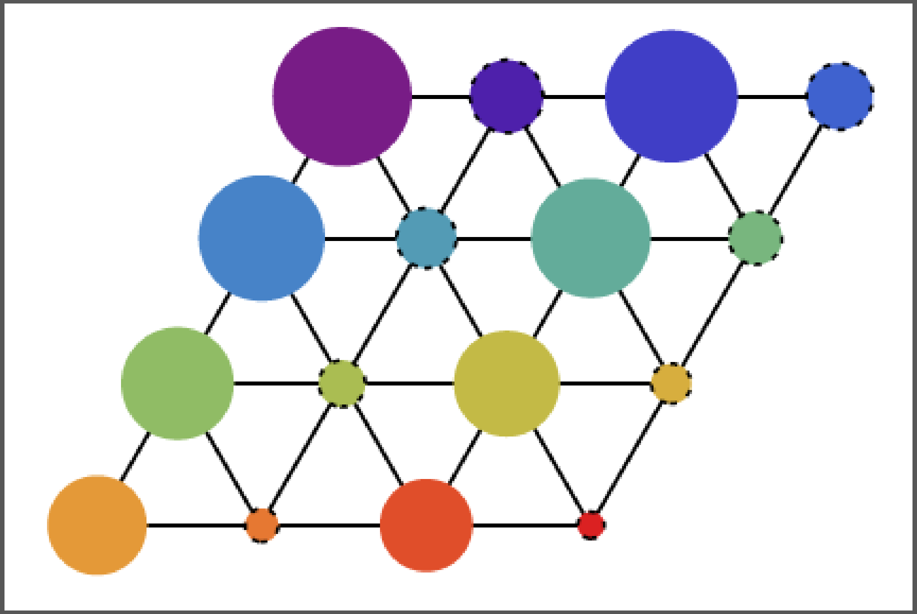


t

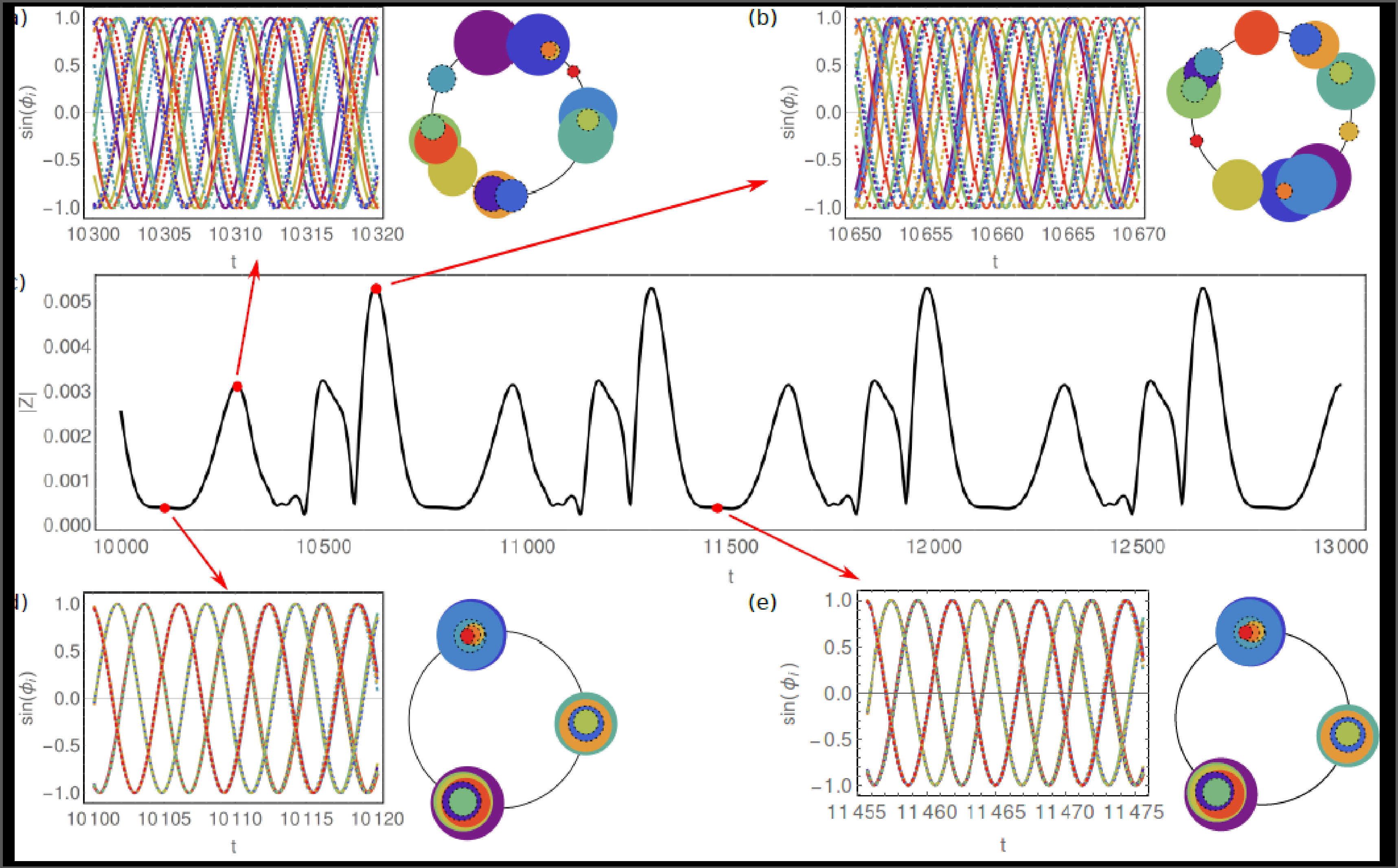
⟹ obvious multistability

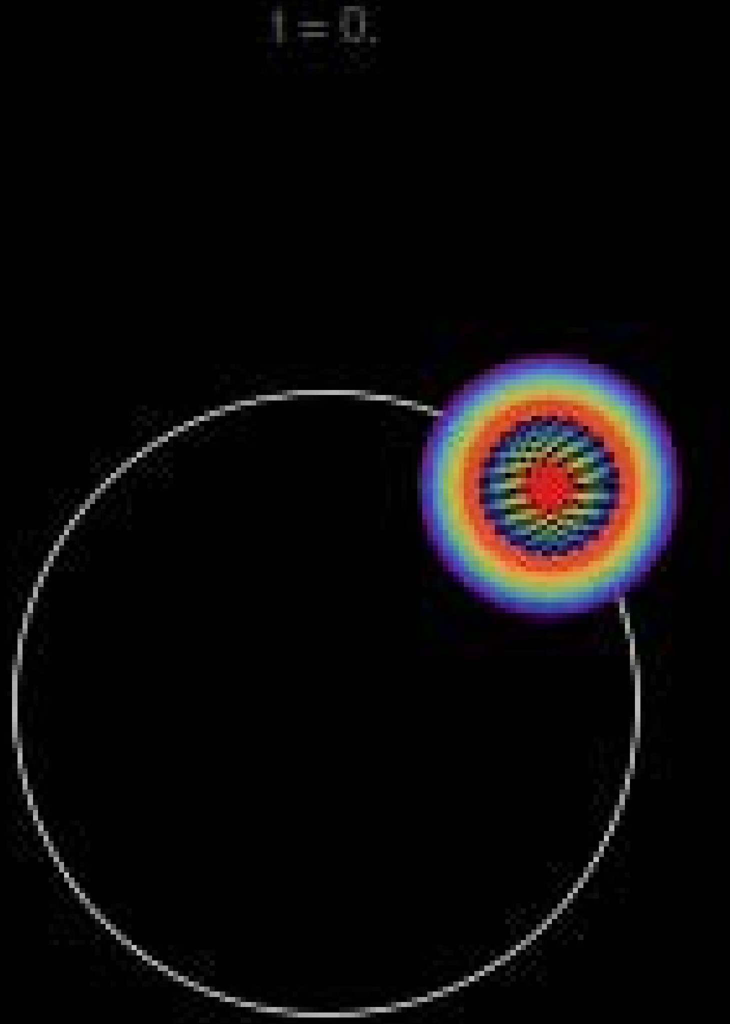
Periods of the order parameter between 500 and 5000 t.u.

Consider in more detail individual phase trajectories, and choose in parallel a representation on the unit circle



What do the phase trajectories look like for selected points of the order parameter?





Heteroclinic orbits connecting unstable limit cycles?

- **Notice: we have an emergence of very long time scales out of short time scales from the individual oscillations.**

- The rather rich attractor space is now deterministically explored along very long closed paths, depending on the initial conditions and/or on the realization of the random distribution of $g(\omega)$.

Histograms of order parameter periods

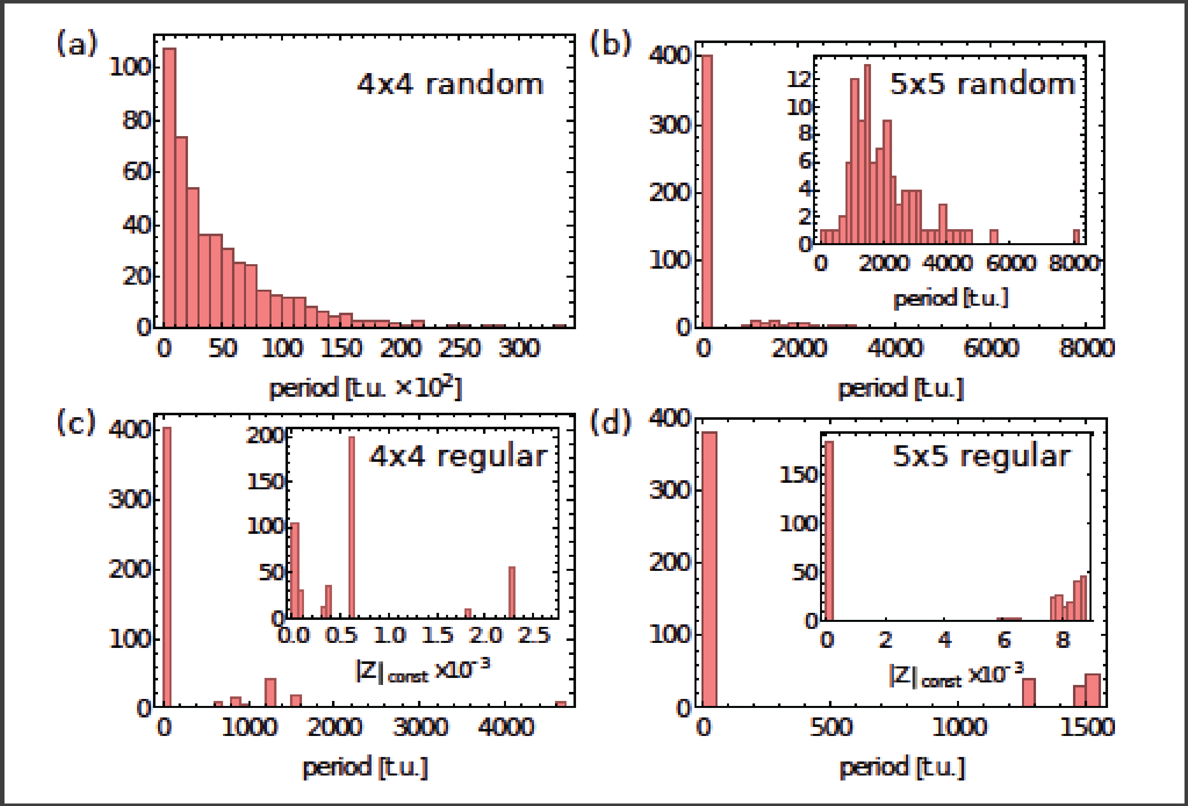
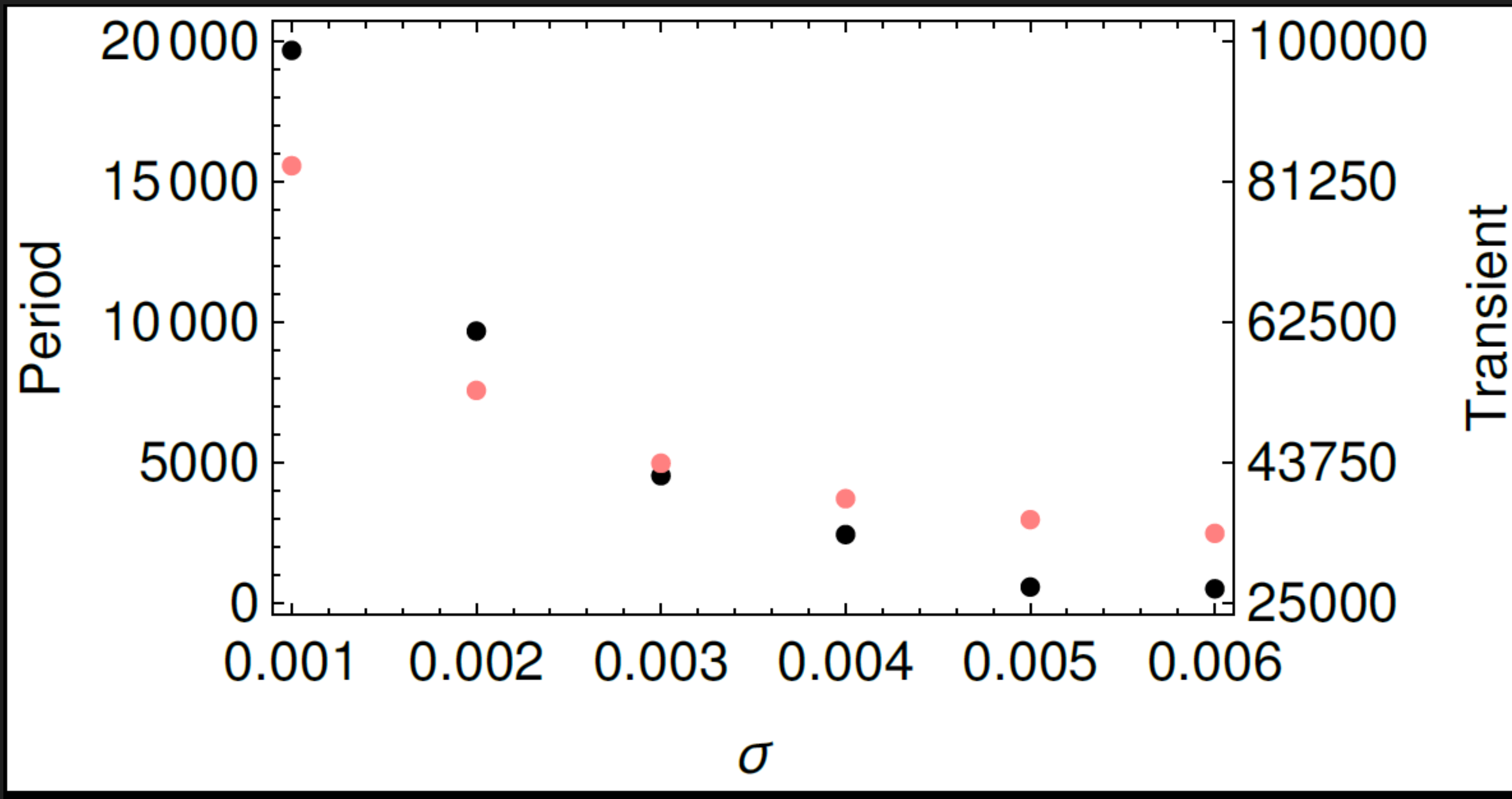


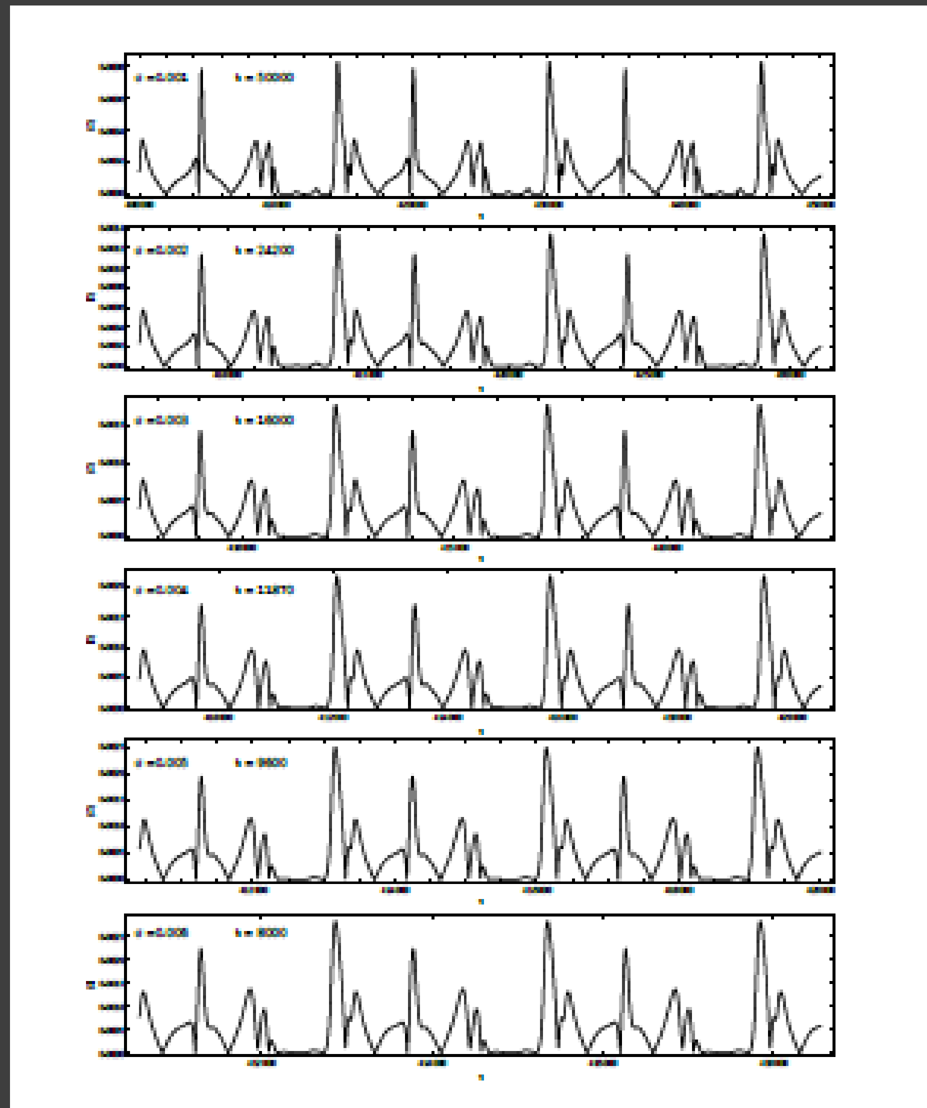
Figure 4. Histograms of order parameter periods as a function of the period length for 500 realizations of $g(\omega)$ for random ω's, and 500 different initial conditions for regular ω's for two different lattice sizes. Parameters are $\mu = 1$, $\sigma = 0.01$, $\kappa = -2$, initial conditions are taken from a vicinity of the fixed point at $\varphi_i = 0.5$.

What is the influence of the width of g(ω) ?



Both the period of the heteroclinic cycle and the transient time towards it increase with decreasing width of regular frequency distributions.

Self-similar temporal sequences



σ=0.001 t=50000

σ=0.002 t=24200

σ=0.003 t=16000

σ=0.004 t=11870

σ=0.005 t= 9600

σ=0.006 t= 8000

Notice: $\boxed{\sigma t \;=\; 48.4 \pm 0.9.}$

In particular we have long transients

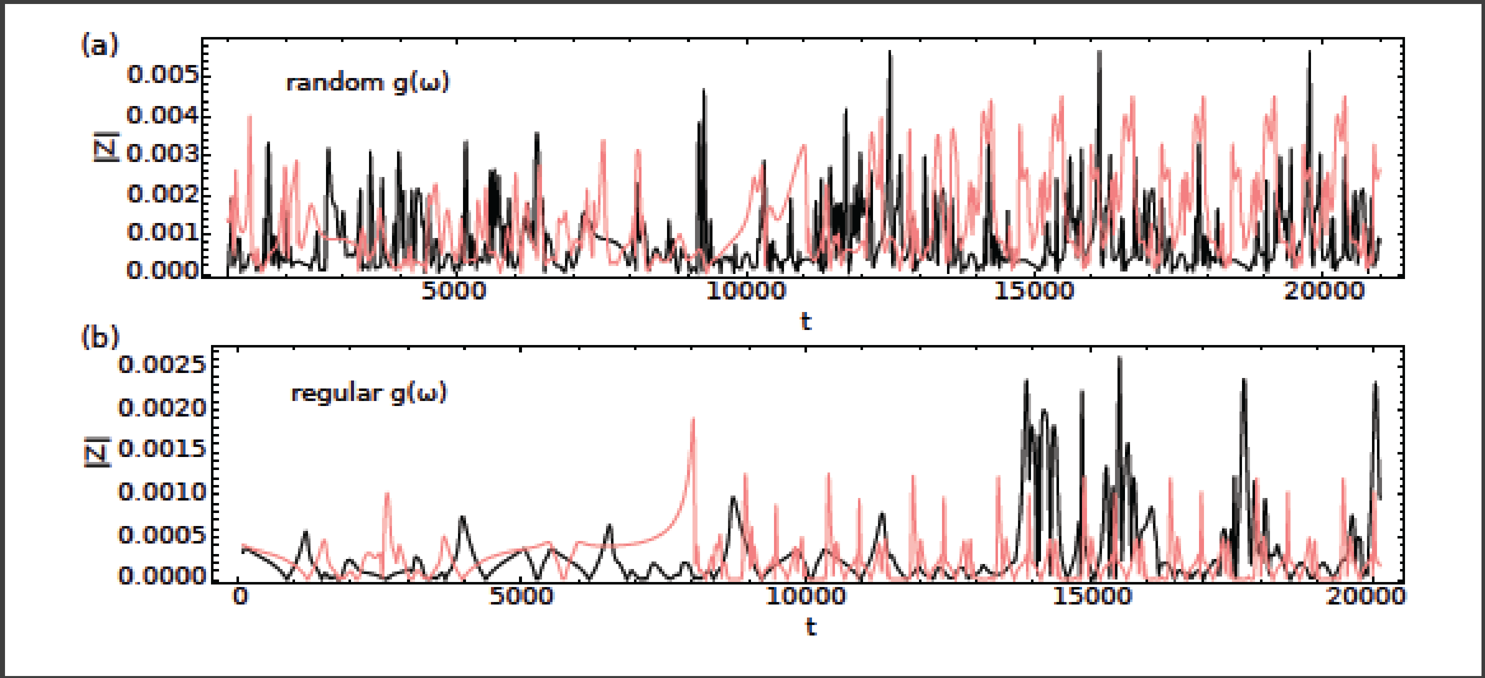


Figure 2. Transients of the order parameter for initial conditions around the fixed point. Two trajectories in black and red are shown for a 4×4 hexagonal lattice for random ω's, corresponding to two different realizations of $g(\omega)$ with the same mean μ and standard deviation σ, and regular ω's, corresponding to two different initial conditions, respectively. All initial conditions are taken from a small radius around $\varphi_i = 0.5\ \forall i$. The parameters are $\mu = 1$, $\sigma = 0.01$, and $\kappa = -2$.

Aging Mechanism in Deterministic Systems:

- Rich attractor space with long and versatile inherent time scales that got explored formerly via noise and now during the deterministic time evolution.

- Depending on the waiting time, a different part gets explored, with which the second measurement in the autocorrelation function correlates.

- The smaller the width of the distributions, the longer on average the periods of the heteroclinic orbit.

D. Labavic & H. M-O, Emerging long orbits and self-similar temporal patterns in classical oscillators, submitted 2017.

Outlook to open questions and next steps

- **Other manifestations of aging in the response to external forces**

- **Physical aging in other oscillatory systems like genetic circuits**

Any relation to biological aging?

physical aging in the sense of an age dependent response to perturbations
biological aging e.g. in the sense of a deterioration of pacemaker cells.

APPLICATIONS OF FRUSTRATION BEYOND SPIN GLASSES AND BIOLOGY

So far to oscillatory and excitable systems, but also to:

- **social science** and the approach of social balance
- **computer science** and satisfiability problems

  F. Radicchi, D. Vilone, S. Yoon and H.M-O, PRE75, 026106 (2007)
- **smart energy grids and the Braess paradox (work by Witthaut and Timme,**

  New J. Phys. 14, 083036 **2012)**

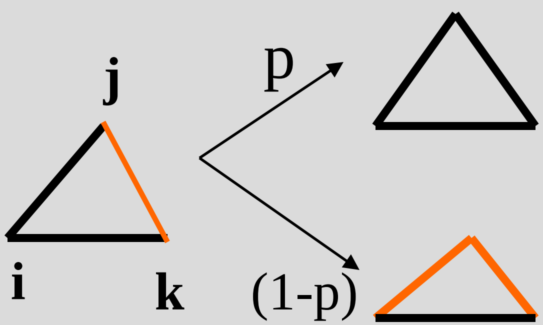
REDUCTION OF FRUSTRATION IN “SPIN-LIKE” SOCIAL SYSTEMS

Social balance and triad dynamics on various given topologies

Network: all-to-all topology
Nodes: individuals
Links: +1 for friends, -1 for enemies

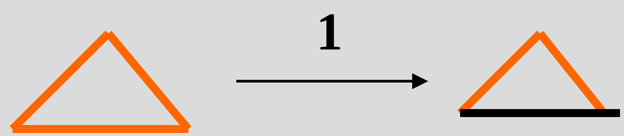
Antal et al., PRE 72,036121 (2005)

Triad dynamics to approach a balanced state (the “paradise”)



imbalanced
frustrated

balanced



imbalanced

balanced

p is the propensity parameter driving a certain phase transition

Consider first artificial networks with triad dynamics where the frustration can be reduced even to zero.

Questions of interest here:

- **What is the time to reach the frustration-free or balanced states?**
- **Can a local stochastic algorithm find these states in a finite time?**
- **How does the time depend on the algorithm that is used?**

Our generalizations: (F. Radicchi, D.Vilone, HMO: PRE**75**, 026106 (2007);PRE **75**, 021118 (2007); J.Stat.Phys.**129**, 593 (2007))

- **From triad to k-cycle dynamics** (distinguish between even and odd k)
- **From all-to-all to random, diluted and to regular topologies**
- **Mapping to optimization problems in computer science**

Comparison between the approach to social balance and SAT

Different: meaning of frustration

In common: formal criterion for frustration
algorithm driven by the reduction of frustration via flipping signs
solution of the problem exists (here ferromagnetic state). Question: how to find it?

Is it possible via a local stochastic algorithm within a finite time?

Whether yes or no depends on

- **the propensity parameter p**
- **the degree of dilution α**
- **the network topology**

Summary and Conclusions

Concept of frustration is a very generic one. It

- can explain the proliferation of attractors and effects going along with frustration like order-by-disorder or aging in oscillatory or excitable systems

- has implications for control and design
  to keep a device flexible
  to avoid conflict like geometric frustration in power grids

- leads to interesting perspectives for out-of-equilibrium physics in a broader class of systems with possible applications to biology and beyond

F. Radicchi, D. Vilone, S. Yoon and H.M-O, PRE75, 026106 (2007);
PRE 75, 021118 (2007); J.Stat.Phys.129, 593 (2007).
P. Kaluza and H.M-O, Chaos 20, 043111 (2010).
F. Ionita, D. Labavic, M. Zaks and H.M-O, Eur. Phys. J. B 86(12), 511(2013)
F. Ionita, and H.M-O, Phys.Rev.Lett.112, 094101 (2014)
D. Labavic & H. M-O, Physical aging and emerging long-period orbits in deterministic oscillatory systems, submitted 2017.

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